HERMITIAN K-THEORY FOR STABLE ∞-CATEGORIES II: COBORDISM CATEGORIES AND ADDITIVITY

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To Andrew Ranicki.

Abstract. We define Grothendieck-Witt spectra in the setting of Poincaré ∞-categories and show that they fit into an extension with a K- and an L-theoretic part. As consequences, we deduce localisation sequences for Verdier quotients and generalisations of Karoubi's fundamental and periodicity theorems for rings in which 2 need not be invertible. Our set-up allows for the uniform treatment of such algebraic examples alongside homotopy-theoretic generalisations: For example, the periodicity theorem holds for complex oriented E_1-rings, and we show that the Grothendieck-Witt theory of parametrised spectra recovers Weiss and Williams' LA-theory.

Our Grothendieck-Witt spectra are defined via a version of the hermitian Q-construction, and a novel feature of our approach is to interpret the latter as a cobordism category. This perspective also allows us to give a hermitian version – along with a concise proof – of the theorem of Blumberg, Gepner and Tabuada, and provides a cobordism theoretic description of the aforementioned LA-spectra.

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INTRODUCTION

Overview. Unimodular symmetric and quadratic forms are ubiquitous objects in mathematics appearing in contexts ranging from norm constructions in number theory to surgery obstructions in geometric topology. Their classification, however, even over simple rings such as the integers, remains out of reach. A simplification, following ideas of Grothendieck for the study of projective modules, suggests to consider for a commutative ring R (for ease of exposition) the abelian group $\mathrm{GW}_0^{\mathrm{q}}(R)$ given as the group completion of the monoid of isomorphism classes of finitely generated projective R-modules P, equipped with a unimodular quadratic (say) form q, with addition the orthogonal sum

$$[P,q]+[P',q'] = [P\oplus P', q\perp q'].$$

This group, commonly known as the (quadratic) Grothendieck-Witt group of R, was given a homotopy-theoretical refinement at the hands of Karoubi and Villamayor in [KV71], by adapting Quillen's approach to higher algebraic K-theory.

For this, one organises the collection of pairs (P,q) of unimodular quadratic forms into a groupoid $\mathrm{Unimod}^{\mathrm{q}}(R)$, which may be viewed as an E_∞-monoid using the symmetric monoidal structure on $\mathrm{Unimod}^{\mathrm{q}}(R)$ arising from the orthogonal sum considered above. One can then take the group completion to obtain an E_∞-group

$$\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R) = \mathrm{Unimod}^{\mathrm{q}}(R)^{\mathrm{grp}},$$

the classical (quadratic) Grothendieck-Witt space, whose group of components is the Grothendieck-Witt group described above. By definition, the higher Grothendieck-Witt groups of R are the homotopy groups of $\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R)$.

There are variants for symmetric bilinear and even forms, and instead of starting with a commutative ring, one can study unimodular hermitian forms valued in an invertible $R\otimes_{\mathbb{Z}} R$-module M equipped with an involution (subject to an invertibility condition) also for non-commutative R; this generality includes both the case of a ring R with anti-involution by considering $M = R$, and also skew-symmetric and skew-quadratic forms by changing the involution on M by a sign. Polarisation in general produces maps

$$\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R,M) \longrightarrow \mathcal{GW}^{\mathrm{ev}}_{\mathrm{cl}}(R,M) \longrightarrow \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R,M)$$

which are equivalences if 2 is a unit in R. In fact, there are further generalisations based on the notion of form parameters, but we will refrain from engaging with that generality in the introduction.

In the present paper, we establish a decomposition of the Grothendieck-Witt space into a K-theoretic and an L-theoretic part, the latter of which is closely related to Witt groups of unimodular forms: For $r \in \{\mathrm{q}, \mathrm{ev}, \mathrm{s}\}$, the Witt group $\mathrm{W}^r(R,M)$ of the pair (R,M) is given by dividing isomorphism classes of unimodular M-valued forms by those admitting a Lagrangian. In low degrees, the relation takes the form of an exact sequence

$$\mathrm{K}_0(R)_{\mathrm{C}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0^r(R, M) \longrightarrow \mathrm{W}^r(R, M) \longrightarrow 0;$$

here, the map labelled hyp assigns to a projective module P its hyperbolisation $P \oplus \mathrm{Hom}_R(P, M)$ equipped with the evaluation form, and the C_2-coinvariants on the group $\mathrm{K}_0(R)$ on the left are formed with respect to the action $P \mapsto \mathrm{Hom}_R(P, M)$. The first goal of the paper is to extend this to a long exact sequence with L-groups playing the role of higher Witt groups. Such results are well-known principally from the work of Karoubi and Schlichting if 2 is a unit in R, and have lead to a good understanding of Grothendieck-Witt theory relative to K-theory for two reasons: Firstly, Witt groups are rather accessible. As an example let us mention that Voevodsky's solution to the Milnor conjecture provides a complete filtration of the Witt group $\mathrm{W}(k)$ for any field k not of characteristic 2 with filtration quotients $\mathrm{H}^*(\mathrm{Gal}(\overline{k}/k), \mathbb{Z}/2)$, and an older result of Kato achieves a similar description in characteristic 2, see [Kat82, Voe03, OVV07]. Secondly, by work of Ranicki [Ran92] the higher L-groups satisfy $\mathrm{L}_{i+2}(R, M) = \mathrm{L}_i(R, -M)$ if 2 is invertible in R, and are thus in particular 4-periodic, which greatly reduces the computational complexity. The second goal of the present paper series is to describe the extent to which such periodicity statements still hold if 2 is not invertible in R. Let us also mention that the K-theoretic part of the description is rather indifferent to the invertibility of 2 in R, so from an understanding of the L-theoretic term, one can often deduce absolute statements about Grothendieck-Witt theory by appealing to the recent progress in the understanding of algebraic K-theory. We will take up this thread in the third instalment of the series.

History and main result. To state our results, let us give a more detailed account of the ingredients. The study of Grothendieck-Witt spaces begins by comparing them to Quillen's algebraic K-theory space $\mathcal{K}(R)$ defined as the group completion of the groupoid of finitely generated projective modules over R. To this end, one has

$$\mathrm{fgt} : \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M) \to \mathcal{K}(R) \quad \text{and} \quad \mathrm{hyp} : \mathcal{K}(R) \to \mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M),$$

the former extracting the underlying module of a unimodular form, the latter induced by the hyperbolisation construction explained above.

In his fundamental papers [Kar80a, Kar80b], Karoubi analysed the case in which 2 is a unit in R (so no distinction between the three flavours of Grothendieck-Witt groups is necessary). He considered the spaces

$$\mathcal{U}_{\mathrm{cl}}(R, M) = \mathrm{fib}(\mathcal{K}(R) \xrightarrow{\mathrm{hyp}} \mathcal{GW}_{\mathrm{cl}}(R, M)) \quad \text{and} \quad \mathcal{V}_{\mathrm{cl}}(R, M) = \mathrm{fib}(\mathcal{GW}_{\mathrm{cl}}(R, M) \xrightarrow{\mathrm{fgt}} \mathcal{K}(R)),$$

produced equivalences

$$\Omega\mathcal{U}_{\mathrm{cl}}(R, -M) \simeq \mathcal{V}_{\mathrm{cl}}(R, M),$$

and moreover showed that the cokernels $\mathrm{W}_i(R, M)$ of $\mathrm{K}_i(R) \xrightarrow{\mathrm{hyp}} \mathrm{GW}_i(R, M)$ satisfy

$$\mathrm{W}_i(R, M)[\tfrac{1}{2}] \cong W_{i+2}(R, -M)[\tfrac{1}{2}]$$

and are in particular 4-periodic up to 2-torsion. In fact, Karoubi shows that this latter statement also holds without the assumption that 2 be invertible in R; in other words, the additional difficulties of Grothendieck-Witt theory as compared to K-theory are concentrated at the prime 2. These results are nowadays known as Karoubi's fundamental and periodicity theorems and form one of the conceptual pillars of hermitian K-theory; they permit one to inductively deduce results on higher Grothendieck-Witt groups from information about algebraic K-theory on the one hand and about $\mathrm{W}_i(R, \pm M)$ for $i = 0, 1$ on the other.

To control the behaviour of the 2-torsion in the cokernel of the hyperbolisation map, Kobal introduced in [Kob99] refinements of the hyperbolic and forgetful maps: By the invertibility assumption on M, the functor taking M-valued duals induces an action of the group C_2 on the algebraic K-theory spectrum, and we denote the arising C_2-spectrum by $\mathrm{K}(R, M)$ and similarly by $\mathcal{K}(R, M)$ the K-theory C_2-space. The maps above then refine to a sequence

$$\mathcal{K}(R, M)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M) \longrightarrow \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M) \xrightarrow{\mathrm{fgt}} \mathcal{K}(R, M)^{\mathrm{hC}_2},$$

whose composite is the norm on $\mathcal{K}(R, M)$. Kobal used these refinements to show that, if 2 is invertible in R, the cofibre of $\mathrm{hyp} : \mathcal{K}(R, M)_{\mathrm{hC}_2} \to \mathcal{GW}_{\mathrm{cl}}(R, M)$ is 4-periodic on the nose.

The next major steps forward were then taken by Schlichting in [Sch17], who introduced (non-connective) Grothendieck-Witt spectra for differential graded categories with duality in which 2 is invertible. He used these to give a new proof of Karoubi's fundamental theorem by first establishing the existence of a fibre sequence

$$\mathrm{GW}_{\mathrm{cl}}(R, M[-1]) \xrightarrow{\mathrm{fgt}} \mathrm{K}(R, M) \xrightarrow{\mathrm{hyp}} \mathrm{GW}_{\mathrm{cl}}(R, M),$$

which he termed the Bott sequence; here $\mathrm{GW}_{\mathrm{cl}}(R, M[i])$ is the Grothendieck-Witt spectrum of the category $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ of bounded complexes of finitely generated projective R-modules with its duality determined by $M[i]$. For $i = 0$ (in which case we suppress it from notation) Schlichting shows that indeed $\Omega^{\infty}\,\mathrm{GW}_{\mathrm{cl}}(R, M) \simeq \mathcal{GW}_{\mathrm{cl}}(R, M)$. The salient feature that relates this sequence to Karoubi's theorem is the existence of an equivalence $\mathrm{GW}_{\mathrm{cl}}(R, M[-2]) \simeq \mathrm{GW}_{\mathrm{cl}}(R, -M)$. Still assuming 2 invertible in R, Schlichting furthermore showed that the (4-periodic) homotopy groups of the cofibre of the refined hyperbolic map $\mathrm{hyp}\colon\ \mathrm{K}(R, M)_{\mathrm{hC}_2} \to \mathrm{GW}_{\mathrm{cl}}(R, M)$ are indeed given by the Witt groups $\mathrm{W}(R, M)$ and $\mathrm{W}(R, -M)$ in even degrees and by Witt groups of formations in odd degrees.

This led to the folk theorem that if 2 is a unit in R, the cofibre of $\mathrm{hyp}\colon\ \mathrm{K}(R, M)_{\mathrm{hC}_2} \to \mathrm{GW}_{\mathrm{cl}}(R, M)$ is given by Ranicki's L-theory spectum $\mathrm{L}(R, M)$ from [Ran92], whose homotopy groups are well-known to match Schlichting's results, though as far as we are aware no account at the level of spectra has appeared in the literature.

Let us not fail to mention that Schlichting also introduced a variant of symmetric Grothendieck-Witt spectra without the assumption that 2 is invertible in R in [Sch10a], that satisfy localisation results by the celebrated [Sch10b]. These are, however, of slightly different flavour in that they should relate to non-connective K-theory, though to the best of our knowledge, this is not developed in the literature. To differentiate, we will refer to them as Karoubi-Grothendieck-Witt spectra, and relegate a thorough discussion to the fourth part of this series of papers.

The strategy described above has for example lead to an almost complete computation of the Grothendieck-Witt groups of $\mathbb{Z}[\frac{1}{2}]$ in [BK05] and to great structural insight by controlling the 2-adic behaviour of the forgetful map $\mathrm{GW}_{\mathrm{cl}} \to \mathrm{K}^{\mathrm{hC}_2}$ in [BKSØ15] under the assumption that 2 is a unit. Without this assumption, however, many of the methods employed break down. In particular, the relation of Grothendieck-Witt-groups to L-groups remained mysterious: In particular, it seems well-known that the cofibre of the hyperbolisation map in the quadratic case cannot directly relate to quadratic L-spectra. If 2 is not invertible in R, there are indeed many flavours of L-groups, and as far as we are aware not even a precise conjecture has been put forward. In contrast to this situation, Karoubi conjectured in [Kar09] a precise form in which his fundamental theorems should extend to general rings: It is not only the sign that changes when passing from $\mathrm{U}(R, M)$ to $\mathrm{V}(R, -M)$ but also the form parameter. A similar suggestion was made by Giffen, see [Wil05]. In what is hopefully evident notation, they predicted

$$\Omega\,\mathrm{U}^{\mathrm{q}}_{\mathrm{cl}}(R, -M) \simeq \mathrm{V}^{\mathrm{ev}}_{\mathrm{cl}}(R, M) \quad \text{and} \quad \Omega\,\mathrm{U}^{\mathrm{ev}}_{\mathrm{cl}}(R, -M) \simeq \mathrm{V}^{\mathrm{s}}_{\mathrm{cl}}(R, M).$$

In this paper series along with its companion [HS21], we entirely resolve these questions. In the present paper, we obtain the extensions of Karoubi's periodicity and fundamental theorem, affirming in particular the conjecture of Karoubi and Giffen described above, and also determine the cofibre of the hyperbolisation map in terms of an L-theory spectrum. In distinction with the variants usually employed for example in geometric topology, the L-spectra appearing are generally not 4-periodic. The third part of this series is devoted to a detailed study of these spectra, and in particular an investigation of their periodicity properties. While the results of that paper are largely specific to the case of discrete rings, the results of the present paper also apply much more generally to schemes, E_{∞}-rings, parametrised spectra among others.

Our approach is based on placing Grothendieck-Witt- and L-theory into a common general framework, namely the setting of Poincaré ∞-categories, introduced by Lurie in his approach to L-theory [Lur11], and developed in detail in the first part of this series. A Poincaré ∞-category is a small stable ∞-category $\mathcal{C}$ together with a certain kind of functor $\Omega\colon\ \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ which encodes the type of form (such as, quadratic, even or symmetric) under consideration. The requirements on Ϙ are such, that it, in particular, yields an associated duality equivalence $\mathrm{D}_{\Omega}\colon\ \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

As mentioned, Lurie defined L-theory for general Poincaré ∞-categories, and it is by now standard to view K-theory as a functor on stable ∞-categories. The duality D_{Ω} induces a C_2-action on the K-spectrum of a Poincaré ∞-category, and we will denote the resulting C_2-spectrum by $\mathrm{K}(\mathcal{C}, Ϙ)$. Adapting the hermitian Q-construction, we here also produce a Grothendieck-Witt spectrum $\mathrm{GW}(\mathcal{C}, Ϙ)$ in this generality. To explain how this generalises the Grothendieck-Witt theory of discrete rings, take $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(R)$, the stable subcategory of the derived ∞-category $\mathcal{D}(R)$ spanned by the perfect complexes over R. As part of Paper [I] we constructed Poincaré structures

$$Ϙ^{\mathrm{q}}_M \Longrightarrow Ϙ^{\mathrm{gq}}_M \Longrightarrow Ϙ^{\mathrm{ge}}_M \Longrightarrow Ϙ^{\mathrm{gs}}_M \Longrightarrow Ϙ^{\mathrm{s}}_M$$

connected by maps as indicated: Roughly, the outer two assign to a chain complex its spectrum of homotopy coherent quadratic or symmetric M-valued forms, whereas the middle three are the more subtle animations, or in more classical terminology non-abelian derivations, of the functors

$$\mathrm{Quad}_M, \mathrm{Ev}_M, \mathrm{Sym}_M : \mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{A}b$$

parametrising ordinary M-valued quadratic, even and symmetric forms, respectively. The comparison maps between these are equivalences if 2 is a unit in R, but in general they are five distinct Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$. Now, essentially by construction, the spectra

$$\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{q}}_M) = \mathrm{L}^{\mathrm{q}}(R, M) \quad \text{and} \quad \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{s}}_M) = \mathrm{L}^{\mathrm{s}}(R, M)$$

are Ranicki's 4-periodic L-spectra, but from the main result of [HS21] we find that it is the middle three Poincaré structures which give rise to the classical Grothendieck-Witt spaces, i.e. we have

$$\Omega^\infty \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gq}}_M) \simeq \mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M), \quad \Omega^\infty \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{ge}}_M) \simeq \mathcal{GW}^{\mathrm{ev}}_{\mathrm{cl}}(R, M)$$

$$\text{and} \quad \Omega^\infty \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gs}}_M) \simeq \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M).$$

This mismatch (which is also the reason for carrying the subscript cl through the introduction) explains much of the subtlety that arose in previous attempts to connect Grothendieck-Witt- and L-theory.

In case 2 is invertible in R, the identification extends to $\mathrm{GW}_{\mathrm{cl}}(R, M) \simeq \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gs}}_M)$ and we will therefore use the names $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}$, $\mathrm{GW}^{\mathrm{ev}}_{\mathrm{cl}}$, and $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}$ also for the Grothendieck-Witt spectra of the Poincaré ∞-categories considered above.

As the main result of the present paper, we provide extensions of Karoubi's periodicity theorem and Schlichting's extension of his fundamental theorem in complete generality:

Main Theorem. *For every Poincaré ∞-category* $(\mathcal{C}, Ϙ)$*, there is a fibre sequence*

$$\mathrm{K}(\mathcal{C}, Ϙ)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, Ϙ) \xrightarrow{\mathrm{bord}} \mathrm{L}(\mathcal{C}, Ϙ),$$

which canonically splits after inverting 2 *and a fibre sequence*

$$\mathrm{GW}(\mathcal{C}, Ϙ^{[-1]}) \xrightarrow{\mathrm{fgt}} \mathrm{K}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, Ϙ).$$

Here, we have used $Ϙ^{[i]}$ to denote the shifted Poincaré structure $\mathbb{S}^i \otimes Ϙ$. As in Schlichting's set-up, this operation satisfies

$$(\mathcal{D}^{\mathrm{p}}(R), (Ϙ^{\mathrm{q}}_M)^{[2]}) \simeq (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{q}}_{-M}) \quad \text{and} \quad (\mathcal{D}^{\mathrm{p}}(R), (Ϙ^{\mathrm{s}}_M)^{[2]}) \simeq (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{s}}_{-M}),$$

so if 2 is a unit in R, we, in particular, recover the periodicity results of Karoubi and Schlichting mentioned above, and extend the identification of the cofibre of the hyperbolisation map to the spectrum level. More importantly, however, if 2 is not invertible, we find

$$(\mathcal{D}^{\mathrm{p}}(R), (Ϙ^{\mathrm{gs}}_M)^{[2]}) \simeq (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{ge}}_{-M}) \quad \text{and} \quad (\mathcal{D}^{\mathrm{p}}(R), (Ϙ^{\mathrm{ge}}_M)^{[2]}) \simeq (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gq}}_{-M}),$$

whence the second part settles the conjecture of Giffen and Karoubi [Kar09, Conjecture 1]. Explicitly, we obtain:

Corollary. *For a discrete ring* R *and an invertible* R*-module* M *with involution, there are canonical equivalences*

$$\mathrm{U}^{\mathrm{q}}_{\mathrm{cl}}(R, -M) \simeq \mathbb{S}^1 \otimes \mathrm{V}^{\mathrm{ev}}_{\mathrm{cl}}(R, M) \quad \textit{and} \quad \mathrm{U}^{\mathrm{ev}}_{\mathrm{cl}}(R, -M) \simeq \mathbb{S}^1 \otimes \mathrm{V}^{\mathrm{s}}_{\mathrm{cl}}(R, M).$$

As a consequence of the first part of our Main Theorem, we obtain a direct relation between the Grothendieck-Witt spectra for different form parameters. As an implementation of Ranicki's L-theoretic periodicity results, Lurie produced canonical equivalences

$$\mathrm{L}(\mathcal{C}, \Qoppa^{[1]}) \simeq \mathbb{S}^1 \otimes \mathrm{L}(\mathcal{C}, \Qoppa).$$

Applying this four times, we obtain a stabilisation map

$$\mathrm{stab} :\ \mathbb{S}^4 \otimes \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M}) \simeq \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gq}}_{M}) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M}),$$

and as another articulation of periodicity we have:

Corollary. *The natural map* $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M) \to \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M)$ *fits into a commutative diagram*

$$\begin{tikzcd}
\mathrm{K}(R)_{\mathrm{hC}_2} \arrow[r, "\mathrm{hyp}"] \arrow[d, "\mathrm{id}"] & \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M) \arrow[r] \arrow[d] & \mathbb{S}^4 \otimes \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M}) \arrow[d, "\mathrm{stab}"] \\
\mathrm{K}(R)_{\mathrm{hC}_2} \arrow[r, "\mathrm{hyp}"] & \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M) \arrow[r, "\mathrm{bord}"] & \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M})
\end{tikzcd}$$

of fibre sequences, i.e. the cofibres of the two hyperbolisation maps differ by a fourfold shift.

If 2 is invertible in R, this shift on the right hand side is invisible since in that case $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M}) = \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{s}}_{M})$ is 4-periodic.

In the body of the text, we shall derive more general versions of these corollaries concerning Grothendieck-Witt groups associated to suitable pairs of form parameters. In particular, that form of the first corollary settles Conjectures 1 and 2 of [Kar09] in full, see the end of §4.3 and §4.5 for details.

Outlook. As mentioned, the main content of the third paper in this series is a detailed investigation of the spectra $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M})$. We show there that in non-negative degrees, $\pi_* \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M})$ is Ranicki's original version of symmetric L-theory from [Ran81], which he eventually abandoned in favour of $\mathrm{L}^{\mathrm{s}}(R, M)$ precisely because in general it lacks the 4-periodicity exhibited by the latter. In particular, the cofibre of the hyperbolisation map $\mathrm{K}(R)_{\mathrm{hC}_2} \to \mathrm{GW}^{\mathrm{gs}}(R, M)$ is not generally 4-periodic if 2 is not invertible in R.

Furthermore, improving a previous bound of Ranicki's, we show there that for R commutative and noetherian of global dimension d, the comparison maps

$$\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gq}}_{M}) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{ge}}_{M}) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_{M}) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{s}}_{M})$$

are equivalences in degrees past $d-2, d$ and $d+2$, respectively. Thus, in sufficiently high degrees, the periodic behaviour of the cofibre of the hyperbolisation map is restored, and surprisingly there is also no difference between the various flavours of Grothendieck-Witt groups. This allows one to use the inductive methods previously only available if 2 is invertible in more general situations. We demonstrate this by giving a solution for number rings of Thomasson's homotopy limit problem [Tho83], asking when the map

$$\mathrm{GW}^{\mathrm{s}}(R, M) \to \mathrm{K}(R, M)^{\mathrm{hC}_2}$$

is a 2-adic equivalence, and an essentially complete computation of $\mathrm{GW}^{r}(\mathbb{Z})$ where $r \in \{\pm\mathrm{s}, \pm\mathrm{q}\}$ (over rings in which 2 is not a zero-divisor, quadratic and even forms turn out to agree, as do skew-symmetric and skew-even ones), affirming a conjecture of Berrick and Karoubi from [BK05].

Before explaining the strategy of proof in the next section, let us finally mention that feeding Poincaré ∞-categories of parametrised spectra into our machinery produces, by our Main Theorem, another set of interesting objects, the LA-spectra introduced by Weiss and Williams in their study of automorphism groups of manifolds [WW14]. In this case, the results of the next section allow for an entirely new interpretation of these spectra, which sheds light on their geometric meaning. In particular, this furthers the program suggested by Williams in [Wil05] to connect the study of manifold topology more intimately with hermitian K-theory. We will spell this out in the third section of this introduction along with further results concerning discrete rings, that require a bit of preparation.

Hermitian K-theory of Poincaré ∞-categories. Let us now sketch in greater detail the road to our main results. Besides the set-up of Poincaré ∞-categories, the main novelty of our approach is its direct connection to the theory of cobordism categories of manifolds. To facilitate the discussion, recall that Cob_d has as objects $(d-1)$-dimensional closed, oriented, smooth manifolds, and cobordisms thereof as morphisms. The celebrated equivalence

$$|\mathrm{Cob}_d| \simeq \Omega^{\infty-1}\mathrm{MTSO}(d),$$

established by Galatius, Madsen, Tillmann and Weiss in [GTMW09] then lies at the heart of much modern work on the homotopy types of diffeomorphism groups [GRW14]; here $\mathrm{MTSO}(d)$ denotes the Thom spectrum of $-\gamma_d \to \mathrm{BSO}(d)$, where γ_d denotes the universal vector bundle over $\mathrm{BSO}(d)$.

Now, a Poincaré ∞-category $(\mathcal{C}, \Omega)$ determines a space of Poincaré objects $\mathrm{Pn}(\mathcal{C}, \Omega)$ to be thought of as the higher categorical generalisation of the groupoid $\mathrm{Unimod}(\boldsymbol{R}, \boldsymbol{M})$ of unimodular forms considered in the case of discrete rings above. Along with the Grothendieck-Witt spectrum, we produce for every Poincaré ∞-category $(\mathcal{C}, \Omega)$ an analogous cobordism ∞-category $\mathrm{Cob}(\mathcal{C}, \Omega) \in \mathrm{Cat}_\infty$ with objects given by $\mathrm{Pn}(\mathcal{C}, \Omega^{[1]})$ and morphisms given by spaces of Poincaré cobordisms, Ranicki style; here our dimension conventions adhere to those of the geometric setting.

As the technical heart of the present paper, we show the following version of the additivity theorem:

Theorem A. *If*

$$(\mathcal{C}, \Omega) \longrightarrow (\mathcal{D}, \Phi) \longrightarrow (\mathcal{E}, \Psi)$$

is a split Poincaré-Verdier sequence, then the second map induces a bicartesian fibration of ∞-categories

$$\mathrm{Cob}(\mathcal{D}, \Phi) \longrightarrow \mathrm{Cob}(\mathcal{E}, \Psi),$$

whose fibre over $0 \in \mathrm{Cob}(\mathcal{E}, \Psi)$ *is* $\mathrm{Cob}(\mathcal{C}, \Omega)$. *In particular, one obtains a fibre sequence*

$$|\mathrm{Cob}(\mathcal{C}, \Omega)| \longrightarrow |\mathrm{Cob}(\mathcal{D}, \Phi)| \longrightarrow |\mathrm{Cob}(\mathcal{E}, \Psi)|$$

of spaces.

Here, a Poincaré-Verdier sequence is a null-composite sequence, which is both a fibre sequence and a cofibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$, the ∞-category of Poincaré ∞-categories; we call it *split* if both underlying functors admit both adjoints. This requirement precisely makes the underlying sequence of stable ∞-categories $\mathcal{C} \to \mathcal{C}' \to \mathcal{C}''$ into a stable recollement. The simplest (and in fact universal) example of such a recollement is the sequence

$$\mathcal{C} \xrightarrow{x \mapsto [x \to 0]} \mathrm{Ar}(\mathcal{C}) \xrightarrow{[x \to y] \mapsto y} \mathcal{C}.$$

By the non-hermitian version of Theorem A due to Barwick [Bar17] (which can in fact also be extracted as special case of Theorem A), it gives rise to a fibre sequence

$$|\operatorname{Span}(\mathcal{C})| \longrightarrow |\operatorname{Span}(\mathrm{Ar}(\mathcal{C}))| \longrightarrow |\operatorname{Span}(\mathcal{C})|$$

which is split by the functor $\mathcal{C} \to \mathrm{Ar}(\mathcal{C})$ taking x to id_x. Taking loopspaces thus results in an equivalence

$$\mathcal{K}(\mathrm{Ar}(\mathcal{C})) \simeq \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{C}),$$

since $\mathcal{K}(\mathcal{C}) \simeq \Omega|\operatorname{Span}(\mathcal{C})|$, which makes Theorem A an hermitian analogue of Waldhausen's additivity theorem.

The simplest example of a split Poincaré-Verdier sequence arises similarly: If $\mathcal{C}$ has a Poincaré structure Ω, then the arrow category of $\mathcal{C}$ refines to a Poincaré ∞-category $\mathrm{Met}(\mathcal{C}, \Omega)$, whose Poincaré objects encode Poincaré objects in $(\mathcal{C}, \Omega)$ equipped with a Lagrangian, or in other words a nullbordism. There results the metabolic Poincaré-Verdier sequence

$$(\mathcal{C}, \Omega^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, \Omega) \xrightarrow{\partial} (\mathcal{C}, \Omega),$$

refining the recollement above. We interpret the cobordism ∞-category $\mathrm{Cob}^\partial(\mathcal{C}, \Omega)$ of its middle term as that of Poincaré objects with boundary in $(\mathcal{C}, \Omega)$. From the additivity theorem, we then find a fibre sequence

$$|\mathrm{Cob}(\mathcal{C}, \Omega)| \longrightarrow |\mathrm{Cob}^\partial(\mathcal{C}, \Omega)| \xrightarrow{\partial} |\mathrm{Cob}(\mathcal{C}, \Omega^{[1]})|,$$

that is entirely analogous to Genauer's fibre sequence

$$|\mathrm{Cob}_d| \longrightarrow |\mathrm{Cob}^\partial_d| \xrightarrow{\partial} |\mathrm{Cob}_{d-1}|$$

from geometric topology [Gen12]. Note, however, that neither of these latter sequences are split (the adjoint functors in a split Poincaré-Verdier sequence need not be compatible with the Poincaré structures), so the name additivity is maybe slightly misleading, but we will stick with it.

Our proof of Theorem A is in fact modelled on the recent proof of Genauer's fibre sequence at the hands of the ninth author [Ste21], and is new even in the context of algebraic K-theory. Similar results are known in varying degrees of generality, see for example [Sch17, HSV19]. The actual additivity theorem we prove is, however, quite a bit more general than Theorem A: We show that in fact every additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$, a mild strenghthening of the requirement that split Poincaré-Verdier sequences are taken to fibre sequences, gives rise to an $\mathcal{F}$-based cobordism ∞-category $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ and that the functor $|\mathrm{Cob}^{\mathcal{F}}|\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ is then also additive. Applied to $\mathcal{F} = \mathrm{Pn}$, this gives the result above, but the statement can now be iterated. Since the functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ (and thus also $\mathcal{GW} = \Omega^\infty \mathrm{GW}$) is defined by an iterated hermitian Q-construction, this generality gives sufficient control to establish:

Theorem B.

i) There is a natural equivalence

$$|\mathrm{Cob}(\mathcal{C}, \Omega)| \simeq \Omega^{\infty-1}\, \mathrm{GW}(\mathcal{C}, \Omega),$$

so, in particular, $\Omega|\mathrm{Cob}(\mathcal{C}, \Omega)| \simeq \mathcal{GW}(\mathcal{C}, \Omega)$.

ii) The functors $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *and* $\mathcal{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{S})$ *are the initial additive functors equipped with a transformation* $\mathrm{Pn} \to \mathcal{GW} \simeq \Omega^\infty\, \mathrm{GW}$, *respectively.*

iii) The functor $\mathrm{L}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *is the initial additive, bordism invariant functor equipped with a transformation* $\mathrm{Pn} \to \Omega^\infty\, \mathrm{L}$.

Here, we call an additive functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ bordism invariant if it vanishes when evaluated on metabolic categories, though there are many other characterisations. Theorem B simultaneously gives the hermitian analogue of the theorem of Blumberg, Gepner and Tabuada from [BGT13], that $\mathrm{K}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}p$ is the initial additive functor with a transformation $\mathrm{Cr} \to \Omega^\infty\, \mathrm{K} = \mathcal{K}$, and of the theorem of Galatius, Madsen, Tillmann and Weiss concerning the homotopy type of the cobordism category [GTMW09]. Just as for the additivity theorem, our cobordism theoretic methods provide a more direct proof of the universal property of algebraic K-theory avoiding all mention of non-commutative motives.

From Theorem B, it is straight-forward to obtain our Main Theorem: The first assertion of the Main Theorem may be restated as the formula

$$\mathrm{cof}(\mathrm{hyp}\colon\ \mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}(\mathcal{C}, \Omega)) \simeq \mathrm{L}(\mathcal{C}, \Omega),$$

and it is somewhat tautologically true that the left hand side is the initial bordism invariant functor under GW, whence the universal properties of GW and L from Theorem B give the claim. For the second statement we take another queue from geometric topology and use Ranicki's algebraic Thom construction to produce an equivalence

$$|\mathrm{Cob}^{\partial}(\mathcal{C}, \Omega)| \simeq |\,\mathrm{Span}(\mathcal{C})| = \Omega^{\infty-1}\, \mathrm{K}(\mathcal{C})$$

which extends to an identification

$$\mathrm{GW}(\mathrm{Met}(\mathcal{C}, \Omega)) \simeq \mathrm{K}(\mathcal{C})$$

for all Poincaré ∞-categories $(\mathcal{C}, \Omega)$. Via Theorem B, the metabolic Poincaré-Verdier sequence then gives rise to the fibre sequence

$$\mathrm{GW}(\mathcal{C}, \Omega) \xrightarrow{\mathrm{fgt}} \mathrm{K}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, \Omega^{[1]}),$$

which we term the Bott-Genauer sequence, bearing witness to its relation with the fibre sequence

$$\mathrm{MTSO}(d) \longrightarrow \mathbb{S}[\mathrm{BSO}(d)] \longrightarrow \mathrm{MTSO}(d-1)$$

first established by Galatius, Madsen, Tillmann, and Weiss, and beautifully explained by Genauer's theorem [Gen12], that

$$|\mathrm{Cob}^{\partial}_d| \simeq \Omega^{\infty-1}\mathbb{S}[\mathrm{BSO}(d)].$$

As explained in the previous section, if 2 is invertible (and the input is sufficiently strict), this sequence is due to Schlichting, but as far as we are aware, its connection with the fibre sequence of Thom spectra above had not been noticed before.

With the Main Theorem established, we observe that since both L- and K-theory are well-known to take arbitrary bifibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$ to fibre sequences (and not just split ones) we obtain:

Corollary C. *The functor* $\mathrm{GW}: \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *is Verdier localising, i.e. it takes arbitrary Poincaré-Verdier sequences*

$$(\mathcal{C}, \Omega) \longrightarrow (\mathcal{D}, \Phi) \longrightarrow (\mathcal{E}, \Psi)$$

to bifibre sequences

$$\mathrm{GW}(\mathcal{C}, \Omega) \longrightarrow \mathrm{GW}(\mathcal{D}, \Phi) \longrightarrow \mathrm{GW}(\mathcal{E}, \Psi)$$

of spectra.

This result is a full hermitian analogue of the localisation theorems available for algebraic K-theory, and (as far as we are aware) subsumes and extends all known localisation sequences for Grothendieck-Witt groups, in particular the celebrated results of [Sch10b]. We explicitly spell out an application to localisations of discrete rings in Corollary F below.

The fibre sequence of the Main Theorem can also be neatly repackaged using equivariant homotopy theory: The assignment $(\mathcal{C}, Ϙ) \mapsto \mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2}$ is another example of a bordism invariant functor, whence Theorem B produces a natural map $\Xi: \mathrm{L}(\mathcal{C}, Ϙ) \to \mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2}$. A version of this map first appeared in the work of Weiss and Williams on automorphisms of manifolds [WW14], and we show that our construction agrees with theirs. Using this map, one can reexpress the fibre sequence from the Main Theorem as a cartesian square

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{C}, Ϙ) \arrow[r, "\mathrm{bord}"] \arrow[d, "\mathrm{fgt}"] & \mathrm{L}(\mathcal{C}, Ϙ) \arrow[d, "\Xi"] \\
\mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{hC}_2} \arrow[r] & \mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2},
\end{tikzcd}$$

which we term the fundamental fibre square.

Now, in [HM], Hesselholt and Madsen promoted the Grothendieck-Witt spectrum $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M)$ into the genuine fixed points of what they termed the real algebraic K-theory $\mathrm{KR}^{\mathrm{s}}_{\mathrm{cl}}(R, M)$, a genuine C_2-spectrum. We similarly produce a functor $\mathrm{KR}: \mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathcal{S}p^{\mathrm{gC}_2}$ using the language of spectral Mackey functors, with the property that the isotropy separation square of $\mathrm{KR}(\mathcal{C}, Ϙ)$ is precisely the fundamental fibre square above, so that in particular

$$\mathrm{KR}(\mathcal{C}, Ϙ)^{\mathrm{gC}_2} \simeq \mathrm{GW}(\mathcal{C}, Ϙ) \quad \text{and} \quad \mathrm{KR}(\mathcal{C}, Ϙ)^{\varphi \mathrm{C}_2} \simeq \mathrm{L}(\mathcal{C}, Ϙ);$$

here $(-)^{\mathrm{gC}_2}$ and $(-)^{\varphi\mathrm{C}_2}: \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$ denote the genuine and geometric fixed points functors, respectively. Combined with the comparison results of [HS21] this affirms the conjecture of Hesselholt and Madsen that the geometric fixed points of the real algebraic K-theory spectrum of a discrete ring are a version of Ranicki's L-theory.

As the ultimate expression of periodicity, we then enhance our extension of Karoubi's periodicity to the following statement in the language of genuine homotopy theory:

Theorem D. *The boundary map of the metabolic Poincaré-Verdier sequence provides a canonical equivalence*

$$\mathrm{KR}(\mathcal{C}, Ϙ^{[1]}) \simeq \mathbb{S}^{1-\sigma} \otimes \mathrm{KR}(\mathcal{C}, Ϙ),$$

where σ denotes the sign representation of C_2.

Passing to geometric fixed points and using $(\mathbb{S}^{1-\sigma} \otimes X)^{\varphi\mathrm{C}_2} \simeq \mathbb{S}^1 \otimes X^{\varphi\mathrm{C}_2}$ recovers the result of Lurie that $\mathrm{L}(\mathcal{C}, Ϙ^{[1]}) \simeq \mathbb{S}^1 \otimes \mathrm{L}(\mathcal{C}, Ϙ)$, whereas the Bott-Genauer sequence and thus the abstract version of Karoubi periodicty, i.e. $\mathrm{U}(\mathcal{C}, Ϙ^{[2]}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{C}, Ϙ)$, and in particular the periodicity results for the classical Grothendieck-Witt spectra of discrete rings, becomes an instance of the general fibre sequence

$$X^{\mathrm{gC}_2} \xrightarrow{\mathrm{fgt}} X \xrightarrow{\mathrm{nm}} (\mathbb{S}^{1-\sigma} \otimes X)^{\mathrm{gC}_2},$$

valid for any genuine C_2-spectrum X.

Further applications to rings and parametrised spectra. We start by specialising the abstract results of the previous section to Grothendieck-Witt spectra of (discrete) rings. In the body of the paper, we derive these for E_1-ring spectra satisfying appropriate assumptions, but aside from a few comments, we refrain from engaging with this generality here.

Suppose then given a ring R and an invertible module with involution M over R, that is an $R \otimes_{\mathbb{Z}} R$-module and a map $\sigma \colon M \to M$ such that the following properties hold:

i) σ is linear over the map $R \otimes_{\mathbb{Z}} R \to R \otimes_{\mathbb{Z}} R$ flipping the two factors, i.e.
$$\sigma((r \otimes s) \cdot m) = (s \otimes r) \cdot \sigma(m)$$
for all $r, s \in R$ and $m \in M$,

ii) $\sigma^2 = \mathrm{id}_M$,

iii) M is projective when regarded as an R-module via the left (say) inclusion $R \to R \otimes_{\mathbb{Z}} R$, and

iv) the map
$$R \to \mathrm{Hom}_R(M, M), \quad r \longmapsto (m \mapsto (1 \otimes r) \cdot m)$$
is an isomorphism, where again M is viewed as an R-module via the left factor.

For R commutative, any $\otimes_R$-invertible R-module M, with the $R \otimes R$-module structure $(r \otimes s) \cdot m = rsm$ and $\sigma = \pm\mathrm{id}_R$ gives an example, as does any ring R equipped with a skew-involution ω and $M = R$, with $(r \otimes s) \cdot t = rt\omega(s)$ and $\sigma = \pm\omega$, and further generalisations of these two ideas, such as Wall's anti-structures.

Associated to this data, we constructed in Paper [I] a sequence of Poincaré structures
$$\Qoppa^{\mathrm{q}}_M = \Qoppa^{\geq\infty}_M \Longrightarrow \cdots \Longrightarrow \Qoppa^{\geq m}_M \Longrightarrow \Qoppa^{\geq m-1}_M \Longrightarrow \cdots \Longrightarrow \Qoppa^{\geq -\infty}_M = \Qoppa^{\mathrm{s}}_M$$
on the stable ∞-category $\mathcal{D}^{\mathrm{p}}(R)$, ultimately coming from the Postnikov filtration of M^{tC_2}. The genuine Poincaré structures from the first section appear as $\Qoppa^{\mathrm{gs}} = \Qoppa^{\geq 0}_M$, $\Qoppa^{\mathrm{ge}}_M = \Qoppa^{\geq 1}_M$, and $\Qoppa^{\mathrm{gq}}_M = \Qoppa^{\geq 2}_M$. Recall that these give rise to the classical symmetric, even and quadratic Grothendieck-Witt spectra of (R, M), whereas essentially by construction $\Qoppa^{\mathrm{q}}_M$ and $\Qoppa^{\mathrm{s}}_M$ give rise to the classical L-spectra of (R, M). This sequence of Poincaré structures collapses in to a single one if 2 is invertible in R, since in that case $M^{\mathrm{tC}_2} \simeq 0$.

Now, extending the discussion after the Main Theorem, we constructed in Paper [I] equivalences of the form
$$(\mathcal{D}^{\mathrm{p}}(R), (\Qoppa^{\geq m}_M)^{[2]}) \simeq (\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\geq m+1}_{-M}).$$
and entirely similar results hold for the stable subcategories $\mathcal{D}^c(R)$ of $\mathcal{D}^{\mathrm{p}}(R)$ spanned by those objects $X \in \mathcal{D}^{\mathrm{p}}(R)$ with $[X] \in c \subseteq \mathrm{K}_0(R)$, whenever c is a subgroup closed under the involution on $K_0(R)$ induced by M. For example $\mathcal{D}^{\mathrm{f}}(R)$, the smallest stable subcategory of $\mathcal{D}^{\mathrm{p}}(R)$ spanned by $R[0]$, corresponds to the image of $\mathbb{Z} \to \mathrm{K}_0(R)$, $1 \mapsto R$. On the L-theory side, the switch between these categories is known as the change of decoration, whereas it does not affect the positive Grothendieck-Witt groups (we will show this in Paper [IV]). Applying Theorem D, we find the following result, which (for 2 invertible in R) proves an unpublished conjecture of Hesselholt-Madsen:

Corollary E (Genuine Karoubi periodicity)**.** *For a (discrete) ring R, an invertible R-module M with involution, a subgroup $c \subseteq \mathrm{K}_0(R)$ closed under the involution induced by M, and $m \in \mathbb{Z} \cup \{\pm\infty\}$, there are canonical equivalences*
$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\geq m}_M) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\geq m-1}_{-M}).$$
In particular, the genuine C_2-spectra
$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{s}}_M) \quad \textit{and} \quad \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{q}}_M)$$
are $(4-4\sigma)$-periodic and even $(2-2\sigma)$-periodic if R has characteristic 2.

We also find
$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gq}}_M) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{ge}}_{-M}) \simeq \mathbb{S}^{4-4\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gs}}_M)$$
for any discrete ring R and invertible R-module with involution M, but often more is true: If for example the norm map $M_{\mathrm{C}_2} \to M^{\mathrm{C}_2}$ is surjective (i.e. $\pi_{2i}(M^{\mathrm{tC}_2}) = 0$ for all $i \in \mathbb{Z}$), we have $\Qoppa^{\geq 2i}_M = \Qoppa^{\geq 2i+1}_M$ and $\Qoppa^{\geq 2i+1}_{-M} = \Qoppa^{\geq 2i+2}_{-M}$, so we obtain
$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gq}}_M) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gq}}_{-M}) \quad \text{and} \quad \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gs}}_M) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gs}}_{-M}).$$

This applies for example whenever M is an invertible module over a commutative ring R without 2-torsion equipped with the sign involution.

In a different direction, the $(4 - 4\sigma)$- or $(2 - 2\sigma)$-fold periodicity of

$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{s}}_M) \quad \text{and} \quad \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{q}}_M)$$

in fact holds for any complex oriented or real oriented E_1-ring R, respectively; we will deduce it in this generality in the body of the paper along with higher periodicity results for other ring spectra such as ko and tmf, see §4.5.

Let us now turn to the behaviour of Grothendieck-Witt spectra under localisations of rings. As one application of Corollary C we find:

Corollary F. *Let R be a (discrete) ring, M an invertible module with involution over R, $c \subseteq \mathrm{K}_0(R)$ a subgroup closed under the involution induced by M, and $f, g \in R$ elements spanning the unit ideal. Then, the square*

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{D}^c(R), \Qoppa^{\geq m}_M) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{D}^{\mathrm{im}(c)}(R[\tfrac{1}{f}]), \Qoppa^{\geq m}_{M[1/f]}) \arrow[d] \\
\mathrm{GW}(\mathcal{D}^{\mathrm{im}(c)}(R[\tfrac{1}{g}]), \Qoppa^{\geq m}_{M[1/g]}) \arrow[r] & \mathrm{GW}(\mathcal{D}^{\mathrm{im}(c)}(R[\tfrac{1}{fg}]), \Qoppa^{\geq m}_{M[1/fg]})
\end{tikzcd}$$

is cartesian.

The case $m = 0$ recovers the affine case of Schlichting's celebrated Mayer-Vietoris principle for Grothendieck-Witt groups of schemes [Sch10b], and the case $m = 1, 2$ extends these results from symmetric to even and quadratic Grothendieck-Witt groups.

Outlook. We will not consider the Grothendieck-Witt theory of schemes in the present paper, as it works more smoothly when considering Karoubi-Grothendieck-Witt spectra, i.e. the variant of Grothendieck-Witt theory that is invariant under idempotent completion, just as non-connective K-spectra are better suited for the study of schemes than connective ones; as explained previously, it is this variant which Schlichting considers in [Sch10b] as well. We will develop this extension in Paper [IV] and give a proof of Nisnevich descent in another upcoming paper [CHN].

We use our main result in the third instalment of this paper series, to deduce dévissage results for the fibres of localisation maps as in the above square if $m = 0$, i.e. for symmetric Grothendieck-Witt groups, under the additional assumption R is a Dedekind domain. In fact, dévissage statements hold naturally for $m = -\infty$, and we transport them to other classical Grothendieck-Witt spectra by a detailed analysis of the L-theory spectra involved.

Lastly, we turn to another class of examples of Poincaré ∞-categories, namely those formed by compact parametrised spectra over a space B. The relevance of these examples is already visible in the equivalences

$$\mathrm{A}(B) \simeq \mathrm{K}((\mathcal{S}p/B)^\omega)$$

describing Waldhausen's K-theory of spaces in the present framework. Given a stable spherical fibration ξ over B, there are three important Poincaré structures on $(\mathcal{S}p/B)^\omega$, the quadratic, symmetric and visible one, all of whose underlying duality is the Costenoble-Waner functor

$$E \mapsto \mathrm{Hom}_B(E \boxtimes E, \Delta_! \xi);$$

here $\boxtimes$ is the exterior tensor product, $\Delta\colon B \to B \times B$ is the diagonal map, and the subscript ! denotes the left adjoint functor to its associated pullback. Then from the isotropy separation square of $\mathrm{KR}((\mathcal{S}p/B)^\omega, \Qoppa^r_\xi)$ with $r \in \{\mathrm{q}, \mathrm{s}, \mathrm{v}\}$ we find:

Corollary G. *There are canonical equivalences*

$$\mathrm{GW}((\mathcal{S}p/B)^\omega, \Qoppa^r_\xi) \simeq \mathrm{LA}^r(B, \xi)$$

and in particular

$$\Omega^{\infty-1}\mathrm{LA}^r(B, \xi) \simeq |\mathrm{Cob}((\mathcal{S}p/B)^\omega, \Qoppa^r_\xi)|$$

for $r \in \{\mathrm{q}, \mathrm{s}, \mathrm{v}\}$.

Here, $\mathrm{LA}^r(B,\xi)$ denotes the spectra constructed by Weiss and Williams (under the names $\mathrm{LA}_\bullet$, $\mathrm{LA}^\bullet$, and VLA) in their pursuit of a combination of surgery theory and pseudo-isotopy theory into a direct description of the spaces $\mathcal{G}(M)/\mathrm{Top}(M)$ for closed manifolds M, see [WW14]. This result unites their work with the recent approaches to the study of diffeomorphism groups at the hands of Galatius and Randal-Williams [GRW14]. In particular, the second part provides a cycle model for the previously rather mysterious spectra $\mathrm{LA}^r(B,\xi)$ that can be used to give a new construction of Waldhausen's map

$$\widetilde{\mathrm{Top}}(M)/\mathrm{Top}(M) \longrightarrow \mathrm{Wh}(M)_{\mathrm{hC}_2}$$

along with a new proof of the index theorems of Weiss and Williams. These results will appear in future work.

In the present paper, we only give a small application of the above equivalence in another direction. We use computations of Weiss and Williams for $B = *$ together with the universal properties of GW and L to determine the automorphism groups of these functors. The result is that

$$\pi_0 \operatorname{Aut}(\mathrm{GW}) \cong (\mathrm{C}_2)^2 \quad \text{and} \quad \pi_0 \operatorname{Aut}(\mathrm{L}) \cong \mathrm{C}_2$$

the former spanned by $-\mathrm{id}_{\mathrm{GW}}$ and $\mathrm{id}_{\mathrm{GW}} - (\mathrm{hyp}\circ\mathrm{fgt})$ and the latter by $-\mathrm{id}_{\mathrm{L}}$.

Remark. During the completion of this work, on the one hand Schlichting announced results similar to the corollaries of our main theorem, and some of the applications we pursue in the third instalment of this series in [Sch19], though as far as we are aware no proofs have appeared yet. On the other hand the draft [HSV19] contains a construction of the real algebraic K-theory spectrum in somewhat greater generality than in the present paper (in particular, for not necessarily stable ∞-categories), with a version of Theorem B, part ii) as their main result, albeit using a weaker notion of additivity than the one we use here (resulting in a logically incomparable result).

However, as far as we are aware, neither of these systematically relates Grothendieck-Witt theory to L-theory, the main thread of our work.

Organisation of the paper. In the next section, we briefly summarise the necessary results of Paper [I], providing in particular a guide to the requisite parts. In §1, we study (co)fibre sequences in $\mathrm{Cat}^{\mathrm{p}}_\infty$ in detail and introduce additive and localising functors. The analogous results in the setting of stable ∞-categories, on which our results are based, are well-known but seem difficult to locate coherently in the literature. We therefore give a systematic account in Appendix A, without any claim of originality.

The real work of the present paper then starts in §2. It contains the definition of the hermitian Q-construction and the algebraic cobordism ∞-category and proves Theorem A as 2.5.1 and 2.5.3. In §3, we then generally analyse the behaviour of additive functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ and $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$. This leads to very general versions of Theorem B in 3.3.6, 3.4.5, our Main Theorem in 3.6.7 and Theorem C in 3.7.7. We then obtain all other results of this introduction as simple consequences in §4, where we specialise the discussion to the Grothendieck-Witt functor.

Finally, there is a second appendix which establishes two comparison results to other Grothendieck-Witt spectra, not immediate from the results of [HS21]. They are not used elsewhere in the paper.

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RECOLLECTION

In the present section, we briefly recall the parts of Paper [I] that are most relevant for the considerations of the present paper. We first summarise the abstract features of the theory, and then spell out some examples.

Poincaré ∞-categories and Poincaré objects. Recall from §[I].1.2 that an hermitian structure on a small stable ∞-category $\mathcal{C}$ is a reduced, quadratic functor $Ϙ: \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$; see Diagram (1) below for a characterisation of such functors. A pair $(\mathcal{C}, Ϙ)$ consisting of this data we call an hermitian ∞-category. These organise into an ∞-category $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ whose morphisms $(\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ consist of what we term hermitian functors, that is pairs (f, η) where $f : \mathcal{C} \to \mathcal{D}$ is an exact functor and $\eta : Ϙ \Rightarrow \Phi \circ f^{\mathrm{op}}$ is a natural transformation.

To such an hermitian ∞-category is associated its ∞-category of hermitian forms $\mathrm{He}(\mathcal{C}, Ϙ)$, whose objects consist of pairs (X, q) where $X \in \mathcal{C}$ and q is a $Ϙ$-hermitian form on X, i.e. a point in $\Omega^{\infty} Ϙ(X)$, see §[I].2.1. Morphisms are maps in $\mathcal{C}$ preserving the hermitian forms. The core of the ∞-category $\mathrm{He}(\mathcal{C}, Ϙ)$ is denoted $\mathrm{Fm}(\mathcal{C}, Ϙ)$, and these assemble into functors

$$\mathrm{He} : \mathrm{Cat}^{\mathrm{h}}_{\infty} \to \mathrm{Cat}_{\infty} \quad \text{and} \quad \mathrm{Fm} : \mathrm{Cat}^{\mathrm{h}}_{\infty} \to \mathcal{S}.$$

In order to impose a non-degeneracy condition on the forms in $\mathrm{Fm}(\mathcal{C}, Ϙ)$, one needs a non-degeneracy condition on the hermitian ∞-category $(\mathcal{C}, Ϙ)$ itself. To this end recall the classification of quadratic functors from Goodwillie calculus: Any reduced quadratic functor uniquely extends to a natural cartesian diagram

$$\begin{tikzcd} Ϙ(X) \arrow[r] \arrow[d] & \Lambda_{Ϙ}(X) \arrow[d] \\ \mathrm{B}_{Ϙ}(X, X)^{\mathrm{hC}_2} \arrow[r] & \mathrm{B}_{Ϙ}(X, X)^{\mathrm{tC}_2} \end{tikzcd} \tag{1}$$

where $\Lambda_{Ϙ} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is linear (i.e. exact), and $\mathrm{B}_{Ϙ} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is bilinear (i.e. exact in each variable) and symmetric (i.e. comes equipped with a refinement to an element of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{S}p)^{\mathrm{hC}_2}$, with C_2 acting by flipping the input variables); see §[I].1.3.

A hermitian structure $Ϙ$ is called Poincaré if there exists an equivalence $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that

$$\mathrm{B}_{Ϙ}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{D}Y)$$

naturally in $X, Y \in \mathcal{C}^{\mathrm{op}}$. By Yoneda's lemma, such a functor D is uniquely determined if it exists, so we refer to it as $\mathrm{D}_{Ϙ}$. By the symmetry of $\mathrm{B}_{Ϙ}$, the functor $\mathrm{D}_{Ϙ}$ then automatically satisfies $\mathrm{D}_{Ϙ} \circ \mathrm{D}_{Ϙ}^{\mathrm{op}} \simeq \mathrm{id}_{\mathcal{C}}$. Any hermitian functor $(F,\eta)\colon (\mathcal{C},\Qoppa) \to (\mathcal{D},\Phi)$ between Poincaré ∞-categories (i.e. hermitian ∞-categories whose hermitian structure is Poincaré) induces a canonical map

$$F \circ \mathrm{D}_{\Qoppa} \Longrightarrow \mathrm{D}_{\Phi} \circ F^{\mathrm{op}};$$

see §[I].1.2. We say that (F,η) is a Poincaré functor if this transformation is an equivalence, and Poincaré ∞-categories together with Poincaré functors form a (non-full) subcategory $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ of $\mathrm{Cat}^{\mathrm{h}}_{\infty}$.

Now, if $(\mathcal{C},\Qoppa)$ is Poincaré, then to any hermitian form $(X,q) \in \mathrm{Fm}(\mathcal{C},\Qoppa)$ there is canonically associated a map

$$q^{\sharp}\colon X \longrightarrow \mathrm{D}_{\Qoppa} X$$

as the image of q under

$$\Omega^{\infty}\Qoppa(X) \longrightarrow \Omega^{\infty}\mathrm{B}_{\Qoppa}(X,X) \simeq \mathrm{Hom}_{\mathcal{C}}(X,\mathrm{D}_{\Qoppa}X)$$

and we say that (X,q) is Poincaré if $q^{\sharp}$ is an equivalence. The full subspace of $\mathrm{Fm}(\mathcal{C},\Qoppa)$ spanned by the Poincaré forms is denoted by $\mathrm{Pn}(\mathcal{C},\Qoppa)$ and provides a functor

$$\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S},$$

which we suggest to view in analogy with the functor $\mathrm{Cr}\colon \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{S}$ taking a stable ∞-category to its groupoid core. Details about this functor are spelled out in §[I].2.1.

The simplest example of a Poincaré ∞-category to keep in mind is $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(R)$, where R is a discrete commutative ring and $\mathcal{D}^{\mathrm{p}}(R)$ is the ∞-category of perfect complexes over R (i.e. finite chain complexes of finitely generated projective R-modules), together with the symmetric and quadratic Poincaré structures given by

$$\Qoppa^{\mathrm{q}}_{R}(X) \simeq \hom_R(X \otimes^{\mathbb{L}}_{R} X, M)_{\mathrm{hC}_2} \quad \text{and} \quad \Qoppa^{\mathrm{s}}_{R}(X) \simeq \hom_R(X \otimes^{\mathbb{L}}_{R} X, M)^{\mathrm{hC}_2},$$

where $\hom_R$ denotes the mapping spectrum of the stable ∞-category $\mathcal{D}^{\mathrm{p}}(R)$ (in other words the spectrum underlying the derived mapping complex $\mathbb{R}\,\mathrm{Hom}_R$). In either case, the bilinear part and duality are given by

$$\mathrm{B}(X,Y) \simeq \hom_R(X \otimes^{\mathbb{L}}_{R} Y, R) \quad \text{and} \quad \mathrm{D}(X) \simeq \mathbb{R}\,\mathrm{Hom}_R(X,R),$$

which makes both $\Qoppa^{\mathrm{s}}_{R}$ and $\Qoppa^{\mathrm{q}}_{R}$ into Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$.

We will discuss further examples in detail below.

Constructions of Poincaré ∞-categories. We next collect a few important structural properties of the ∞-categories $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ and $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. First of all, by the results of §[I].6.1 they are both complete and cocomplete, and the inclusion $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{h}}_{\infty}$ is conservative, i.e. it detects equivalences among Poincaré ∞-categories. Furthermore, the forgetful functors

$$\mathrm{Cat}^{\mathrm{p}}_{\infty} \longrightarrow \mathrm{Cat}^{\mathrm{h}}_{\infty} \longrightarrow \mathrm{Cat}^{\mathrm{ex}}_{\infty}$$

both possess both adjoints, so preserve both limits and colimits; these are constructed in §[I].7.2 and §[I].7.3. For the right hand functor, the adjoints simply equip a stable ∞-category $\mathcal{C}$ with the trivial hermitian structure 0. For the left hand functor, the left and right adjoints are related by a shift: Denoting the right adjoint functor by $(\mathcal{C},\Qoppa) \mapsto \mathrm{Pair}(\mathcal{C},\Qoppa)$, the left adjoint is given by $(\mathcal{C},\Qoppa) \mapsto \mathrm{Pair}(\mathcal{C},\Qoppa^{[-1]})$, where generally $\Qoppa^{[i]}$ denotes the hermitian structure $\Sigma^{i} \circ \Qoppa$. We refrain at this place from giving the explicit construction of the ∞-category $\mathrm{Pair}(\mathcal{C},\Qoppa)$ since it is somewhat involved, and we shall not need it here.

The following two special cases of this construction will be of great importance. By the above discussion, the left and right adjoint of the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$ agree. They are given by the hyperbolic construction $\mathcal{C} \to \mathrm{Hyp}(\mathcal{C})$ with underlying ∞-category $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ and Poincaré structure $\hom_{\mathcal{C}}\colon (\mathcal{C} \times \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{S}p$, see §[I].2.2. The associated duality is given by $(X,Y) \mapsto (Y,X)$, and there is a natural equivalence

$$\mathrm{Cr}\,\mathcal{C} \simeq \mathrm{Pn}\,\mathrm{Hyp}(\mathcal{C})$$

implemented by $X \mapsto (X,X)$. We denote by

$$f_{\mathrm{hyp}}\colon \mathrm{Hyp}(\mathcal{C}) \to (\mathcal{D},\Qoppa') \quad \text{and} \quad f^{\mathrm{hyp}}\colon (\mathcal{C},\Qoppa) \to \mathrm{Hyp}(\mathcal{D})$$

the Poincaré functors obtained through these adjunctions from a exact functor $f\colon \mathcal{C} \to \mathcal{D}$.

The other important case is the composite of the inclusion $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ with its left adjoint. This assigns to a Poincaré ∞-category $(\mathcal{C}, Ϙ)$ the metabolic ∞-category $\mathrm{Met}(\mathcal{C}, Ϙ)$, whose underlying ∞-category is the arrow category $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$ and whose Poincaré structure is given by

$$Ϙ^{\mathrm{met}}(X \to Y) \simeq \mathrm{fib}(Ϙ(Y) \to Ϙ(X)),$$

see §[I].2.3. The associated duality is

$$\mathrm{D}_{Ϙ^{\mathrm{met}}}(X \to Y) \simeq \mathrm{fib}(\mathrm{D}_Ϙ(Y) \to \mathrm{D}_Ϙ(X)) \longrightarrow \mathrm{D}_Ϙ(Y).$$

The Poincaré objects in $\mathrm{Met}(\mathcal{C}, Ϙ)$ are best thought of as Poincaré objects with boundary in the Poincaré ∞-category $(\mathcal{C}, Ϙ^{[-1]})$, which embeds into $\mathrm{Met}(\mathcal{C}, Ϙ)$ via $X \mapsto (X \to 0)$, i.e. as the objects with trivial boundary.

From the various adjunction units and counits there then arises a diagram

$$\begin{tikzcd}
& \mathrm{Hyp}(\mathcal{C}) \arrow[dl, equal] \arrow[dd, "\mathrm{can}"] \arrow[dr, "\mathrm{hyp}"] & \\
\mathrm{Hyp}(\mathcal{C}) & & (\mathcal{C}, Ϙ) \\
& \mathrm{Met}(\mathcal{C}, Ϙ) \arrow[ul, "\mathrm{lag}"] \arrow[ur, "\mathrm{met}"'] &
\end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{p}}_\infty$ for every Poincaré ∞-category; the underlying functors pointing to the right are given by

$$\mathrm{met}(X \to Y) = Y \quad \text{and} \quad \mathrm{hyp}(X, Y) = X \oplus \mathrm{D}_Ϙ Y,$$

whereas the other two are given by extending the source and identity functors

$$s\colon\ \mathrm{Ar}(\mathcal{C}) \longrightarrow \mathcal{C} \quad \text{and} \quad \mathrm{id}\colon\ \mathcal{C} \longrightarrow \mathrm{Ar}(\mathcal{C})$$

using the adjunction properties of Hyp. Regarding the induced maps after applying Pn, one finds that an element in $(X, q) \in \pi_0\mathrm{Pn}(\mathcal{C}, Ϙ)$ is in the image of met if it admits a Lagrangian, that is a map $f\colon L \to X$ such that there is an equivalence $f^*q \simeq 0$, whose associated nullhomotopy of the composite

$$L \xrightarrow{f} X \simeq \mathrm{D}_Ϙ X \xrightarrow{\mathrm{D}_Ϙ f} \mathrm{D}_Ϙ L$$

makes this sequence into a fibre sequence in $\mathcal{C}$. Similarly, (X, q) lies in the image of hyp if there is an equivalence $X \simeq L \oplus \mathrm{D}_Ϙ L$ which translates the form q into the canonical evaluation form on the target.

Thus the Poincaré categories $\mathrm{Hyp}(\mathcal{C})$ and $\mathrm{Met}(\mathcal{C}, Ϙ)$ encode the theory of metabolic and hyperbolic forms in $(\mathcal{C}, Ϙ)$, and the remainder of the diagram witnesses that any hyperbolic form has a canonical Lagrangian, from which it can be reconstructed.

One further property of these constructions that we shall need is that the duality $\mathrm{D}_Ϙ$ equips the underlying ∞-category of $(\mathcal{C}, Ϙ)$ with the structure of a homotopy fixed point in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ under the C_2-action given by taking $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$, or in other words, the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ is C_2-equivariant for the trivial action on the source and the opponing action on the target; see §[I].7.2. As a formal consequence, its adjoint Hyp is equivariant as well, and thus the composite

$$\mathrm{Cat}^{\mathrm{p}}_\infty \xrightarrow{\mathrm{fgt}} \mathrm{Cat}^{\mathrm{ex}}_\infty \xrightarrow{\mathrm{Hyp}} \mathrm{Cat}^{\mathrm{p}}_\infty$$

lifts to a functor $\mathcal{H}\mathrm{yp}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to (\mathrm{Cat}^{\mathrm{p}}_\infty)^{\mathrm{hC}_2} = \mathrm{Fun}(\mathrm{BC}_2, \mathrm{Cat}^{\mathrm{p}}_\infty)$, the ∞-category of (naive) C_2-objects in $\mathrm{Cat}^{\mathrm{p}}_\infty$. The action map on $\mathcal{H}\mathrm{yp}(\mathcal{C}, Ϙ)$ is given by the composite

$$\mathrm{Hyp}(\mathcal{C}) \xrightarrow{\mathrm{flip}} \mathrm{Hyp}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{Hyp}(\mathrm{D}_Ϙ)} \mathrm{Hyp}(\mathcal{C})$$

and the functor $\mathrm{hyp}\colon\ \mathrm{Hyp}(\mathcal{C}) \to (\mathcal{C}, Ϙ)$ is invariant under the action on the source.

We recall from §[I].5.2 that the ∞-category $\mathrm{Cat}^{\mathrm{h}}_\infty$ admits a symmetric monoidal structure making the functor $\mathrm{fgt}\colon\ \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ symmetric monoidal for Lurie's tensor product of stable ∞-categories on the target. While we do not use the monoidal structure in the present paper, we heavily exploit the following: The monoidal structure on $\mathrm{Cat}^{\mathrm{h}}_\infty$ is cartesian closed, i.e. $\mathrm{Cat}^{\mathrm{h}}_\infty$ admits internal function objects, and also both tensors and cotensors over Cat_∞, see §[I].6.2, §[I].6.4 and §[I].6.3, respectively. More explicitly, to hermitian ∞-categories $(\mathcal{C}, \Qoppa)$ and $(\mathcal{D}, \Phi)$ and an ordinary category $\mathcal{J}$, there are associated hermitian ∞-categories

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)), \quad (\mathcal{C}, \Qoppa)_{\mathcal{J}} \quad \text{and} \quad (\mathcal{C}, \Qoppa)^{\mathcal{J}}.$$

connected by natural equivalences

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa)_{\mathcal{J}}, (\mathcal{D}, \Phi)) \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi))^{\mathcal{J}} \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)^{\mathcal{J}})$$

The underlying ∞-categories in the outer cases are given by

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad \mathrm{Fun}(\mathcal{J}, \mathcal{C})$$

and their hermitian structures $\mathrm{nat}^{\Phi}_{\Qoppa}$ and $\Qoppa^{\mathcal{J}}$ are given by

$$f \longmapsto \mathrm{nat}(\Qoppa, \Phi \circ f^{\mathrm{op}}) \quad \text{and} \quad f \longmapsto \lim_{\mathcal{J}^{\mathrm{op}}} \Qoppa \circ f^{\mathrm{op}}.$$

This results in particular in equivalences

$$\mathrm{Fm}\,\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)) \simeq \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{h}}_{\infty}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)), \quad \mathrm{Pn}\,\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)) \simeq \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi))$$

$$\text{and} \quad \mathrm{He}((\mathcal{C}, \Qoppa)^{\mathcal{J}}) \simeq \mathrm{Fun}(\mathcal{J}, \mathrm{He}(\mathcal{C}, \Qoppa)),$$

though Poincaré objects in $(\mathcal{C}, \Qoppa)^{\mathcal{J}}$ are not generally easy to describe. Furthermore, the tensoring construction is unfortunately far less explicit, and as we only need few concrete details, let us refrain from spelling it out here; for $\mathcal{J}$ a finite poset it is described explicitly in Proposition [I].6.5.8. Finally, we note that neither the tensor nor cotensor construction generally preserve Poincaré ∞-categories, though Lurie established sufficient criteria which we recorded in §[I].6.6.

Examples of Poincaré ∞-categories. Finally, we discuss the most important examples in detail: Poincaré structures on module ∞-categories and parametrised spectra.

We start with the former. Fix therefore an E_1-algebra A over a base E_∞-ring spectrum k and a subgroup $c \subseteq \mathrm{K}_0(A)$. Consider then the ∞-category of compact A-module spectra Mod^{ω}_A or more generally its full subcategory Mod^c_A spanned by all those $X \in \mathrm{Mod}^{\omega}_A$ with $[X] \in c \subseteq \mathrm{K}_0(A)$. For example, $\mathrm{Mod}^{\mathrm{f}}_A = \mathrm{Mod}^{\langle A \rangle}_A$ is the stable subcategory of Mod_A spanned by A itself. For the reader mostly interested in the applications to discrete rings, we recall that any discrete ring R gives rise to such data, via the Eilenberg-Mac Lane functor $\mathrm{H} \colon \mathcal{A}b \longrightarrow \mathcal{S}p$, which is lax symmetric monoidal and therefore induces a functor

$$\mathrm{H} \colon \mathrm{Ring} \longrightarrow \mathrm{Alg}_{\mathrm{E}_1}(\mathrm{Mod}\,\mathrm{H}\mathbb{Z})$$

which allows us to view any discrete ring may as an E_1-algebra over $\mathrm{H}\mathbb{Z}$; indeed, both versions of the functor H are equivalences onto the full subcategory of the target with vanishing homotopy groups outside degree 0. There are, furthermore, equivalences

$$\mathrm{Mod}^{\omega}_{\mathrm{H}R} \simeq \mathcal{D}^{\mathrm{p}}(R) \quad \text{and} \quad \mathrm{Mod}^{\mathrm{f}}_{\mathrm{H}R} \simeq \mathcal{D}^{\mathrm{f}}(R),$$

where $\mathcal{D}^{\mathrm{p}}(R)$ denotes the full subcategory of the derived ∞-category $\mathcal{D}(R)$ of R spanned by the perfect complexes, i.e. finite chain complexes of finitely generated projective R-modules and $\mathcal{D}^{\mathrm{f}}(R)$ is the full subcategory spanned by the finite chain complexes of finite free R-modules. In this regime the reader should keep in mind, that terms such as $\otimes_{\mathrm{H}\mathbb{Z}}$ or $\mathrm{Hom}_{\mathrm{H}R}$ will evaluate to the functors $\otimes^{\mathbb{L}}_{\mathbb{Z}}$ and $\mathbb{R}\,\mathrm{Hom}_R$.

Hermitian structures on the ∞-categories Mod^c_A are generated by A-modules with genuine involution (M, N, α); let us go through these ingredients one by one; compare §[I].3.2. The first entry M is what we term an A-module with (naive) involution: An $A \otimes_k A$-module, equipped with the structure of a homotopy fixed point in the ∞-category $\mathrm{Mod}_{A \otimes_k A}$ under the C_2-action flipping the two factors; see §[I].3.1.

In the case of a discrete ring R, the simplest examples of such a structure is given by a discrete $R \otimes_{\mathbb{Z}} R$-module M, and a selfmap $M \to M$, that squares to the identity on M, and is semilinear for the flip map of $R \otimes_{\mathbb{Z}} R$. If R is a ring equipped with an anti-involution σ, then $M = R$ is a valid choice by using σ to turn the usual $R \otimes_{\mathbb{Z}} R^{\mathrm{op}}$-module structure on R into an $R \otimes_{\mathbb{Z}} R$-module structure. The involution on R can then be chosen as σ or $-\sigma$ (or $\epsilon\sigma$ for any other central unit ϵ with $\sigma(\epsilon) = \epsilon^{-1}$).

The additional data of a module with genuine involution consists of an A-module spectrum N and an A-linear map $\alpha \colon N \to M^{\mathrm{tC}_2}$; to make sense of the latter term, note that upon forgetting the $A \otimes_k A$-action, the involution equips M with the structure of a (naive) C_2-spectrum (or even k-module spectrum). The spectrum M^{tC_2} then becomes an $(A \otimes_k A)^{tC_2}$-module via the lax monoidality of the Tate construction and from here obtains an A-module structure on M^{tC_2} by pullback along the Tate diagonal $A \to (A \otimes_k A)^{tC_2}$, which is a map of E_1-ring spectra; see [NS18, Chapter III.1] for an exposition of the Tate diagonal in the present language. Let us immediately warn the reader that the Tate diagonal is not generally k-linear for the k-module structure on $(A \otimes_k A)^{tC_2}$ arising from the unit map $k \to (k \otimes_k k)^{tC_2} = k^{tC_2}$, as this map is usually different from the Tate-diagonal of k (in particular, this is the case for $k = H\mathbb{Z}$ by [NS18, Theorem III.1.10]).

Even if only interested in discrete R, one therefore has to leave not only the realm of discrete R-modules to form the Tate construction, but even the realm of derived categories, as no replacement for the Tate diagonal can exist in that regime.

The hermitian structure associated to a module with genuine involution (M, N, α) as described above is given by the pullback

$$\begin{tikzcd}
\Qoppa^{\alpha}_{M}(X) \arrow[rr] \arrow[d] & & \hom_A(X, N) \arrow[d, "\alpha_*"] \\
\hom_{A\otimes_k A}(X \otimes_k X, M)^{hC_2} \arrow[r] & \hom_{A\otimes_k A}(X \otimes_k X, M)^{tC_2} \arrow[r, "\simeq"] & \hom_A(X, M^{tC_2})
\end{tikzcd}$$

where the C_2-action on $\hom_{A\otimes_k A}(X \otimes_k X, M)$ is given by flipping the factors in the source and the involution on M. It is a Poincaré structure on Mod^{ω}_A if M restricts to an object of Mod^{ω}_A under either inclusion $A \to A \otimes_k A$, and furthermore M is invertible, i.e. the natural map

$$A \to \hom_A(M, M)$$

is an equivalence. In this case, the associated duality is given by $X \mapsto \hom_A(X, M)$ regarded as an A-module via the extraneous A-module structure on M; see again §[I].3.1. Given a subgroup $c \in K_0(A)$, one obtains a Poincaré structure on Mod^{c}_A if in addition c is closed under the duality on $K_0(A)$ induced by M. In the example of Mod^{f}_A this translates to $M \in \mathrm{Mod}^{f}_A$.

With the preliminaries established, let us give some concrete examples. We shall restrict to the special case of discrete rings here for ease of exposition. So assume given a discrete ring R and a discrete invertible $R \otimes_{\mathbb{Z}} R$-module M with involution, that is finitely generated projective (or stably free, as appropriate) when regarded as an element of $\mathcal{D}(R)$ via either inclusion of R into $R \otimes_{\mathbb{Z}} R$.

Generalising the simple case discussed in the first part, associated to this data are most easily defined the quadratic and symmetric Poincaré structures $\Qoppa^{q}_{M}$ and $\Qoppa^{s}_{M}$ given by

$$\Qoppa^{q}(X) = \hom_{R \otimes^{\mathbb{L}}_{\mathbb{Z}} R}(X \otimes^{\mathbb{L}}_{\mathbb{Z}} X, M)_{hC_2} \quad \text{and} \quad \Qoppa^{s}(X) = \hom_{R \otimes^{\mathbb{L}}_{\mathbb{Z}} R}(X \otimes^{\mathbb{L}}_{\mathbb{Z}} X, M)^{hC_2}$$

which correspond to the modules with genuine involution

$$(M, 0, 0) \quad \text{and} \quad (M, M^{tC_2}, \mathrm{id}),$$

respectively. Interpolating between these, we have the genuine family of Poincaré structures $\Qoppa^{\geq i}_{M}$ corresponding to the modules with genuine involution $(M, \tau_{\geq i} M^{tC_2}, \tau_{\geq i} M^{tC_2} \to M^{tC_2})$ for $i \in \mathbb{Z}$. As already done in the introduction, we shall often include the quadratic and symmetric structures via $i = \pm\infty$ to facilitate uniform statements. These intermediaries are important mostly since they contain the following examples: The functors

$$\mathrm{Quad}_M, \quad \mathrm{Ev}_M, \quad \text{and} \quad \mathrm{Sym}_M : \mathrm{Proj}(R)^{\mathrm{op}} \longrightarrow \mathcal{A}b$$

assigning to a finitely generated projective module its abelian group of M-valued quadratic, even or symmetric forms, respectively, admit animations (or non-abelian derived functors in more classical terminology) which we term

$$\Qoppa^{gq}_{M}, \quad \Qoppa^{ge}_{M} \quad \text{and} \quad \Qoppa^{gs}_{M} : \mathcal{D}^{p}(R)^{\mathrm{op}} \longrightarrow \mathcal{S}p,$$

respectively. One of the main results of Paper [I] are equivalences

$$\Qoppa^{gq}_{M} \simeq \Qoppa^{\geq 2}_{M}, \quad \Qoppa^{ge}_{M} \simeq \Qoppa^{\geq 1}_{M} \quad \text{and} \quad \Qoppa^{gs}_{M} \simeq \Qoppa^{\geq 0}_{M},$$

see §[I].4.2. It is also not difficult to see that no further members of the genuine family arise as animations of functors $\mathrm{Proj}(R) \to \mathcal{A}b$.

Turning to a different kind of example, consider the ∞-categories $\mathcal{S}p_B = \mathrm{Fun}(B, \mathcal{S}p)$ for some $B \in \mathcal{S}$. Entirely parallel to the discussion above, one can derive hermitian structures on the compact objects of $\mathcal{S}p_B$ from triples (M, N, α) with $M \in (\mathcal{S}p_{B\times B})^{\mathrm{hC}_2}$ and $\alpha\colon N \to (\Delta^* M)^{\mathrm{tC}_2}$ a map in $\mathcal{S}p_B$, where $\Delta\colon B \to B{\times}B$ is the diagonal, §[I].4.4. The most important examples of such functors are the visible Poincaré structures $\Omega^{\mathrm{v}}_{\xi}$ given by the triples

$$(\Delta_!\xi, \xi, u\colon \xi \to (\Delta^*\Delta_!\xi)^{\mathrm{tC}_2}),$$

where $\xi\colon B \to \mathrm{Pic}(\mathbb{S})$ is some stable spherical fibration over B, where $\Delta_!\colon \mathcal{S}p_B \to \mathcal{S}p_{B\times B}$ is the left adjoint to Δ^* and where u is the unit of this adjunction (which factors through $\xi \to (\Delta^*\Delta_!\xi)^{\mathrm{hC}_2}$ since Δ is invariant under the C_2-action on $B \times B$). These hermitian structures are automatically Poincaré with associated duality given by

$$X \longmapsto \hom_B(X, \Delta_!\xi),$$

the Costenoble-Waner duality functor twisted by ξ. The reason these are so important is that any closed manifold M, in fact every Poincaré complex, defines an very interesting element, its visible symmetric signature, in $\mathrm{Pn}(\mathcal{S}p^{\omega}_M, \Omega^{\mathrm{v}}_{\nu})$, where $\nu\colon M \to \mathrm{Pic}(\mathbb{S})$ is its stable normal bundle of a manifold or more generally the Spivak fibration of a Poincaré complex.

As a common special case of the previous examples, let us finally mention the universal Poincaré structure Ω^{u} on $\mathcal{S}p^{\omega} = \mathrm{Mod}^{\omega}_{\mathbb{S}}$ from §[I].4.1: It is associated to the triple $(\mathbb{S}, \mathbb{S}, \mathbb{S} \to \mathbb{S}^{\mathrm{tC}_2})$, with structure map the unit of $\mathbb{S}^{\mathrm{tC}_2}$, which happens to agree with the Tate diagonal in this special case. The Poincaré ∞-category $(\mathcal{S}p^{\omega}, \Omega^{\mathrm{u}})$ represents the functors Pn and Fm, i.e. for every Poincaré ∞-category $(\mathcal{C}, \Omega)$ and every hermitian ∞-category $(\mathcal{D}, \Phi)$, there are equivalences

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}((\mathcal{S}p^{\omega}, \Omega^{\mathrm{u}}), (\mathcal{C}, \Omega)) \simeq \mathrm{Pn}(\mathcal{C}, \Omega) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{h}}_{\infty}}((\mathcal{S}p^{\omega}, \Omega^{\mathrm{u}}), (\mathcal{D}, \Phi)) \simeq \mathrm{Fm}(\mathcal{D}, \Phi)$$

natural in the input.

1. POINCARÉ-VERDIER SEQUENCES AND ADDITIVE FUNCTORS

In this section, we study the analogue of (split) Verdier sequences in the context of Poincaré ∞-categories, as well as their analogue for idempotent complete Poincaré ∞-categories, which, following a suggestion of Clausen and Scholze, we call *Karoubi sequences*. In particular, our terminology differs from that of Blumberg-Gepner-Tabuada [BGT13]; see Appendix A for a thorough discussion.

After developing the example of module ∞-categories in some detail, we proceed to introduce the notions of *additive*, *Verdier-localising*, and *Karoubi-localising* functors $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$, encoding the preservation of an increasing number of such sequences, or rather, for not necessarily stable $\mathcal{E}$, of a mild generalisation thereof in the form of certain cartesian and cocartesian squares in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. These three notions we introduce correspond loosely to satisfying Waldhausen's additivity theorem, Quillen's localisation theorem and Bass' strengthening thereof.

The notion of an additive functor from $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ to $\mathcal{S}p$ is central in our work, since it essentially abstracts the minimal property enjoyed by our main subject of interest, the functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ (only to be defined in Definition 4.2.1), that permits us to develop a general theory decomposing it into simpler pieces; furthermore GW is characterised as the universal such additive functor with a transformation from the functor Pn, space of Poincaré forms. Analogously to K-theory, the functor GW turns out to be furthermore Verdier-localising, justifying as well the study of that notion. Finally, just as non-connective K-theory relates to K-theory, the search for a Karoubi-localising approximation of GW will yield in Paper [IV] the Karoubi-Grothendieck-Witt spectrum functor $\mathbb{GW}$.

In the present section, we only give the very basic properties of such functors, as the only immediately interesting examples are the space valued functors Cr and Pn. After §2 introduces more interesting examples, we return to a detailed study of additive functors in §3. The study of Karoubi-localising functors will be taken up in Paper [IV].

1.1. **Poincaré-Verdier sequences.** As the basis for our study we require a rather detailed analysis of Verdier sequences in the set-up of stable ∞-categories. Essentially all of the results we need seem well-known, but have not been coherently organised. To keep the exposition brief we have largely collected such statements and their proofs into Appendix A, the focus of the present section being on incorporating Poincaré structures.

A sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \tag{2}$$

in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ with vanishing composite is a *Verdier sequence* (Definition A.1.1) if it is both a fibre and a cofibre sequence in $\mathrm{Cat}_\infty^{\mathrm{ex}}$, in which case we refer to f as a *Verdier inclusion* and to p as a *Verdier projection*. We also say that (2) is *split* (Definition A.2.4) if p or equivalently f admits *both* adjoints.

1.1.1. **Definition.** A sequence

$$(\mathcal{C}, \Omega) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\vartheta)} (\mathcal{E}, \Psi) \tag{3}$$

of Poincaré functors with vanishing composite is called a *Poincaré-Verdier sequence* if it is both a fibre sequence and a cofibre sequence in $\mathrm{Cat}_\infty^{\mathrm{p}}$, in which case we call (f,η) a *Poincaré-Verdier inclusion* and (p,ϑ) a *Poincaré-Verdier projection*. We shall say that (3) is *split* if the underlying Verdier sequence splits.

1.1.2. **Remark.** As explained in Remark A.1.2, the (pointwise) condition of the composite vanishing implies that sequence (2) extends to a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\ \{0\} \arrow[r] & \mathcal{E} \end{tikzcd}$$

in an essentially unique manner and the condition that it forms a Verdier sequence amounts to this square being both cartesian and cocartesian in $\mathrm{Cat}_\infty^{\mathrm{ex}}$. If Ω, Φ and Ψ are now Poincaré structures on $\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$ respectively, then, since $\Psi(0) \simeq 0 \in \mathcal{S}p$, any null functor carries an essentially unique hermitian structure, and this hermitian structure is automatically Poincaré since the duality on $\mathcal{E}$ preserves zero objects. Thus, a sequence of Poincaré functors with null composite uniquely extends to a square as above of Poincaré ∞-categories, and the condition of being Poincaré-Verdier is the condition that this square is cartesian and cocartesian in $\mathrm{Cat}_\infty^{\mathrm{p}}$.

1.1.3. **Observation.** *Since the forgetful and hyperbolic functors are both-sided adjoints to one another, we immediately find that the underlying sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of a (split) Poincaré-Verdier sequence is a (split) Verdier sequence, and that the hyperbolisation of any (split) Verdier sequence is a (split) Poincaré-Verdier sequence.*

We now proceed to consider Poincaré-Verdier sequences more closely. To begin, recall that the inclusion $\mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ preserves both limits and colimits (Proposition [I].6.1.4), and since it is also conservative we get that it detects limits and colimits. We may hence test if a given sequence of Poincaré ∞-categories is a (co)fibre sequence at the level of $\mathrm{Cat}_\infty^{\mathrm{h}}$. In addition, the projection $\mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ preserves small limits and colimits (Lemma [I].6.1.2), and is a bicartesian fibration with backwards transition maps given by restriction and forward transition maps given by left Kan extensions. This means that limits in $\mathrm{Cat}_\infty^{\mathrm{h}}$ are computed by first taking the limit $\mathcal{D}$ of underlying stable ∞-categories, then pulling back all the quadratic functors to $\mathcal{D}^{\mathrm{op}}$, and finally calculating the limit of the resulting diagram in the ∞-category of quadratic functors on $\mathcal{D}^{\mathrm{op}}$. Similarly, colimits are computed by first computing the colimit $\mathcal{D}$ of underlying stable ∞-categories, then left Kan extending all the quadratic functors to $\mathcal{D}^{\mathrm{op}}$, and finally calculating the colimit of the resulting diagram in the ∞-category of quadratic functors on $\mathcal{D}^{\mathrm{op}}$. We also note that limits and colimits in $\mathrm{Fun}^{\mathrm{q}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}p)$, i.e. of quadratic functors, can be computed in $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}p)$, see Remark [I].1.1.15.

1.1.4. **Proposition.** *Let*

$$(\mathcal{C}, \Omega) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\vartheta)} (\mathcal{E}, \Psi) \tag{4}$$

be a sequence in $\mathrm{Cat}_\infty^{\mathrm{p}}$ *with vanishing composite. Then the following holds:*

i) The sequence (4) *is a fibre sequence in* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *if and only if its image in* $\mathrm{Cat}^{\mathrm{ex}}_\infty$ *is a fibre sequence and* $\eta\colon Ϙ \to f^*\Phi$ *is an equivalence.*

ii) The sequence (4) *is a cofibre sequence in* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *if and only if its image in* $\mathrm{Cat}^{\mathrm{ex}}_\infty$ *is a cofibre sequence and* $\vartheta\colon \Phi \to p^*\Psi$ *exhibits* $\Psi\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{S}p$ *as the left Kan extension of* Φ *along* p^{op}.

iii) It is a Poincaré-Verdier sequence if and only if its image in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ *is a Verdier sequence, and the Poincaré structures on* $\mathcal{C}$ *and* $\mathcal{E}$ *are obtained from that of* $\mathcal{D}$ *by pullback and left Kan extension, respectively.*

Proof. Specialising the preceding discussion to the case of squares with one corner the zero Poincaré ∞-category gives that (4) is a fibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$ if and only if its image in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is a fibre sequence and $Ϙ \to f^*\Phi \to f^*p^*\Psi$ is a fibre sequence in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p)$, which, since $f^*p^*\Psi \simeq 0$, just means that the map $Ϙ \to f^*\Phi$ is an equivalence. This proves i).

Similarly, (4) is a cofibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$ if and only if its image in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is a cofibre sequence and $p_!f_!Ϙ \to p_!\Phi \to \Psi$ is a cofibre sequence of quadratic functors, which, since $p_!f_!Ϙ'' \simeq 0$ just means that the map $p_!\Phi \to \Psi$ is an equivalence, so that we get ii). □

Combining this Proposition with Proposition A.1.9 which states that an exact functor $\mathcal{C} \to \mathcal{D}$ between stable ∞-categories is a Verdier inclusion if and only if it is fully faithful and its essential image is closed under retracts in $\mathcal{D}$, we get:

1.1.5. **Corollary.** *A Poincaré functor* $(f, \eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ *is a Poincaré-Verdier inclusion if and only if* f *is fully faithful, its essential image is closed under retracts, and the map* $\eta\colon Ϙ \to f^*\Phi$ *is an equivalence.*

To state the analogous corollary concerning Poincaré-Verdier projections, let us first stress that we take the localisation $\mathcal{D}[W^{-1}]$ of an ∞-category $\mathcal{D}$ at a set W of morphisms to mean the initial ∞-category under $\mathcal{D}$ in which the morphisms from W become invertible. Beware that we differ in our use of terminology here from Lurie, who requires the existence of adjoints to the functor $\mathcal{D} \to \mathcal{D}[W^{-1}]$. See Lemma A.2.3 for the precise relationship between the two notions of localisations.

Given an exact functor $f\colon \mathcal{C} \to \mathcal{D}$, the *Verdier quotient* $\mathcal{D}/\mathcal{C}$ of $\mathcal{D}$ by $\mathcal{C}$ is the localisation of $\mathcal{D}$ with respect to the collection of maps whose fibre lies in the smallest stable subcategory containing the essential image of f (see Definition A.1.3).

By [NS18, Theorem I.3.3(i)], $\mathcal{D}/\mathcal{C}$ is again a stable ∞-category and the canonical functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact. For a further discussion of Verdier quotients, we refer the reader to §A.1. The main output of the discussion there is Proposition A.1.6, which shows that an exact functor is a Verdier projection if and only if it is a localisation. Combining this with Proposition 1.1.4, we get:

1.1.6. **Corollary.** *A Poincaré functor* $(p, \vartheta)\colon (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ *is a Poincaré-Verdier projection if and only if* $p\colon \mathcal{D} \to \mathcal{E}$ *is a localisation and* $\Phi \to p^*\Psi$ *exhibits* Ψ *as the left Kan extension of* Φ *along* p.

1.1.7. **Example.** If $p\colon \mathcal{D} \to \mathcal{E}$ is a Verdier projection and Φ is a Poincaré structure on $\mathcal{D}$, then the hermitian structure $p_!\Phi$ on $\mathcal{E}$ and the canonical hermitian refinement of p are Poincaré if and only if $\ker(p)$ is invariant under the duality, and in this case

$$(\ker(p), \Phi) \longrightarrow (\mathcal{D}, \Phi) \longrightarrow (\mathcal{E}, p_!\Phi).$$

is a Poincaré-Verdier sequence.

Indeed, if $p_!\Phi$ and p are Poincaré, then it is immediate that $\ker(p)$ is closed under the duality. Conversely if $\ker(p)$ is closed under the duality, then, since the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ preserves colimits, the cofibre of the inclusion $(\ker(p), \Phi) \to (\mathcal{D}, \Phi)$ in $\mathrm{Cat}^{\mathrm{p}}_\infty$ must be equivalent to a Poincaré ∞-category of the form $(\mathcal{E}, \Psi)$ for some Poincaré structure on $\mathcal{E}$ equipped with a Poincaré functor $(p, \vartheta)\colon (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$. The latter is then a Poincaré-Verdier projection by construction, and by Proposition 1.1.4 ii) the natural transformation $p_!\Phi \to \Psi$ determined by ϑ must be an equivalence, and so the desired properties of $p_!\Phi$ follow.

1.1.8. **Remark.** Whenever a functor $p\colon \mathcal{C} \to \mathcal{D}$ admits a left adjoint g, the left Kan extension $p_!Ϙ$ of a functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ along (the opposite of) p is given by $g^*Ϙ$ (along with the transformation $Ϙ \to p^*g^*Ϙ$ induced by the co-unit of the adjunction).

For exact p, the left Kan extension $p_!\Qoppa$ can often be computed by the following trick, even if p is does not have a left adjoint (cf. Lemma [I].1.4.1): Consider the square

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r] \arrow[d, "p^{\mathrm{op}}"'] & \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \arrow[d, "\mathrm{Ind}(p^{\mathrm{op}})"] \\
\mathcal{D}^{\mathrm{op}} \arrow[r] & \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})
\end{tikzcd}$$

Since p is exact, the functor $\mathrm{Ind}(p^{\mathrm{op}})\colon\ \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})$ preserves all colimits and as its target is presentable, it admits a right adjoint $\widetilde{g}\colon\ \mathrm{Ind}(\mathcal{D}^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ and the left Kan extension $p_!\Qoppa\colon\ \mathcal{D} \to \mathcal{S}p$ is given by the composite

$$\mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Ind}(\mathcal{D}^{\mathrm{op}}) \xrightarrow{\widetilde{g}} \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{Ind}(\Qoppa)} \mathrm{Ind}(\mathcal{S}p) \xrightarrow{\mathrm{colim}} \mathcal{S}p;$$

indeed, since $\mathcal{D}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{D}^{\mathrm{op}})$ is fully faithful $p_!\Qoppa$ is the restriction of left Kan extension of $\Qoppa$ to $\mathrm{Ind}(\mathcal{D}^{\mathrm{op}})$. By commutativity of the above square, this is equivalent to the left Kan extension along $\mathrm{Ind}(p^{\mathrm{op}})$ of the left Kan extension $\widetilde{\Qoppa}\colon\ \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{S}p$ of $\Qoppa$ to $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$, which in turn is given explicitly as the composite

$$\widetilde{\Qoppa}\colon\ \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{Ind}(\Qoppa)} \mathrm{Ind}(\mathcal{S}p) \xrightarrow{\mathrm{colim}} \mathcal{S}p.$$

Finally, left Kan extensions along $\mathrm{Ind}(p^{\mathrm{op}})$ are given by restriction along $\widetilde{g}$ by adjunction, which results in the claimed formula.

For example, in the special case where $p\colon\ \mathcal{C} \to \mathcal{D}$ is a Verdier projection, the composite $\mathcal{D}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ takes $p(c)$ to $\mathrm{colim}_{x\in(ker(p)_{c/})^{\mathrm{op}}}\,\mathrm{fib}(c\to x)$ by [NS18, Theorem I.3.3], where the fibre is formed in $\mathcal{C}$ (as opposed to $\mathcal{C}^{\mathrm{op}}$). Ultimately the above procedure therefore results in the formula

$$(p_!\Qoppa)(p(c)) \simeq \operatorname*{colim}_{c'\in(\ker(p)_{c/})^{\mathrm{op}}} \Qoppa(\mathrm{fib}(c \to c'))$$

for the left Kan extension of $\Qoppa$ along a Verdier projection, though this is of course also a simple consequence of the pointwise formulae for Kan extensions.

Finally, we record:

1.1.9. **Proposition.** *Pullbacks of Poincaré-Verdier projections are Poincaré-Verdier projections.*

Proof. At the level of underlying ∞-categories, this is A.1.11, so we are left to show that given a cartesian square

$$\begin{tikzcd}
(\mathcal{C},\Qoppa) \arrow[r, "f"] \arrow[d, "p"] & (\mathcal{D},\Phi) \arrow[d, "q"] \\
(\mathcal{C}',\Qoppa') \arrow[r, "g"] & (\mathcal{D}',\Phi')
\end{tikzcd}$$

in which q is a Verdier projection and the natural map $q_!\Phi \Rightarrow \Phi'$ is an equivalence, also the map $p_!\Qoppa \Rightarrow \Qoppa'$ is one. To this end, we use the formula from the previous remark to compute

$$\begin{aligned}
p_!\Qoppa(p(c)) &\simeq \operatorname*{colim}_{c'\in(\ker(p)_{c/})^{\mathrm{op}}} \Qoppa(\mathrm{fib}(c \to c')) \\
&\simeq \operatorname*{colim}_{d\in(\ker(q)_{f(c)/})^{\mathrm{op}}} \Phi(\mathrm{fib}(f(c) \to d)) \times_{\Phi'(qf(c))} \Qoppa'(p(c)) \\
&\simeq (q_!\Phi)(qf(c)) \times_{\Phi'(qf(c))} \Qoppa'(p(c))
\end{aligned}$$

from which the claim is immediate. □

1.2. **Split Poincaré-Verdier sequences and Poincaré recollements.** We turn to split Poincaré-Verdier sequences, which are by definition Poincaré-Verdier sequences in which the underlying Verdier sequence is split. Let us therefore mention from Lemma A.2.5 that a sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \tag{5}$$

in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$ with vanishing composite is a split Verdier sequence if and only if it is a fibre sequence and p admits fully faithful left and right adjoints, if and only if it is a cofibre sequence and f is fully faithful and admits left and right adjoints. Furthermore, this notion is equivalent to that of a stable recollement.

In the context of Poincaré ∞-categories, the existence of one of the adjoints, in fact, implies that of the other:

1.2.1. **Observation.** *The underlying functor p of a Poincaré functor admits a left adjoint if and only if it admits a right adjoint.*

For a left or right adjoint to p gives a right or left adjoint to p^{op}, respectively, but p and p^{op} are naturally equivalent by means of the dualities in source and target.

With this at hand, we derive the following criterion to recognise split Poincaré-Verdier sequences.

1.2.2. **Proposition.** *Let*

$$(\mathcal{C}, \Omega) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\vartheta)} (\mathcal{E}, \Psi) \tag{6}$$

be a sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$ with vanishing composite. Then the following holds:

- *i) Suppose that (6) is a fibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$. Then (6) is a split Poincaré-Verdier sequence if and only if p admits a fully faithful left adjoint g and the transformation*
$$g^*\Phi \overset{g^*\vartheta}{\Longrightarrow} g^*p^*\Psi \overset{u^*}{\Longrightarrow} \Psi$$
is an equivalence, where $u\colon \mathrm{id}_{\mathcal{C}} \Rightarrow pg$ denotes an adjunction unit.
- *ii) Suppose that (6) is a cofibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$. Then (6) is a split Poincaré-Verdier sequence if and only if f is fully faithful, $\eta\colon \Omega \to f^*\Phi$ is an equivalence, and f admits a right adjoint.*

Proof. Assume that (6) is a fibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$, hence its image in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is a fibre sequence as well. By the previous observation, the existence of a left adjoint to p implies that of a right adjoint, so the underlying sequence of stable ∞-categories is a split Verdier-sequence if and only if p admits a fully faithful left adjoint $g\colon \mathcal{E} \to \mathcal{D}$. In this case, it follows from Remark 1.1.8 that $g^*\Phi$ is a left Kan extension of Φ, and the transformation from the statement is the extension of ϑ. Thus, Ψ is a left Kan extension of Φ if and only if it is an equivalence, which gives the claim by Proposition 1.1.4.

The second item is immediate from Observation 1.2.1 and Proposition 1.1.4 i). ☐

1.2.3. **Corollary.**

- *i) A Poincaré functor $(f,\eta)\colon (\mathcal{C},\Omega) \to (\mathcal{D},\Phi)$ is a split Poincaré-Verdier inclusion if and only if f is fully faithful, admits a right adjoint, and the map $\eta\colon \Omega \to f^*\Phi$ is an equivalence.*
- *ii) A Poincaré functor $(p,\vartheta)\colon (\mathcal{D},\Phi) \to (\mathcal{E},\Psi)$ is a split Poincaré-Verdier projection if and only if p admits a fully faithful left adjoint g and the composite transformation $g^*\Phi \overset{g^*\vartheta}{\Longrightarrow} g^*p^*\Psi \overset{u^*}{\Longrightarrow} \Psi$ is an equivalence.*

1.2.4. **Remark.** By means of the equivalence $g^*\Phi \simeq \Psi$, the left adjoint g to a Poincaré-Verdier projection $p\colon (\mathcal{D},\Phi) \to (\mathcal{E},\Psi)$ automatically becomes an hermitian functor $(\mathcal{E},\Psi) \to (\mathcal{D},\Phi)$ (which is usually not Poincaré). One readily checks that the unit gives an equivalence of hermitian functors $\mathrm{id}_{(\mathcal{E},\Psi)} \Rightarrow pg$, making g a section of p in $\mathrm{Cat}^{\mathrm{h}}_\infty$.

In fact, granting that the ∞-categories $\mathrm{He}(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{E},\Psi)))$ provide a Cat_∞-enrichment to $\mathrm{Cat}^{\mathrm{h}}_\infty$ (a fact we will neither prove nor even make precise here), the adjunction between g and p is an enriched one, i.e. its unit $\mathrm{id}_{\mathcal{E}} \Rightarrow gp$ and counit $pg \Rightarrow \mathrm{id}_{\mathcal{D}}$ canonically promote to objects in $\mathrm{He}(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{E},\Psi),(\mathcal{E},\Psi)))$ and $\mathrm{He}(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{D},\Phi)))$, such that the triangle identities hold in these ∞-categories.

Conversely, the existence of such an enriched left adjoint to p, whose unit is an equivalence, is readily checked to amount precisely to the conditions of Corollary 1.2.3 ii).

Similarly, the existence of an enriched right adjoint with counit an equivalence boils down to precisely the conditions in i) above, and therefore detects split Poincaré-Verdier inclusions; in particular, the counit always provides the right adjoint to a Poincaré-Verdier inclusion with an hermitian structure (which is again usually not Poincaré).

We warn the reader that the analogous statements involving the right adjoint to a Poincaré-Verdier projection and the left adjoint to a Poincaré-Verdier inclusion fail entirely; for instance in the metabolic Poincaré-Verdier sequence of Example 1.2.5 below, the only hermitian refinement of the right adjoint to the projection is null, and so certainly does not give rise to a splitting of p.

The following is the most important example of a split Poincaré-Verdier sequence. It is in fact universal by Theorem 1.2.9 below and will be fundamental to several results we prove:

1.2.5. **Example.** For any Poincaré ∞-category $(\mathcal{C}, Ϙ)$, the sequence

$$(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, Ϙ) \xrightarrow{\mathrm{met}} (\mathcal{C}, Ϙ)$$

is a split Poincaré-Verdier sequence, the *metabolic fibre sequence*; the left hand Poincaré functor is given by sending x to $x \to 0$, together with the identification $\Omega Ϙ(X) \simeq \mathrm{fib}(Ϙ(0) \to Ϙ(x))$.

Proof. The underlying sequence of stable ∞-categories, described in detail in Proposition A.2.12, is a split Verdier sequence. The sequence is a fibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$ by Proposition 1.1.4 i). To see that it is a split Poincaré-Verdier sequence, apply Proposition 1.2.2 using the fully faithful left adjoint to met given by the exact functor $g\colon \mathcal{C} \to \mathrm{Met}(\mathcal{C})$ sending x to $0 \to x$. ☐

We have systematically collected examples into Section 1.4 below, and encourage the reader yearning for them to jump ahead to that section. In the remainder of this section, we provide an analogue of the classification of split Verdier-projections, i.e. that they arise as pullbacks of the target functor $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$. The role of this universal split Verdier projection is played by the metabolic Poincaré-Verdier sequence above. To this end we first record:

1.2.6. **Corollary.** *A pullback of a split Poincaré-Verdier projection is again a split Poincaré-Verdier projection.*

Proof. From Corollary A.2.7, we know that the underlying functor of the pullback is again a split Verdier projection. Thus, it remains to analyse the Poincaré structures, where the claim is a straight-forward consequence of Corollary 1.2.3 or even 1.1.9. ☐

Now, recall that for a split Verdier sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

with adjoints $g \dashv f \dashv g'$ and $q \dashv p \dashv q'$, the (co)units fit into fibre sequences

$$fg' \Longrightarrow \mathrm{id}_{\mathcal{D}} \Longrightarrow q'p \quad \text{and} \quad qp \Longrightarrow \mathrm{id}_{\mathcal{D}} \Longrightarrow fg;$$

see Lemma A.2.5. Furthermore, there is a canonical equivalence $gq' \simeq \Sigma_{\mathcal{C}} g'q$ and denoting this functor $c\colon \mathcal{E} \to \mathcal{C}$, there results a cartesian square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "g \to cp"] \arrow[d, "p"] & \mathrm{Ar}(\mathcal{C}) \arrow[d, "\mathrm{t}"] \\ \mathcal{E} \arrow[r, "c"] & \mathcal{C}, \end{tikzcd} \tag{7}$$

cf. Proposition A.2.12. We now set out to show that this diagram canonically upgrades to a pullback in $\mathrm{Cat}^{\mathrm{p}}_\infty$, when extracted from a Poincaré-Verdier sequence

$$(\mathcal{C}, Ϙ) \xrightarrow{f} (\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi).$$

We need:

1.2.7. **Lemma.** *For a left split Verdier sequence* $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *and an hermitian structure* Φ *on* $\mathcal{D}$ *such that* $\mathrm{B}_\Phi(q(e), f(c)) \simeq 0$ *for every* $c \in \mathcal{C}$ *and* $e \in \mathcal{E}$*, the fibre sequence*

$$qp(d) \longrightarrow d \longrightarrow fg(d)$$

induces a fibre sequence

$$\Phi(fg(d)) \longrightarrow \Phi(d) \longrightarrow \Phi(qp(d))$$

of spectra.

The assumption of the lemma is satisfied for all Poincaré-Verdier sequences $(\mathcal{C}, Ϙ) \xrightarrow{f} (\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi)$ since then

$$\mathrm{B}_\Phi(q(e), f(c)) \simeq \mathrm{Hom}_{\mathcal{D}}(q(e), \mathrm{D}_\Phi f(c)) \simeq \mathrm{Hom}_{\mathcal{D}}(q(e), f(\mathrm{D}_Ϙ c)) \simeq \mathrm{Hom}_{\mathcal{E}}(e, pf(\mathrm{D}_Ϙ c)) \simeq 0.$$

Proof. From Example [I].1.1.21, we find the fibre of $\Phi(d) \to \Phi(qp(d))$ equivalent to the total fibre of the diagram

$$\begin{tikzcd}
\Phi(d) \arrow[r] \arrow[d] & \Phi(qp(d)) \arrow[d] \\
\mathrm{B}_\Phi(qp(d), d) \arrow[r] & \mathrm{B}_\Phi(qp(d), qp(d)).
\end{tikzcd}$$

The fibre of the lower horizontal map is $\mathrm{B}_\Phi(qp(d), fg(d))$ which vanishes by assumption. □

We next equip the horizontal functors of (7) with hermitian structures.

1.2.8. **Construction.** Given a split Verdier sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ and a hermitian structure Φ on $\mathcal{D}$ such that $\mathrm{B}_\Phi(q(e), f(c)) \simeq 0$, denote by $\Qoppa$ its restriction to $\mathcal{C}$ and by Ψ its left Kan extension to $\mathcal{E}$. We thus find

$$\Phi(fg(d)) \simeq \Qoppa(g(d)) \quad \text{and} \quad \Phi(qp(d)) \simeq \Psi(p(d))$$

so that the fibre sequence of lemma 1.2.7 gives a natural equivalence

$$\Phi \simeq \mathrm{fib}\left(p^*\Psi \to g^*\Qoppa^{[1]}\right).$$

Applying the unit transformation $u : \mathrm{id}_\mathcal{D} \to q'p$ we obtain a diagram

$$\begin{tikzcd}
\Phi(q'p(d)) \arrow[r] \arrow[d] & \Psi(pq'p(d)) \arrow[r] \arrow[d] & \Qoppa^{[1]}(gq'p(d)) \arrow[d] \\
\Phi(d) \arrow[r] & \Psi(p(d)) \arrow[r] & \Qoppa^{[1]}(g(d))
\end{tikzcd} \tag{8}$$

whose rows are fibre sequences. By the triangle identities, the unit $p(d) \to pq'p(d)$ is an equivalence, since it is a one-sided inverse to the counit, which is an equivalence as q' is fully faithful. Thus, the middle vertical arrow in (8) is an equivalence.

It follows that the natural transformation $p^*\Psi \to g^*\Qoppa^{[1]}$ factors naturally through the maps $(gq'p)^*\Qoppa^{[1]} \to g^*\Qoppa^{[1]}$ induced by the unit of $p \vdash q'$. But since p^{op} is a localisation this factorisation

$$\Psi \circ p^{\mathrm{op}} = p^*\Psi \longrightarrow (gq'p)^*\Qoppa^{[1]} = \left((gq')^*\Qoppa^{[1]}\right) \circ p^{\mathrm{op}}$$

can be regarded as a natural transformation

$$\eta : \Psi \to (gq')^*\Qoppa^{[1]}$$

providing the desired hermitian structure to the functor $c = gq' : \mathcal{E} \to \mathcal{C}$.

The diagram (8) also provides an equivalence

$$\mathrm{cof}\left[(q'p)^*\Phi \Rightarrow \Phi\right] \simeq \mathrm{fib}\left[(gq'p)^*\Qoppa^{[1]} \Rightarrow g^*\Qoppa^{[1]}\right],$$

so in particular, a natural transformation

$$\xi : \Phi \to (g \to cp)^*(\Qoppa^{[1]})^{\mathrm{met}}.$$

Furthermore, the diagram

$$\begin{tikzcd}
\Phi \arrow[r, "\xi"] \arrow[d] & (g \to cp)^*(\Qoppa^{[1]})^{\mathrm{met}} \arrow[d, "\mathrm{met}"] \\
p^*\Psi \arrow[r, "\eta"] & (cp)^*\Qoppa^{[1]}
\end{tikzcd} \tag{9}$$

commutes by construction.

In total, we obtain a diagram

$$\begin{tikzcd}
(\mathcal{D}, \Phi) \arrow[r, "(g \to cp, \xi)"] \arrow[d, "p"] & \mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}) \arrow[d, "\mathrm{met}"] \\
(\mathcal{E}, \Psi) \arrow[r, "(c, \eta)"] & (\mathcal{C}, \Qoppa^{[1]})
\end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{h}}_\infty$. At the level of underlying stable ∞-categories, it is cartesian by Proposition A.2.12. Furthermore, the diagram (9) is also cartesian: By (8), both vertical cofibres are given by $g^*\Qoppa^{[1]}$, connected by the identity. We conclude that the diagram above is cartesian in $\mathrm{Cat}^{\mathrm{h}}_\infty$.

The following is then the main result of the present section:

1.2.9. **Theorem.** *The square*

$$\begin{tikzcd}
(\mathcal{D},\Phi) \arrow[r, "(g\to cp{,}\xi)"] \arrow[d, "p"] & \mathrm{Met}(\mathcal{C},\Qoppa^{[1]}) \arrow[d, "\mathrm{met}"] \\
(\mathcal{E},\Psi) \arrow[r, "(c{,}\eta)"] & (\mathcal{C},\Qoppa^{[1]})
\end{tikzcd}$$

is cartesian in $\mathrm{Cat}^{\mathrm{p}}_\infty$ *for every split Poincaré-Verdier sequence*

$$(\mathcal{C},\Qoppa)\xrightarrow{f}(\mathcal{D},\Phi)\xrightarrow{p}(\mathcal{E},\Psi).$$

Proof of Theorem 1.2.9. As limits in $\mathrm{Cat}^{\mathrm{p}}_\infty$ are detected in $\mathrm{Cat}^{\mathrm{h}}_\infty$ by Proposition [I].6.1.4, it only remains to show that the horizontal arrows are Poincaré functors, i.e. that they preserve the dualities. It suffices to treat the top arrow, since the lower one is obtained by forming cofibres (with respect to the canonical maps from $(\mathcal{C},\Qoppa)$), and $\mathrm{Cat}^{\mathrm{p}}_\infty$ is closed under colimits in $\mathrm{Cat}^{\mathrm{h}}_\infty$ by Proposition [I].6.1.4. Recall then that generally

$$\mathrm{D}_{\Qoppa^{\mathrm{met}}}(f : x \to y) \simeq \left[\mathrm{D}_\Qoppa \operatorname{cof}(f) \to \mathrm{D}_\Qoppa y\right],$$

whence it remains to check that the maps

$$g(\mathrm{D}_\Phi d) \longrightarrow \mathrm{D}_{\Qoppa^{[1]}} \operatorname{cof}(g(d) \to cp(d)) \quad \text{and} \quad cp(\mathrm{D}_\Phi d) \longrightarrow \mathrm{D}_{\Qoppa^{[1]}}(cp(d))$$

induced by ξ are equivalences. But through the fibre sequence $fg' \Rightarrow \mathrm{id}_{\mathcal{D}} \Rightarrow q'p$, the target of the left hand map becomes

$$\mathrm{D}_\Qoppa gfg'(d) \simeq g'fg(\mathrm{D}_\Phi d),$$

and unwinding definitions, the map induced by ξ is given by the unit of $f \vdash g'$, which is an equivalence since f is fully faithful. Similarly, the target of the second map is given by

$$\Sigma_{\mathcal{C}}\mathrm{D}_\Qoppa gq'p(d) \simeq \Sigma_{\mathcal{C}}g'qp(\mathrm{D}_\Phi d)$$

and the map in question unwinds to an instance of the natural equivalence $gq' \Rightarrow \Sigma_{\mathcal{C}}g'q$ constructed before Proposition A.2.12. □

1.2.10. **Remark.** Using the identification $c \simeq \Sigma_{\mathcal{C}}g'q$, a lengthy diagram chase shows that the composite

$$\Psi \xrightarrow{\eta} c^*\Qoppa^{[1]} \simeq \Sigma_{\mathcal{S}p} \circ (g'q)^*\Qoppa \circ \Omega_{\mathcal{C}^{\mathrm{op}}} \longrightarrow (g'q)^*\Qoppa$$

is given by the composite of the two canonical hermitian structures carried by the functors g' and q, see Remark 1.2.4.

The uniqueness of the classifying map in Theorem 1.2.9 is implied by the following hermitian analogue of Proposition A.2.14:

1.2.11. **Proposition.** *Let us consider a split Verdier sequence* $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ *and a hermitian structure* Φ *on* $\mathcal{D}$ *such that* $\mathrm{B}_\Phi(q(e), f(c)) \simeq 0$ *for all* $c \in \mathcal{C}$ *and* $e \in \mathcal{E}$*. Then for every hermitian ∞-category* $(\mathcal{C}',\Qoppa')$ *the full subcategory of* $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),\mathrm{Met}(\mathcal{C}',\Qoppa'^{[1]}))$ *spanned by the pairs* (F,η) *that give rise to adjointable squares*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "F"] \arrow[d] & \mathrm{Ar}(\mathcal{C}') \arrow[d, "t"] \\
\mathcal{E} \arrow[r, "\overline{F}"] & \mathcal{C}'
\end{tikzcd}$$

on underlying ∞-categories is equivalent to $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Qoppa),(\mathcal{C}',\Qoppa'))$ *as a hermitian ∞-category via restriction to horizontal fibres, where* $\Qoppa$ *denotes the restriction of* Φ *to* $\mathcal{C}$*.*

Here, adjointability refers to the diagrams formed by passing to vertical left or right adjoints commuting, see [Lur09a, §7.3.1] for a detailed discussion of such squares.

Given Proposition A.2.14, one might expect a hermitian version of adjointability to appear in the present statement; this is in fact simply implied by the adjointability at the level of underlying ∞-categories, essentially since a morphism in $\mathrm{Fun}^{\mathrm{h}}((\mathcal{C},\Qoppa),(\mathcal{D},\Phi))$ is invertible if and only if its image in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D})$ is.

1.2.12. **Corollary.** *The horizontal maps in Theorem 1.2.9 are determined up to contractible choice by yielding a pullback on underlying stable ∞-categories and inducing the identity functor on the vertical fibre* $(\mathcal{C},\Qoppa)$.

Put differently, $\mathrm{met}\colon \mathrm{Met}(\mathcal{C},\Qoppa^{[1]}) \to (\mathcal{C},\Qoppa^{[1]})$ is the universal Poincaré-Verdier projection with fibre $(\mathcal{C},\Qoppa)$.

Proof. Note first that the lower horizontal map in Theorem 1.2.9 is uniquely determined by the upper one through the universal property of Poincaré-Verdier quotients. Thus, to apply Proposition 1.2.11 it only remains to note that cartesian squares with vertical Verdier projections are adjointable. This is easy to check directly and also contained in Proposition A.3.15. □

Proof of Proposition 1.2.11. On underlying ∞-categories, the restriction functor is an equivalence by Proposition A.2.14. It therefore suffices to show that the restriction map

$$\mathrm{nat}\left(\Phi, \Qoppa'^{[1]}_{\mathrm{met}}\circ F^{\mathrm{op}}\right) \longrightarrow \mathrm{nat}\left(\Qoppa, \Qoppa'\circ F^{\mathrm{op}}\right)$$

is an equivalence of spectra for every $F\colon \mathcal{D} \to \mathrm{Ar}(\mathcal{C}')$. To see this, note that adjointability naturally identifies F with the functor taking d to the arrow $Gg(d) \to Gcp(d)$, where $G\colon \mathcal{C} \to \mathcal{C}'$ is the functor induced by F on vertical fibres. Since therefore

$$\Qoppa'^{[1]}_{\mathrm{met}}F(d) \simeq \mathrm{fib}\left(\Qoppa^{[1]}(Gcp(d)) \to \Qoppa^{[1]}(Gg(d))\right),$$

the source of the map in question is equivalent to the fibre of

$$\mathrm{nat}\left(\Phi, \Qoppa'^{[1]}\circ(Gcp)^{\mathrm{op}}\right) \longrightarrow \mathrm{nat}\left(\Phi, \Qoppa'^{[1]}\circ(Gg)^{\mathrm{op}}\right).$$

Writing $cp = \mathrm{cof}(g' \Rightarrow g)$, we can use Example [I].1.1.21 to express $\Qoppa'^{[1]}Gcp(d)^{\mathrm{op}}$ as the total fibre of

$$\begin{tikzcd}
\Qoppa'^{[1]}Gg(d) \arrow[r] \arrow[d] & \Qoppa'^{[1]}Gg'(d) \arrow[d, "t"] \\
\mathrm{B}_{\Qoppa'^{[1]}}(Gg(d), Gg'(d)) \arrow[r] & \mathrm{B}_{\Qoppa'^{[1]}}(Gg'(d), Gg'(d)).
\end{tikzcd}$$

This results in a cartesian square

$$\begin{tikzcd}
\mathrm{nat}\left(\Phi, \Qoppa'^{[1]}_{\mathrm{met}}\circ F^{\mathrm{op}}\right) \arrow[r] \arrow[d] & \mathrm{nat}\left(\Phi, \Qoppa'\circ(Gg')^{\mathrm{op}}\right) \arrow[d] \\
\mathrm{nat}\left(\Phi, \mathrm{B}_{\Qoppa'^{[1]}}\circ(Gg, Gg')^{\mathrm{op}}\circ\Delta_{\mathcal{D}^{\mathrm{op}}}\right) \arrow[r] & \mathrm{nat}\left(\Phi, \mathrm{B}_{\Qoppa'^{[1]}}\circ(Gg', Gg')^{\mathrm{op}}\circ\Delta_{\mathcal{D}^{\mathrm{op}}}\right).
\end{tikzcd}$$

Now by adjunction the top right corner is equivalent to

$$\mathrm{nat}\left(\Phi\circ f^{\mathrm{op}}, \Qoppa'\circ G^{\mathrm{op}}\right) \simeq \mathrm{nat}\left(\Qoppa, \Qoppa'\circ G^{\mathrm{op}}\right)$$

and unwinding definitions shows that this identifies the top horizontal map with the restriction in question. We therefore have to show that the lower horizontal map is an equivalence. By Lemma [I].1.1.7, this map identifies with

$$\mathrm{nat}\left(\mathrm{B}_{\Phi}, \mathrm{B}_{\Qoppa'}\circ(Gg, Gg')^{\mathrm{op}}\right) \longrightarrow \mathrm{nat}\left(\mathrm{B}_{\Phi}, \mathrm{B}_{\Qoppa'}\circ(Gg', Gg')^{\mathrm{op}}\right)$$

whose fibre is $\mathrm{nat}(\mathrm{B}_{\Phi}, \mathrm{B}_{\Qoppa'}\circ(Gcp, Gg')^{\mathrm{op}})$, which we will show vanishes. Separating the variables using $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}\times\mathcal{D}^{\mathrm{op}}, \mathcal{S}p) \simeq \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}p))$ yields equivalences

$$\begin{aligned}
\mathrm{nat}\left(\mathrm{B}_{\Phi}, \mathrm{B}_{\Qoppa'}\circ(Gcp, Gg')^{\mathrm{op}}\right) &\simeq \mathrm{nat}\left(\mathrm{B}_{\Phi}\circ(\mathrm{id}, f)^{\mathrm{op}}, \mathrm{B}_{\Qoppa'}\circ(Gcp, G)^{\mathrm{op}}\right) \\
&\simeq \mathrm{nat}\left(((cp)^{\mathrm{op}}\times\mathrm{id}_{\mathcal{C}^{\mathrm{op}}})_!(\mathrm{B}_{\Phi}\circ(\mathrm{id}, f)^{\mathrm{op}}), \mathrm{B}_{\Qoppa'}\circ(G, G)^{\mathrm{op}}\right)
\end{aligned}$$

by adjunction. We now claim that already $(p^{\mathrm{op}}\times \mathrm{id}_{\mathcal{C}^{\mathrm{op}}})_!(\mathrm{B}_\Phi\circ(\mathrm{id},f)^{\mathrm{op}}\simeq 0$: The left Kan extension is obtained by pullback along the right adjoint $(q,\mathrm{id}_{\mathcal{C}})^{\mathrm{op}}$ of $(p,\mathrm{id}_{\mathcal{C}})^{\mathrm{op}}$ and we precisely assumed that $\mathrm{B}_\Phi\circ(q,f)^{\mathrm{op}}\simeq 0$. □

1.3. **Poincaré-Karoubi sequences.**

In this section, we study Poincaré-Karoubi sequences, the analogues of Poincaré-Verdier sequences in the setting of idempotent complete Poincaré ∞-categories. On the one hand, these are important in their own right when considering the hermitian analogue of non-connective K-theory (we will do this in Paper [IV]), on the other, it is often easier to establish Poincaré-Verdier sequences in a two-step process: First one constructs a Poincaré-Karoubi sequence using the Thomason-Neeman localisation theorem A.3.11, or in modern guise, the equivalence between small stable ∞-categories, and compactly generated stable ∞-categories, and then in a second step isolates subcategories forming Poincaré-Verdier sequences; see Proposition 1.4.5 for an example.

We will, in fact, see that every Poincaré-Verdier sequence is a Poincaré-Karoubi sequence (Proposition 1.3.8), and establish a simple criterion for a Poincaré-Karoubi sequence to be a Poincaré-Verdier sequence (Corollary 1.3.10).

Let us establish some terminology: We denote by $\mathcal{C}^\natural$ the idempotent completion of a small ∞-category $\mathcal{C}$ and refer the reader to [Lur09a, §5.1.4] for its construction. The ∞-category $\mathcal{C}^\natural$ is stable if $\mathcal{C}$ is, and the natural functor $i\colon \mathcal{C}\to\mathcal{C}^\natural$ is fully faithful, exact and has dense essential image, where a full subcategory $\mathcal{D}\subseteq\mathcal{C}$ is called *dense* if every object of $\mathcal{C}$ is a retract of one in $\mathcal{D}$. Recall also that we call a functor a *Karoubi equivalence* if it is fully faithful with dense essential image, in other words if it induces an equivalence on the idempotent completions (cf. Definition A.3.1).

1.3.1. **Remark.** We avoid the common term Morita equivalence for what we call a Karoubi equivalence, since it conflicts with the notion of Morita equivalence of (discrete) rings: The very fact that invariants such as K-, L- and Grothendieck-Witt spectra of a ring are defined via its (derived) module categories makes them invariant under Morita equivalences in the latter sense, whereas invariance under Karoubi equivalences is an additional feature, that for example separates connective and non-connective K-theory.

1.3.2. **Definition.** A Poincaré ∞-category is *idempotent complete* if its underlying stable ∞-category is. We denote by $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\subseteq\mathrm{Cat}^{\mathrm{p}}_\infty$ the full subcategory spanned by the idempotent complete Poincaré ∞-categories. A Poincaré functor $(f,\eta)\colon(\mathcal{C},\Omega)\to(\mathcal{D},\Phi)$ is a *Karoubi equivalence* if f is a Karoubi equivalence and $\eta\colon \Omega\to f^*\Phi$ is an equivalence.

1.3.3. **Proposition.** *Let $(\mathcal{C},\Omega)$ be a Poincaré ∞-category and $i\colon\mathcal{C}\to\mathcal{C}^\natural$ its idempotent completion. Then the left Kan extension $i_!\Omega\colon(\mathcal{C}^\natural)^{\mathrm{op}}\to\mathcal{S}p$ is a Poincaré structure on $\mathcal{C}^\natural$ and the canonical hermitian functor $(\mathcal{C},\Omega)\to(\mathcal{C}^\natural,i_!\Omega)$ is Poincaré and a Karoubi equivalence.*

Moreover, for any idempotent-complete Poincaré ∞-category $(\mathcal{D},\Phi)$, the pullback functor

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}^\natural,i_!\Omega),(\mathcal{D},\Phi))\to\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))$$

is an equivalence of Poincaré ∞-categories. In particular, the inclusion of $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\subseteq\mathrm{Cat}^{\mathrm{p}}_\infty$ of idempotent-complete Poincaré ∞-categories has a left adjoint sending $(\mathcal{C},\Omega)$ to $(\mathcal{C}^\natural,i_!\Omega)$.

We often write $(\mathcal{C},\Omega)^\natural$ for this left adjoint.

Proof. By Lemma [I].1.4.1 and Proposition [I].1.4.3, the functor $i_!\Omega$ is quadratic with bilinear part $(i\times i)_!\mathrm{B}_\Omega$. To see that this is perfect, note first that it restricts back to B_Ω since i is fully faithful. Now, the idempotent completion of the equivalence $\mathrm{D}_\Omega\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$ is another equivalence $\mathrm{D}\colon(\mathcal{C}^\natural)^{\mathrm{op}}\simeq(\mathcal{C}^{\mathrm{op}})^\natural\to\mathcal{C}^\natural$, and by the previous observation, the functors

$$\mathrm{Hom}_{\mathcal{C}^\natural}(-,\mathrm{D}-)\quad\text{and}\quad\mathrm{B}_{i_!\Omega}$$

agree on $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}}$, and therefore on all of $(\mathcal{C}^\natural)^{\mathrm{op}}\times(\mathcal{C}^\natural)^{\mathrm{op}}$ by [Lur09a, Proposition 5.1.4.9]. This shows that both $i_!\Omega$ and i are Poincaré.

Finally, let us fix $(\mathcal{D},\Phi)$ an idempotent-complete Poincaré ∞-category and consider the Poincaré functor

$$i^*\colon\ \mathrm{Fun}^{\mathrm{ex}}\left((\mathcal{C}^\natural,i_!\Omega),(\mathcal{D},\Phi)\right)\to\mathrm{Fun}^{\mathrm{ex}}\left((\mathcal{C},\Omega),(\mathcal{D},\Phi)\right).$$

By another application of [Lur09a, Proposition 5.1.4.9], this is an equivalence of the underlying stable ∞-categories, so it suffices to show that it induces also an equivalence on the corresponding quadratic functors.

But for an exact functor $f : \mathcal{C}^\natural \to \mathcal{D}$, this map is precisely the canonical equivalence $\mathrm{nat}(i_!\Qoppa, f^*\Phi) \to \mathrm{nat}(\Qoppa, i^*f^*\Phi)$. □

1.3.4. **Remark.** The adjunction $i_! : \mathrm{Fun}^{\mathrm{q}}(\mathcal{C}) \underset{\perp}{\rightleftarrows} \mathrm{Fun}^{\mathrm{q}}(\mathcal{C}^\natural) : i^*$ between hermitian structures on $\mathcal{C}$ and hermitian structures on $\mathcal{C}^\natural$ is an equivalence, since $i_!$ is fully faithful and i^* is conservative by the density of i. By Proposition 1.3.3, this equivalence restricts to an equivalence between Poincaré structures on $\mathcal{C}$ and Poincaré structures on $\mathcal{C}^\natural$ whose associated duality preserves $\mathcal{C}$.

1.3.5. **Proposition.** *The localisation of $\mathrm{Cat}^{\mathrm{p}}_\infty$ at the Karoubi equivalences admits both a left and a right adjoint, the right adjoint is given by $(\mathcal{C}, \Qoppa) \mapsto (\mathcal{C}, \Qoppa)^\natural$, and the left adjoint by $(\mathcal{C}, \Qoppa) \mapsto (\mathcal{C}^{\min}, j^*\Qoppa)$, where $\mathcal{C}^{\min}$ is the full subcategory of $\mathcal{C}$ spanned the objects $X \in \mathcal{C}$ with $0 = [X] \in \mathrm{K}_0(\mathcal{C})$, and j is its inclusion into $\mathcal{C}$.*

In particular, the idempotent completion functor $(-)^\natural : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ preserves both limits and colimits.

The analogous statement for the underlying stable ∞-categories is Proposition A.3.3.

Proof. We first note that $\mathcal{C}^{\min} \subseteq \mathcal{C}$ is closed under the duality of $\mathcal{C}$, since the duality acts by a group homomorphism on K_0, and so $(\mathcal{C}^{\min}, j^*\Qoppa)$ is Poincaré by Observation [I].1.2.19.

Now, according to Lemma A.2.1, we have to verify that $\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{C}^{\min}, j^*\Qoppa), -)$ and $\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{p}}_\infty}(-, (\mathcal{C}^\natural, i_!\Qoppa))$ invert Karoubi equivalences of Poincaré ∞-categories. Both of these follow from their non-Poincaré counterparts established in Proposition A.3.3 by considering the induced maps on the cartesian squares

$$\begin{tikzcd}
\mathrm{Nat}(\Phi, F^*\Psi) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{D}, \Phi), (\mathcal{E}, \Psi)) \arrow[d] \\
\Delta^0 \arrow[r, "F"] & \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{ex}}_\infty}(\mathcal{D}, \mathcal{E}).
\end{tikzcd}$$

For either $(\mathcal{D}, \Phi) = (\mathcal{C}^{\min}, j^*\Qoppa)$ or $(\mathcal{E}, \Psi) = (\mathcal{C}^\natural, i_!\Qoppa)$ a Karoubi equivalence in the other variable induces an equivalence by Lemma A.2.1 and Proposition A.3.3, and the induced map in the top left corner is an equivalence by [Lur09a, Proposition 5.1.4.9], since $\mathcal{S}p$ is idempotent complete.

The final clause follows since the adjoints are both automatically fully faithful by yet another application of Lemma A.2.1. □

Recall that a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ of exact functors with vanishing composite is a *Karoubi sequence* (Definition A.3.5) if the sequence

$$\mathcal{C}^\natural \to \mathcal{D}^\natural \to \mathcal{E}^\natural$$

is both a fibre and a cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$. In this case, we refer to f as a *Karoubi inclusion* and to p as a *Karoubi projection*. In the same spirit, we put:

1.3.6. **Definition.** A sequence

$$(\mathcal{C}, \Qoppa) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\theta)} (\mathcal{E}, \Psi)$$

of Poincaré functors with vanishing composite is a *Poincaré-Karoubi sequence* if

$$(\mathcal{C}, \Qoppa)^\natural \xrightarrow{(f,\eta)^\natural} (\mathcal{D}, \Phi)^\natural \xrightarrow{(p,\vartheta)^\natural} (\mathcal{E}, \Psi)^\natural$$

is both a fibre sequence and a cofibre sequence in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$. We then call (f, η) a *Poincaré-Karoubi inclusion* and (p, ϑ) a *Poincaré-Karoubi projection*.

We warn the reader that, contrary to the situation for (Poincaré-)Verdier sequences, a (Poincaré-)Karoubi sequence is determined by its inclusion or its projection only up to idempotent completion of the third term.

We record a few simple consequences of the definition.

1.3.7. **Observation.** *Since the forgetful and hyperbolic functors commute with idempotent completion by inspection, the sequence of stable ∞-categories underlying a Poincaré-Karoubi sequence is a Karoubi sequence and the hyperbolisation of a Karoubi sequence is a Poincaré-Karoubi sequence.*

By Proposition A.3.7, any Verdier sequence is a Karoubi sequence. Analogously:

1.3.8. **Proposition.** *Every Poincaré-Verdier sequence is a Poincaré-Karoubi sequence.*

Proof. A bifibre sequence in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ remains so in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ after idempotent completion by Proposition 1.3.5. ☐

1.3.9. **Proposition.** *Let*

$$(\mathcal{C}, Ϙ)\xrightarrow{(f,\eta)}(\mathcal{D},\Phi)\xrightarrow{(p,\vartheta)}(\mathcal{E},\Psi)$$

be a sequence of Poincaré functors with vanishing composite. Then:

- *i) Its idempotent completion is a fibre sequence in* $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ *if and only if the idempotent completion of its underlying sequence is a fibre sequence in* $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ *and* η *induces an equivalence* $Ϙ \Rightarrow f^*\Phi$*.*
- *ii) Its idempotent completion is a cofibre sequence in* $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ *if and only if the idempotent completion of its underlying sequence is a cofibre sequence in* $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ *and* ϑ *exhibits* Ψ *as the left Kan extension of* Φ *along* p*.*
- *iii) It is a Poincaré-Karoubi sequence if and only if its underlying sequence is a Karoubi sequence and both* η *induces an equivalence* $Ϙ \Rightarrow f^*\Phi$ *and* ϑ *exhibits* Ψ *as the left Kan extension of* Φ *along* p*.*

Proof. By Proposition 1.3.5, fibres in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ are computed in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ while cofibres are computed as idempotent completions of cofibres in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. Thus, i) and ii) follow from Proposition 1.1.4 i) and ii), respectively, using the equivalence between quadratic functors on $\mathcal{C}$ and on $\mathcal{C}^{\natural}$ explained in Remark 1.3.4 (as well as the ones for $\mathcal{D}$ and $\mathcal{E}$). Part iii) is i) and ii) put together. ☐

In particular, comparing Proposition 1.3.9 with Proposition 1.1.4 and investing Corollary A.1.10 for a concrete description, we obtain:

1.3.10. **Corollary.** *A Poincaré-Karoubi sequence is a Poincaré-Verdier sequence if and only if its underlying (Karoubi) sequence is a Verdier sequence, i.e. concretely, the image of the inclusion is closed under retracts, and the projection is essentially surjective.*

Combining Proposition 1.3.9 with the concrete characterisation of Karoubi sequences given in Proposition A.3.7, we also have:

1.3.11. **Corollary.** *A sequence of Poincaré functors*

$$(\mathcal{C}, Ϙ)\xrightarrow{(f,\eta)}(\mathcal{D},\Phi)\xrightarrow{(p,\vartheta)}(\mathcal{E},\Psi)$$

with vanishing composite is a Poincaré-Karoubi sequence if and only if both

- *i)* f *is fully faithful and the induced map* $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ *is fully faithful with dense essential image and*
- *ii) the map* $\eta\colon Ϙ \Rightarrow f^*\Phi$ *is an equivalence, as is the induced map* $\vartheta\colon p_!\Phi \Rightarrow \Psi$*.*

Similarly, using Corollary A.3.8, we obtain:

1.3.12. **Corollary.**

- *i) A Poincaré functor* $(f,\eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{D},\Phi)$ *is a Poincaré-Karoubi inclusion if and only if* f *is fully faithful and the map* $\eta\colon Ϙ \Rightarrow f^*\Phi$ *is an equivalence.*
- *ii) A Poincaré functor* $(p,\vartheta)\colon (\mathcal{D},\Phi) \to (\mathcal{E},\Psi)$ *is a Poincaré-Karoubi projection if and only if* p *has dense essential image, the induced functor* $\mathcal{D} \to p(\mathcal{D})$ *is a localisation and* $\vartheta\colon \Phi \Rightarrow p^*\Psi$ *exhibits* Ψ *as the left Kan extension of* Φ *along* p*.*

Since Poincaré structures uniquely extend to idempotent completions, we also find the following as a direct consequence of A.3.10 and 1.1.9:

1.3.13. **Corollary.** *Poincaré-Karoubi projections are closed under pullback.*

Finally, let us establish an analogue of the classification of Verdier and Karoubi projections from Proposition A.3.14, compare also [Nik20] for a slightly different treatment.

To this end, we recall from Proposition A.3.14, for a stable ∞-category $\mathcal{C}$, the Verdier sequence

$$\mathcal{C}^{\natural} \to \mathrm{Latt}(\mathcal{C}) \to \mathrm{Tate}(\mathcal{C}),$$

which is universal in the sense that any Verdier sequence $\mathcal{C}^\natural \to \mathcal{D} \to \mathcal{E}$ pulls back from it through an essentially unique exact functor $\mathcal{E} \to \mathrm{Tate}(\mathcal{C})$. We will use the characterisation

$$\mathrm{Latt}(\mathcal{C}) = \mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \mathrm{Pro}(\mathcal{C}), \hom_{\mathrm{Pro\,Ind}\,\mathcal{C}})$$

from Example A.3.17, where Pair denotes the ∞-category of pairings from §[I].7.1, and consequently that $\mathrm{Tate}(\mathcal{C})$ is the Verdier quotient $\mathrm{Latt}(\mathcal{C})/\mathcal{C}$; see the Appendix for a more explicit description of the ∞-categories $\mathrm{Latt}(\mathcal{C})$ and $\mathrm{Tate}(\mathcal{C})$.

To equip $\mathrm{Latt}(\mathcal{C})$ with a hermitian structure, we observe that for a hermitian category $(\mathcal{C}, \Qoppa)$, the inductive completion $\mathrm{Ind}(\mathcal{C})$ canonically inherits a hermitian structure

$$\tilde{\Qoppa}\colon\ \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{Pro}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{Pro}(\Qoppa)} \mathrm{Pro}(\mathcal{S}p) \xrightarrow{\lim} \mathcal{S}p.$$

Note, however, that $(\mathrm{Ind}(\mathcal{C}), \tilde{\Qoppa})$ is usually not Poincaré even if $(\mathcal{C}, \Qoppa)$ is: Instead, the duality functor $\mathrm{D}_\Qoppa\colon\ \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ extends to an equivalence $\mathrm{Pro}(\mathrm{D}_\Qoppa)\colon\ \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{Pro}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Pro}(\mathcal{C})$; it is of course also not a small ∞-category, but never mind this (the reader averse to this may choose a sufficiently large cut-off cardinal for the inductive and projective completions).

Now recall from §[I].7.3 that the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ admits a right adjoint $\mathrm{Pair}\colon\ \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$, such that the underlying ∞-category of $\mathrm{Pair}(\mathcal{C}, \Qoppa)$ is $\mathrm{Pair}(\mathcal{C}, \mathcal{C}^{\mathrm{op}}, \mathrm{B}_\Qoppa)$. We then put:

1.3.14. **Definition.** For a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, we define

$$\mathrm{Latt}(\mathcal{C}, \Qoppa) = \mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \tilde{\Qoppa}).$$

To see that the underlying ∞-category is indeed $\mathrm{Latt}(\mathcal{C})$, simply note that

$$\mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})^{\mathrm{op}}, \mathrm{B}_{\tilde{\Qoppa}}) \simeq \mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \mathrm{Pro}(\mathcal{C}), \hom_{\mathrm{Pro\,Ind}(\mathcal{C})})$$

via $\mathrm{Pro}(\mathrm{D}_\Qoppa)$, since we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \arrow[r, "\mathrm{B}_{\tilde{\Qoppa}}"] \arrow[d, "\mathrm{id}\times\mathrm{Pro}(\mathrm{D}_\Qoppa)"'] & \mathcal{S}p \\
\mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Pro}(\mathcal{C}) \arrow[ru, "\hom_{\mathrm{Pro\,Ind}(\mathcal{C})}"'] &
\end{tikzcd}$$

because both functors are compatible in both variables with filtered limits, and on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$ the claim holds by definition of $\mathrm{D}_\Qoppa$.

We next set out to refine the ∞-category $\mathrm{Tate}(\mathcal{C})$ to a Poincaré ∞-category. This requires a few preliminary observations: Generally, if $(f, \eta)\colon\ (\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$ is a hermitian embedding in the sense that f is fully faithful and $\eta\colon\ \Qoppa \to f^*\Qoppa'$ is an equivalence, and the domain is Poincaré, then the Poincaré functor $(\mathcal{C}, \Qoppa) \to \mathrm{Pair}(\mathcal{C}', \Qoppa')$ induced by (f, η) is also a hermitian embedding because it may be written as a composite

$$(\mathcal{C}, \Qoppa) \to \mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}) \simeq \mathrm{Pair}(\mathcal{C}, \Qoppa) \to \mathrm{Pair}(\mathcal{C}', \Qoppa')$$

of hermitian embeddings.

Thus, the hermitian embedding $(\mathcal{C}, \Qoppa)^\natural \to (\mathrm{Ind}(\mathcal{C}), \tilde{\Qoppa})$ extends to a Poincaré-Verdier inclusion $i\colon\ (\mathcal{C}, \Qoppa)^\natural \to \mathrm{Latt}(\mathcal{C}, \Qoppa)$.

1.3.15. **Definition.** For a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, we put

$$\mathrm{Tate}(\mathcal{C}, \Qoppa) = \mathrm{Latt}(\mathcal{C}, \Qoppa^{[-1]})/(\mathcal{C}, \Qoppa^{[-1]}).$$

In total, we thus obtain a Poincaré-Verdier sequence

$$(\mathcal{C}, \Qoppa)^\natural \xrightarrow{i} \mathrm{Latt}(\mathcal{C}, \Qoppa) \xrightarrow{q} \mathrm{Tate}(\mathcal{C}, \Qoppa^{[1]}),$$

and by inspection, the sequence of underlying stable ∞-categories agrees with the universal Verdier sequence with fibre $\mathcal{C}^\natural$ from A.3.14.

We now argue that the Poincaré-Verdier sequence just constructed is the universal Poincaré-Verdier sequence with fibre $(\mathcal{C}, \Qoppa)^\natural$.

1.3.16. **Construction.** Given a Poincaré-Verdier sequence

$$(\mathcal{C}, Ϙ) \xrightarrow{f} (\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi),$$

the functor $\mathrm{Ind}(f)$ admits a right adjoint (compare 1.1.8) whose restriction to $\mathcal{D}$ we call the Ind-adjoint $R\colon \mathcal{D} \to \mathrm{Ind}(\mathcal{C})$ of f. It canonically promotes to a hermitian functor $(\mathcal{D}, \Phi) \to (\mathrm{Ind}(\mathcal{C}), \tilde{Ϙ})$: This is given as the composite of the hermitian inclusion $(\mathcal{D}, \Phi) \to (\mathrm{Ind}(\mathcal{D}), \tilde{\Phi})$ and the canonical hermitian functor $(\mathrm{Ind}(\mathcal{D}), \tilde{\Phi}) \to (\mathrm{Ind}(\mathcal{C}), (g')_! \tilde{\Phi})$ where g' is the right adjoint of $\mathrm{Ind}(f)$, and the hermitian structure on the target may be computed by pulling back along $\mathrm{Ind}(f)$ (compare again 1.1.8).

By adjunction, this hermitian functor canonically extends to a Poincaré functor

$$\gamma\colon (\mathcal{D}, \Phi) \to \mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \tilde{Ϙ}) = \mathrm{Latt}(\mathcal{C}, Ϙ)$$

which we call the *classifying map* the Poincaré-Verdier sequence. By construction, the restriction of γ along f agrees with the restriction of i to $\mathcal{C} \subset \mathcal{C}^\natural$ so we obtain a transformation of Poincaré-Verdier sequences

$$\begin{tikzcd}
(\mathcal{C}, Ϙ) \arrow[r] \arrow[d, "f"] & (\mathcal{C}, Ϙ)^\natural \arrow[d, "i"] \\
(\mathcal{D}, \Phi) \arrow[r, "\gamma"] \arrow[d, "p"] & \mathrm{Latt}(\mathcal{C}, Ϙ) \arrow[d, "q"] \\
(\mathcal{E}, \Psi) \arrow[r, "\bar{\gamma}"] & \mathrm{Tate}(\mathcal{C}, Ϙ^{[1]}),
\end{tikzcd}$$

1.3.17. **Theorem.** *For any Poincaré-Verdier sequence*

$$(\mathcal{C}, Ϙ) \xrightarrow{f} (\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi),$$

with $\mathcal{C}$ idempotent complete, the lower square of the transformation of Poincaré-Verdier sequences just constructed is cartesian.

We also record the uniqueness of the classifying map. To state the result, we extend the notion of adjointability to non-split Verdier projections by requiring the diagrams

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r, "i"] \arrow[d, "p"] & \mathrm{Ind}(\mathcal{C}') \arrow[d, "p'"] \\
\mathrm{Ind}(\mathcal{D}) \arrow[r, "j"] & \mathrm{Ind}(\mathcal{D}')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Pro}(\mathcal{C}) \arrow[r, "i"] \arrow[d, "p"] & \mathrm{Pro}(\mathcal{C}') \arrow[d, "p'"] \\
\mathrm{Pro}(\mathcal{D}) \arrow[r, "j"] & \mathrm{Pro}(\mathcal{D}')
\end{tikzcd}$$

to be left and right adjointable, respectively. We shall refer to such squares as Pro/Ind-adjointable.

1.3.18. **Corollary.** *For every Poincaré-Verdier sequence $(\mathcal{C}, Ϙ) \xrightarrow{f} (\mathcal{D}, \Phi) \xrightarrow{p} (\mathcal{E}, \Psi)$ and every idempotent complete Poincaré ∞-category $(\mathcal{C}', Ϙ')$, the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}, \Phi), \mathrm{Latt}(\mathcal{C}', Ϙ'))$ spanned by the functors φ that give rise to* Pro/Ind-*adjointable squares*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\varphi"] \arrow[d, "p"] & \mathrm{Latt}(\mathcal{C}') \arrow[d, "q"] \\
\mathcal{E} \arrow[r, "\overline{\varphi}"] & \mathrm{Tate}(\mathcal{C}')
\end{tikzcd}$$

is equivalent to $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{C}', Ϙ'))$ *via restriction to horizontal fibres.*

In particular, the classifying morphism in Theorem 1.3.17 is determined up to contractible choice by yielding a cartesian square and inducing the identity on vertical fibres.

Proof of Theorem 1.3.17. By inspection, on underlying ∞-categories this transformation of Verdier sequences agrees with the one constructed in A.3.14, and therefore is a pullback. For the Poincaré structures, we then need to show that the square of functors $\mathcal{D}^{\mathrm{op}} \to \mathcal{S}p$,

$$\begin{tikzcd}
\Phi \arrow[r] \arrow[d] & \gamma^*\Qoppa_{\mathrm{Latt}(\mathcal{C},\Qoppa)} \arrow[d] \\
p^*\Psi \arrow[r] & \gamma^*\pi^*\Qoppa^{[1]}_{\mathrm{Tate}(\mathcal{C},\Qoppa)}
\end{tikzcd}$$

is cartesian.

After restricting along i, the diagram becomes cartesian, for the upper horizontal map becomes an equivalence and the lower terms vanish. We next show that the square under consideration also becomes cartesian after applying $p_!$. The left map is adjoint to the equivalence $p_!\Phi \to \Psi$. Here, p^* is fully faithful because p is a localisation; it follows that the counit transformation $p_!p^* \to \mathrm{id}$ is an equivalence and from the triangle identity that $p_!$ of the left map is an equivalence.

For the right vertical map, we claim existence of a natural equivalence $p_!\gamma^* \simeq \bar{\gamma}^*\pi_!$: This follows from the explicit description of left Kan extensions through pull-back along Ind-adjoints from 1.1.8 and the fact the lower square in the transformation of Verdier sequences is Ind-adjointable by A.3.15. Using this equivalence, $p_!$ of the right vertical map identifies with $\bar{\gamma}^*$ of the canonical map

$$\pi_!\Qoppa_{\mathrm{Latt}(\mathcal{C},\Qoppa)} \to \pi_!\pi_*\Qoppa^{[1]}_{\mathrm{Tate}(\mathcal{C},\Qoppa)}$$

and therefore is an equivalence by the same argument as for the left vertical map.

Now, generally a quadratic functor Φ on $\mathcal{D}$ vanishes if (i) $f^*\Phi \simeq 0$, (ii) $p_!\Phi \simeq 0$, and (iii) $(f \times \mathrm{id}_{\mathcal{D}})^*\mathrm{B}_\Phi \simeq 0$: The last condition guarantees that the bilinear part descends to $\mathcal{E} \times \mathcal{E}$ where it vanishes by (ii) (using Proposition [I].1.4.3); so the functor is linear and also descends to $\mathcal{E}$ by (i) and so vanishes by (ii) again.

Applying this to the total fibre of the above square, we need to show that the diagram of bilinear parts is cartesian on objects of the form $(f(c), d)$. In this situation, the lower terms vanish and the upper map evaluates to

$$\mathrm{Hom}_{\mathcal{D}}(f(c), \mathrm{D}_\Phi(d)) \to \mathrm{Hom}_{\mathrm{Latt}(\mathcal{C})}(i(c), \mathrm{D}_{\Qoppa_{\mathrm{Latt}}}(\gamma(d)))$$

which, using the Ind-adjoint R of f and Ind-adjointability, is equivalent to the identity map on $\mathrm{Hom}_{\mathcal{C}}(c, Rd)$. (Here we use Ind-adjointability of the upper square in the transformation, which follows from Ind-adjointability of the lower square by the explicit description of the adjoint transformations from Lemma A.2.5 and the fact that Ind preserves Verdier sequences, see Theorem A.3.11.) ☐

Proof of Corollary 1.3.18. To obtain an inverse, we consider the hermitian embedding

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Qoppa),(\mathcal{C}',\Qoppa')) \to \mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathrm{Ind}(\mathcal{C}'),\widetilde{\Qoppa'})), \quad F \mapsto \mathrm{Ind}(F)\circ R,$$

where R is the Ind-adjoint of f. By adjunction we obtain a Poincaré functor

$$h\colon\ \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Qoppa),(\mathcal{D},\Phi)) \to \mathrm{Pair}(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathrm{Ind}(\mathcal{C}'),\widetilde{\Qoppa'}))) \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Qoppa),\mathrm{Latt}(\mathcal{C}',\Qoppa'))$$

which is again a hermitian embedding.

Explicitly, on underlying ∞-categories h sends $F\colon \mathcal{C} \to \mathcal{D}$ to the functor sending c to $(RF(c) \to LF(c)) \in \mathrm{Latt}(\mathcal{D})$, where R and L are the Ind- and Pro-adjoints of f. This is the same formula as the inverse functor from Proposition A.3.15, so by the latter result, h is an equivalence onto the full subcategory of Pro/Ind-adjointable functors. ☐

To conclude this discussion, let us describe the Poincaré structures on $\mathrm{Latt}(\mathcal{C})$ and $\mathrm{Tate}(\mathcal{C})$ in terms of their original definitions.

1.3.19. **Proposition.** *Under* $\mathrm{Tate}(\mathcal{C}) \subseteq \mathrm{Pro}\,\mathrm{Ind}(\mathcal{C})$*, the Poincaré structure* $\mathrm{Tate}(\Qoppa)$ *on* $\mathrm{Tate}(\mathcal{C})$ *is restricted from the hermitian structure*

$$\hat{\Qoppa}\colon\ (\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}))^{\mathrm{op}} = \mathrm{Ind}\,\mathrm{Pro}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{Ind}\,\mathrm{Pro}(\Qoppa)} \mathrm{Ind}\,\mathrm{Pro}(\mathrm{Sp}) \xrightarrow{\mathrm{colim}} \mathrm{Pro}(\mathrm{Sp}) \xrightarrow{\lim} \mathrm{Sp}$$

on $\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C})$*. Furthermore,* $\mathrm{Latt}(\Qoppa)$ *is given by the composite*

$$\mathrm{Latt}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Ar}(\mathrm{Tate}(\mathcal{C}))^{\mathrm{op}} \xrightarrow{\mathrm{Tate}(\Qoppa)_{\mathrm{ar}}} \mathrm{Sp}.$$

For the arrow construction on Poincaré ∞-categories, we refer the reader to Section [I].2.4. The equivalence between the arrow and metabolic construction in [I].2.4.5 in particular gives us the lower half of the diagram

$$\begin{tikzcd}
(\mathcal{C},\Qoppa)^{\natural} \arrow[r] \arrow[d] & \mathrm{Met}(\mathcal{C},\Qoppa^{[1]})^{\natural} \arrow[r] \arrow[d] & (\mathcal{C},\Qoppa^{[1]})^{\natural} \arrow[d] \\
(\mathcal{C},\Qoppa)^{\natural} \arrow[r] \arrow[d] & \mathrm{Latt}(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & \mathrm{Tate}(\mathcal{C},\Qoppa^{[1]}) \arrow[d] \\
\mathrm{Tate}(\mathcal{C},\Qoppa) \arrow[r] & \mathrm{Met}(\mathrm{Tate}(\mathcal{C},\Qoppa)^{[1]}) \arrow[r] & \mathrm{Tate}(\mathcal{C},\Qoppa^{[1]})
\end{tikzcd}$$

whereas the upper half comes from the inclusion $\mathcal{C}\subseteq\mathrm{Ind}(\mathcal{C})$ under the identification $\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})=\mathrm{Pair}(\mathcal{C},\Qoppa)$ for $(\mathcal{C},\Qoppa)$ Poincaré from [I].7.3.3. In this diagram, all horizontal sequences are Poincaré-Verdier, and all vertical maps are Poincaré-Verdier inclusions (and two are the obvious identities).

Proof. We recall that the embedding $\mathrm{Latt}(\mathcal{C})=\mathrm{Pair}(\mathrm{Ind}(\mathcal{C}))\to\mathrm{Ar}(\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}))$ is given by

$$(i,j,b)\mapsto(b_\sharp : i\to\mathrm{Pro}(\mathrm{D}_\Qoppa)(j))$$

where $b_\sharp$ corresponds to b under the equivalence

$$\mathrm{B}_{\hat{\Qoppa}}(i,j)\simeq\hom_{\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C})}(i,\mathrm{Pro}(\mathrm{D}_\Qoppa)(j))$$

from above. The construction of a hermitian structure on the arrow category $\mathrm{Ar}(\mathcal{C})$ from section [I].2.4 clearly makes sense also for hermitian categories $(\mathcal{C},\Qoppa)$, rather than just Poincaré categories. Thus we obtain a hermitian structure $\hat{\Qoppa}_{\mathrm{ar}}$ on $\mathrm{Ar}(\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}))$, and its restriction under the above embedding is given by the pullback of

$$\mathrm{B}_{\hat{\Qoppa}}(i,\mathrm{Pro}(\mathrm{D}_\Qoppa)(j))\xrightarrow{(b_\sharp)^*}\mathrm{B}_{\hat{\Qoppa}}(i,i)\leftarrow\hat{\Qoppa}(i).$$

For the inductive systems i and j, we compute

$$\mathrm{B}_{\hat{\Qoppa}}(i,\mathrm{Pro}(\mathrm{D}_\Qoppa)(j))\simeq\lim_c\operatorname*{colim}_d\mathrm{B}_{\mathcal{C}}(i(c),\mathrm{D}_\Qoppa j(d))\simeq\lim_c\operatorname*{colim}_d\hom_{\mathcal{C}}(i(c),j(d))\simeq\hom_{\mathrm{Ind}(\mathcal{C})}(i,j).$$

Thus, the restricted hermitian structure agrees with the limit of

$$\hom_{\mathrm{Ind}(\mathcal{C})}(i,j)\overset{\mathrm{B}}{\longrightarrow}_{\tilde{\Qoppa}}(i,i)\leftarrow\tilde{\Qoppa}(i)$$

where the left map sends f to f^*b. But this is the definition of the hermitian structure on pairing categories.

This identifies the hermitian structure restricted from $\hat{\Qoppa}$ with the hermitian structure on $\mathrm{Latt}(\mathcal{C})$. Consider now the commutative diagram in $\mathrm{Cat}^{\mathrm{h}}_\infty$:

$$\begin{tikzcd}
(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & (\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}),\hat{\Qoppa}) \arrow[d] \\
\mathrm{Latt}(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & \mathrm{Ar}(\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}),\hat{\Qoppa}) \arrow[d] \\
\mathrm{Tate}(\mathcal{C},\Qoppa)^{[1]} \arrow[r] & (\mathrm{Pro}\,\mathrm{Ind}(\mathcal{C}),\hat{\Qoppa}^{[1]}).
\end{tikzcd}$$

The left sequence is the cofibre sequence defining the lower term, and the right one is the metabolic sequence, after applying the equivalence $\mathrm{Ar}(\mathcal{C},\Qoppa)\simeq\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})$ from Lemma [I].2.4.5 which still exists in the hermitian (non-Poincaré) setting. The middle horizontal embedding is the hermitian embedding explained above, which induces the other two horizontal hermitian functors. We will show that the lower one is also a hermitian embedding, finishing the proof.

As above, we may compute the Kan extension on the left as pullback along the Pro-adjoint. It thus suffices to prove that the lower square is Pro-adjointable, which is equivalent to Pro-adjointability of the upper square. The Pro-adjoint of the upper left map may be computed from the Ind-adjoint by conjugating with the dualities and is given by $(i,j,b)\mapsto\mathrm{Pro}(\mathrm{D}_\Qoppa)(j)$, while the left adjoint of the upper right map is given by $(c\to d)\mapsto d$. □

1.4. **Examples of Poincaré-Verdier sequences.**

In this section, we consider various examples of interest of Poincaré-Verdier and Poincaré-Karoubi sequences. As one of the most important examples, we already gave the metabolic fibre sequence in Example 1.2.5, which is at the core of our deduction of the main results from the additivity theorem in the next section. We repeat the statement for the reader's convenience: Given a Poincaré ∞-category, the metabolic fibre sequence

$$(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, \Qoppa) \xrightarrow{\mathrm{met}} (\mathcal{C}, \Qoppa)$$

is a split Poincaré-Verdier sequence; its left hand Poincaré functor is given by sending x to $x \to 0$, together with the identification $\Omega\Qoppa(X) \simeq \mathrm{fib}(\Qoppa(0) \to \Qoppa(x))$.

Next, we give a simple recognition criterion for Poincaré-Verdier sequences involving hyperbolic Poincaré ∞-categories. Recall from Corollary [I].7.2.21 and Remark [I].7.2.22 that Hyp is both left and right adjoint to the underlying ∞-category functor $U : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$.

1.4.1. **Lemma.** *Let $g : \mathcal{C} \to \mathcal{D}$ be an exact functor. Then, for a Poincaré structure $\Qoppa$ on $\mathcal{C}$, the functor*

$$g^{\mathrm{hyp}} : (\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Hyp}(\mathcal{D})$$

obtained by right adjointness of Hyp *is a split Poincaré-Verdier projection if and only if g is a split Verdier projection and the restrictions of $\Qoppa$ to both the essential images of l^{op} and $\mathrm{D}^{\mathrm{op}}_{\Qoppa} \circ r$ vanish, where l and r denote the adjoints of g.*

Similarly, for a Poincaré structure Φ on $\mathcal{D}$, the functor

$$g_{\mathrm{hyp}} : \mathrm{Hyp}(\mathcal{C}) \longrightarrow (\mathcal{D}, \Phi)$$

obtained by left adjointness of Hyp *is a split Poincaré-Verdier inclusion if and only if g is a split Verdier inclusion and the restrictions of $\Qoppa$ to both the essential images of g^{op} and $\mathrm{D}^{\mathrm{op}}_{\Phi} \circ g$ vanish.*

Proof. Let us prove the first statement, the second is entirely analogous. It is easy to check that the functor g^{hyp}, which is given by

$$(g, g^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\Qoppa}) : \mathcal{C} \longrightarrow \mathcal{D} \oplus \mathcal{D}^{\mathrm{op}}$$

admits both adjoints if and only if g does; in this case the left adjoint l' to g_{hyp} is given by $(d, d') \mapsto ld \oplus \mathrm{D}_{\Qoppa}(rd')$, and the right adjoint by switching the roles of l and r.

Similarly, one checks that the unit of the adjunction $l' \vdash g^{\mathrm{hyp}}$ is given by

$$(d, d') \xrightarrow{((u,0),(0,c))} (gld \oplus g\mathrm{D}_{\Qoppa}(rd'), g\mathrm{D}_{\Qoppa}(lx) \oplus grd'),$$

where u is the unit of the adjunction $l \vdash g$ and c the counit of $g \vdash r$. If this unit is an equivalence then so are u and c making g into a split Verdier projection by Corollary A.2.6. Conversely, if g is a Verdier projection both u and c are equivalences and it remains to check that g vanishes on the essential images of both $\mathrm{D}^{\mathrm{op}}_{\Qoppa} \circ r$ and $\mathrm{D}^{\mathrm{op}}_{\Qoppa} \circ l$, but this is implied by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(g\mathrm{D}_{\Qoppa}(rd'), d) &\simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{D}_{\Qoppa}(rd'), rd) \\ &\simeq \mathrm{B}_{\Qoppa}(\mathrm{D}_{\Qoppa}(rd'), \mathrm{D}_{\Qoppa}(rd)) \\ \mathrm{Hom}_{\mathcal{D}}(d, g\mathrm{D}_{\Qoppa}(ld')) &\simeq \mathrm{Hom}_{\mathcal{C}}(ld, \mathrm{D}_{\Qoppa}(ld')) \\ &\simeq \mathrm{B}_{\Qoppa}(ld, ld') \end{aligned}$$

both of which vanish by the assumption on $\Qoppa$.

Finally by Corollary 1.2.3, assuming the existence and full faithfulness of a left adjoint, g^{hyp} is a Poincaré-Verdier projection if and only if the map

$$\begin{aligned} \Qoppa(ld \oplus \mathrm{D}_{\Qoppa}(rd')) \longrightarrow \mathrm{B}_{\Qoppa}(ld \oplus \mathrm{D}_{\Qoppa}(rd'), ld \oplus \mathrm{D}_{\Qoppa}(rd')) \simeq \mathrm{Hom}_{\mathcal{C}}(ld \oplus \mathrm{D}_{\Qoppa}(rd'), \mathrm{D}_{\Qoppa}(ld) \oplus rd') \\ \xrightarrow{p} \mathrm{Hom}_{\mathcal{D}}(d \oplus p\mathrm{D}_{\Qoppa}(rd'), p\mathrm{D}_{\Qoppa}(ld) \oplus d') \xrightarrow{((\mathrm{id}_d,0)^*,(0,\mathrm{id}_{d'})_*)} \mathrm{Hom}_{\mathcal{D}}(d, d') \end{aligned}$$

is an equivalence. Under the equivalence

$$\Qoppa(ld \oplus \mathrm{D}_{\Qoppa}(rd')) \simeq \Qoppa(ld) \oplus \Qoppa(\mathrm{D}_{\Qoppa}(rd')) \oplus \mathrm{Hom}_{\mathcal{C}}(ld, rd')$$

this map becomes the projection to the last summand followed by the natural equivalence

$$\mathrm{Hom}_{\mathcal{D}}(ld, r'd) \simeq \mathrm{Hom}_{\mathcal{C}}(gld, d') \simeq \mathrm{Hom}_{\mathcal{C}}(d, d').$$

Thus, it is an equivalence if and only if $\Qoppa(ld)$ and $\Qoppa(\mathrm{D}_{\Qoppa}(rd'))$ both vanish, which is precisely our assumption. □

We next work out the more substantial example of module ∞-categories in detail, where the hermitian structure is defined by means of a module with genuine involution, introduced in §[I].3.2 (compare also the recollection section). We do so first in the generality of a map of E_1-algebras $\phi\colon A \to B$ over some base E_∞-ring k together with a map η of modules with genuine involution $\phi_!(M, N, \alpha) \to (M', N', \beta)$ over B, and eventually specialise to Ore localisations of discrete rings with skew-involution in Corollary 1.4.9. The reader only interested in this case is invited to take k the (Eilenberg-Mac Lane spectrum of the) integers and A and B (and even M and M') discrete from the start, though this does not simplify the discussion. Furthermore, it is important to allow N and N' to be non-discrete, so as to capture the genuine Poincaré structures.

Throughout, unmarked tensor products are always over k, and in case $k = \mathrm{H}R$ translate to the derived tensor product $\otimes^{\mathbb{L}}_{\mathbb{R}}$.

We want to establish general conditions on ϕ under which the hermitian functor $(\phi_!, \eta)$ becomes a Poincaré-Verdier or Poincaré-Karoubi projection. To obtain a Verdier sequence on the underlying stable ∞-categories, the following conditions are necessary and sufficient: A map $\phi\colon A \to B$ of E_1-ring spectra is said to be a *localisation* if the map

$$B \otimes_A B \to B,$$

induced by the multiplication of B is an equivalence of spectra. For such a localisation of ring spectra denote by $I \in \mathrm{Mod}_A$ its fibre. Straight from the definition one finds that I belongs to $(\mathrm{Mod}_A)_B$, the kernel of $\phi_!\colon \mathrm{Mod}_A \to \mathrm{Mod}_B$. We say that ϕ has *perfectly generated fibre* if I belongs to the smallest full subcategory of $(\mathrm{Mod}_A)_B$ containing $\mathrm{Mod}^\omega_A \cap (\mathrm{Mod}_A)_B$ and closed under colimits.

Summarising the discussion of Appendix A.4, we have by Proposition A.4.4 that if $\phi\colon A \to B$ is localisation of E_1-rings with perfectly generated fibre, then for any subgroup $\mathrm{c} \subseteq \mathrm{K}_0(A)$ the induction functors

$$\phi^\omega_!\colon \mathrm{Mod}^\omega_A \to \mathrm{Mod}^\omega_B \quad \text{and} \quad \phi^{\mathrm{c}}_!\colon \mathrm{Mod}^{\mathrm{c}}_A \to \mathrm{Mod}^{\phi(\mathrm{c})}_B$$

are Karoubi and Verdier projections, respectively; here $\mathrm{Mod}^{\mathrm{c}}_A$ denotes the full subcategory of Mod^ω_A spanned by all those A-modules with $[A] \in \mathrm{c} \subseteq \mathrm{K}_0(A)$.

We warn the reader explicitly that when applied to the Eilenberg-Mac Lane spectra of discrete rings, this notion of localisation differs from that in ordinary algebra: If $A \to B$ is a localisation of discrete rings, then $\mathrm{H}A \to \mathrm{H}B$ is a localisation in the sense above if and only if additionally $\mathrm{Tor}^A_i(B, B) = 0$ for all $i > 0$. This is automatic for commutative rings, or more generally if the localisation satisfies an Ore condition, but not true in general. Moreover, there are quotient maps $A \to A/I$ of commutative rings such that $\mathrm{H}A \to \mathrm{H}A/I$ is a localisation. When specialising to the case of discrete rings, we therefore call a map $A \to B$ a *derived localisation* if $\mathrm{H}A \to \mathrm{H}B$ is a localisation in the sense above. We do not know of a simple ring theoretic characterisation of this condition; see §A.4 for a more thorough discussion.

The following example essentially covers all of our applications:

1.4.2. **Example.** If A is an E_1-ring spectrum and $S \subseteq \pi_* A$ is a multiplicatively closed subset of homogeneous elements, which satisfies the left or right Ore condition, then the structure map $\phi\colon A \to A[S^{-1}]$ (see [Lur17, §7.2.3]) is a localisation by Lemma A.4.1, since the forgetful functor $\mathrm{Mod}_{A[S^{-1}]} \to \mathrm{Mod}_A$ is fully faithful. In this case the modules $A/s = \mathrm{cof}[\cdot s\colon \mathbb{S}^n \otimes A \to A]$ for $s \in S$ and $n \in \mathbb{Z}$ form a system of generators for $(\mathrm{Mod}_A)_B$ under shifts and colimits, see [Lur17, Lemma 7.2.3.13], so in particular ϕ has perfectly generated fibre.

Thus $\phi^\omega_!\colon \mathrm{Mod}^\omega_A \to \mathrm{Mod}^\omega_{A[S^{-1}]}$ is a Karoubi projection and $\phi^{\mathrm{c}}_!\colon \mathrm{Mod}^{\mathrm{c}}_A \to \mathrm{Mod}^{\mathrm{im(c)}}_{A[S^{-1}]}$ is a Verdier projection for any $\mathrm{c} \subseteq \mathrm{K}_0(A)$.

Let us now introduce hermitian structures into the picture. As discussed in section §[I].3.2 and the last part of the recollection section, an invertible module with genuine involution $(M, N, \alpha\colon N \to M^{\mathrm{tC}_2})$ over A gives rise to a Poincaré structure $\Qoppa^\alpha_M$ on Mod^ω_A; it restricts to a Poincaré structure on $\mathrm{Mod}^{\mathrm{f}}_A$ provided that M belongs to $\mathrm{Mod}^{\mathrm{c}}_A$ and provided c is closed under the involution on $\mathrm{K}_0(A)$ induced by M. For example, if c is the image of the canonical map $\mathbb{Z} \to \mathrm{K}_0(A), 1 \mapsto A$, then $\mathrm{Mod}^{\mathrm{c}}_A = \mathrm{Mod}^{\mathrm{f}}_A$ and this assumption is satisfied if also $M \in \mathrm{Mod}^{\mathrm{f}}_A$. We computed the left Kan extension of this Poincaré structure along the functor $\phi_!^\omega \colon \mathrm{Mod}_A^\omega \to \mathrm{Mod}_B^\omega$ in Corollary [I].3.4.1: It is the hermitian structure associated to the module with genuine involution

$$\phi_!(M, N, \alpha) = ((B \otimes B) \otimes_{A\otimes A} M, B \otimes_A N, \beta), \tag{10}$$

over B; here, β is the composition

$$B \otimes_A N \xrightarrow{\Delta\otimes\alpha} (B \otimes B)^{\mathrm{tC}_2} \otimes_A M^{\mathrm{tC}_2} \to ((B \otimes B) \otimes_{A\otimes A} M)^{\mathrm{tC}_2}$$

where Δ is the Tate diagonal. For example, by Remark 1.1.8, the same formula then applies for the Kan extension along $\phi_!^{\mathrm{c}} \colon \mathrm{Mod}_A^{\mathrm{c}} \to \mathrm{Mod}_B^{\phi(\mathrm{c})}$.

In order to obtain Poincaré-Karoubi projections, we need a compatibility condition between $\phi \colon A \to B$ and the module with involution M over A:

1.4.3. **Definition.** An invertible module with involution M over A is called *compatible* with a localisation of E_1-rings $A \to B$ if the composite

$$B \otimes_A M \simeq (B \otimes A) \otimes_{A\otimes A} M \longrightarrow (B \otimes B) \otimes_{A\otimes A} M$$

is an equivalence.

1.4.4. **Example.**

i) If A is an E_∞-ring and M an invertible A-module with A-linear involution (regarded as an $A \otimes A$-module via the multiplication map $A \otimes A \to A$), then compatibility is automatic, since in this case the map in question identifies with the evident one $B \otimes_A M \to B \otimes_A B \otimes_A M$ which is an equivalence by the assumption that $A \to B$ is a localisation.

ii) If M is the module with involution over A associated to a Wall anti-structure (ϵ, σ) on a discrete ring A as in Example [I].3.1.13 (i.e. $M = A$ regarded as an $A \otimes A$-module using the involution σ, and then equipped with the involution $\epsilon\sigma$, where $\epsilon \in A^*$) and

$$\phi \colon (A, \epsilon, \sigma) \longrightarrow (B, \delta, \tau)$$

is a map of rings with anti-structure, then M is also automatically compatible with ϕ if the latter is a derived localisation: For in this case, it is readily checked that the maps

$$b \otimes b' \otimes a \longmapsto ba \otimes \tau(b') \quad \text{and} \quad b \otimes b' \longmapsto b \otimes b' \otimes 1$$

give inverse equivalences

$$B \otimes B \otimes_{A\otimes A} A \simeq B \otimes_A B,$$

which translates the map in Definition 1.4.3 to the unit map $B \to B \otimes_A B$ which is an equivalence since ϕ is a derived localisation.

iii) If ϕ is an Ore localisation at the set $S \subseteq \pi_*(A)$, and M is an invertible module with involution over A, then M is compatible with ϕ if after inverting the action of S on M using the first A-module structure, S operates invertibly through the second one.

iv) Combining the two previous examples, if M is the A-module associated to a Wall anti-structure (ϵ, σ) on A, and $S \subseteq A$ satisfies the Ore condition and is closed under the involution σ, then M is compatible with the localisation map $A \to A[S^{-1}]$.

1.4.5. **Proposition.** *Let $\phi \colon A \to B$ be a localisation of E_1-ring spectra, with perfectly generated fibre and let (M, N, α) be an invertible module with genuine involution over A, such that M is compatible with ϕ.*

Then $\phi_!(M, N, \alpha)$ is invertible and the associated functor

$$\phi_!^\omega \colon (\mathrm{Mod}_A^\omega, Ϙ_M^\alpha) \to (\mathrm{Mod}_B^\omega, Ϙ_{\phi_! M}^{\phi_!\alpha})$$

is a Poincaré-Karoubi projection. It restricts to a Poincaré-Verdier projection

$$\phi_!^{\mathrm{c}} \colon (\mathrm{Mod}_A^{\mathrm{c}}, Ϙ_M^\alpha) \to (\mathrm{Mod}_B^{\phi(\mathrm{c})}, Ϙ_{\phi_! M}^{\phi_!\alpha}).$$

if $\mathrm{c} \subseteq \mathrm{K}_0(A)$ is closed under the involution induced by M.

Proof. The natural map

$$B \otimes_A \hom_A(X, M) \longrightarrow \hom_B(B \otimes_A X, B \otimes_A M)$$

is an equivalence for $X = A$ and thus for every compact A-module X, in particular for $X = M$, which shows that $B \otimes_A M$ has B as its B-linear endomorphisms, and therefore by assumption $(B \otimes B) \otimes_{A \otimes A} M$ is invertible (or alternatively, one can apply Remark 1.1.7 together with Proposition [I].3.1.6). Both functors $\phi_!^\omega$ and $\phi_!^c$ are then Poincaré by Lemma [I].3.4.3.

By Corollary 1.1.6, the functor $\phi_!^c$ is a Poincaré-Verdier projection since the underlying map on module categories is a Verdier projection and by definition the Poincaré structure on the target is the left Kan extension of that on the source. Similarly, the functor $\phi_!^\omega$ is a Poincaré-Karoubi projection by Corollary 1.3.12. ☐

This criterion applies immediately in the case of quadratic Poincaré structures:

1.4.6. **Corollary.** *Let $\phi : A \to B$ be a localisation of E_1-ring spectra, with perfectly generated fibre and let M be an invertible module with involution over A that is compatible with ϕ.*

Then

$$\phi_!^\omega : (\mathrm{Mod}_A^\omega, Ϙ_M^{\mathrm{q}}) \to (\mathrm{Mod}_B^\omega, Ϙ_{\phi_! M}^{\mathrm{q}}) \quad \textit{and} \quad \phi_!^c : (\mathrm{Mod}_A^c, Ϙ_M^{\mathrm{q}}) \to (\mathrm{Mod}_B^{\phi(c)}, Ϙ_{\phi_! M}^{\mathrm{q}})$$

are a Poincaré-Karoubi and Poincaré-Verdier projection (for $\mathrm{c} \subseteq \mathrm{K}_0(A)$ closed under the duality), respectively.

Symmetric Poincaré structures are not, however, generally preserved by left Kan extension:

1.4.7. **Example.** The map $p : \mathbb{S} \to \mathbb{S}[\frac{1}{2}]$ does *not* induce an equivalence $p_! Ϙ_{\mathbb{S}}^{\mathrm{s}} \simeq Ϙ_{\mathbb{S}[\frac{1}{2}]}^{\mathrm{s}}$ and consequently the functor

$$p_! : (\mathrm{Mod}_{\mathbb{S}}^\omega, Ϙ^{\mathrm{s}}) \longrightarrow (\mathrm{Mod}_{\mathbb{S}[\frac{1}{2}]}^\omega, Ϙ^{\mathrm{s}}),$$

is not a Poincaré-Karoubi projection: By Lin's theorem the linear part of $p_! Ϙ^{\mathrm{s}}$ is classified by $\mathbb{S}[\frac{1}{2}] \otimes \mathbb{S}_2^\wedge \simeq \mathrm{H}\mathbb{Q}_2$, whereas $\mathbb{S}[\frac{1}{2}]^{\mathrm{tC}_2} \simeq 0$ gives the linear part of the symmetric Poincaré structure on the target.

In the discrete case, an additional flatness assumption excludes such examples, as we will see in the next proposition.

Recall that for discrete (or more generally connective) A, and M an invertible module with (non-genuine) involution over A, we defined in §[I].3.2 the genuine family of Poincaré structures $Ϙ_M^{\geq m}$ for $m \in \mathbb{Z}$ as the Poincaré structures associated to the modules with genuine involution $(M, \tau_{\geq m} M^{\mathrm{tC}_2}, \alpha)$ where $\alpha : \tau_{\geq m} M^{\mathrm{tC}_2} \to M^{\mathrm{tC}_2}$ is the canonical map; the quadratic and symmetric Poincaré structures $Ϙ_M^{\mathrm{q}}$ and $Ϙ_M^{\mathrm{s}}$ are included in the genuine family as $m = -\infty$ and $m = \infty$, respectively.

1.4.8. **Proposition.** *Let $\phi : A \to B$ be a derived localisation between discrete rings with perfectly generated fibre, that furthermore makes B into a flat right module over A and let M be a discrete invertible module with involution over A that is compatible with ϕ. Then for arbitrary $m \in \mathbb{Z} \cup \{\pm\infty\}$, the maps*

$$\phi_!^\omega : (\mathcal{D}^{\mathrm{p}}(A), Ϙ_M^{\geq m}) \to (\mathcal{D}^{\mathrm{p}}(B), Ϙ_{\phi_! M}^{\geq m}) \quad \textit{and} \quad \phi_!^c : (\mathcal{D}^c(A), Ϙ_M^{\geq m}) \to (\mathcal{D}^{\phi(c)}(B), Ϙ_{\phi_! M}^{\geq m}).$$

are a Poincaré-Karoubi and a Poincaré-Verdier projection, respectively, for every $\mathrm{c} \subseteq \mathrm{K}_0(A)$ closed under the duality.

Proof. We use the following two inputs: firstly, B being a flat A^{op}-module implies that it can be written as filtered colimit of finitely generated free A^{op}-modules B_i and secondly, Tate cohomology commutes with filtered colimits of discrete modules in the coefficients. The former statement is a classical theorem of Lazard, see e.g. [Laz69, Théorème 1.2] or [SP18, Tag 058G], and the second statement (for group cohomology) was discovered by Brown in [Bro75, Theorem 3] for groups admitting a classifying space of finite type; given the 2-periodicity of Tate cohomology for C_2 the case at hand also follows immediately from the same statement for group homology, which is obvious from the definitions.

Now, recall the description of $\phi_! \Qoppa^{\geq m}_M$ via (10). We start by considering the case $m = \infty$; in which case we need to show that $B \otimes_A M^{\mathrm{tC}_2} \to ((B \otimes B) \otimes_{A \otimes A} M)^{\mathrm{tC}_2}$ is an equivalence. We can regard this as a natural transformation between spectrum valued functors

$$X \longmapsto X \otimes_A M^{\mathrm{tC}_2} \quad \text{and} \quad X \longmapsto \big((X \otimes X) \otimes_{A \otimes A} M\big)^{\mathrm{tC}_2}$$

on both the category of discrete A^{op}-modules and $\mathcal{D}(A^{\mathrm{op}})$. From the latter case, we obtain that it is an equivalence for every perfect X, as both sides are exact functors and the claim is evidently true for $X = A$. In particular, the claim is true for all finitely generated projective A^{op}-modules since these are perfect when regarded in $\mathcal{D}(A^{\mathrm{op}})$. Since filtered colimits are in particular sifted (i.e. the diagonal of a filtered category is cofinal), regarding the two assignments as functors on the category of discrete A^{op}-modules the second fact makes them commute with filtered colimits of finitely generated free A^{op}-modules (for X a finitely generated free A^{op}-module, $X \otimes X$ is a finitely generated free $(A \otimes A)^{\mathrm{op}}$-module, so $(X \otimes X) \otimes_{A \otimes A} M$ remains discrete, despite $X \otimes X$ and $A \otimes A$ potentially having higher homotopy). Taken together, the transformation is an equivalence for all flat A^{op}-modules, so in particular for $X = B$ as desired.

To obtain the case of the genuine Poincaré structures, just observe that the flatness of B also guarantees that the functor $B \otimes_A - \colon \mathcal{D}(A) \to \mathcal{D}(B)$ commutes with the connective cover functors $\tau_{\geq m}$ for all $m \in \mathbb{Z}$.

The case of the quadratic Poincaré structure is 1.4.6 above. ☐

As a special case we obtain:

1.4.9. **Corollary.** *Let (A, ϵ, σ) a ring with Wall anti-structure, and $S \subseteq A$ a multiplicative subset satisfying the left Ore condition and closed under the involution σ. Then if M denotes the module with involution over A given by endowing A with the $A \otimes A$-module structure arising from σ and the involution $\epsilon\sigma$, we find for all $m \in \mathbb{Z} \cup \{\pm\infty\}$ a Poincaré-Karoubi sequence*

$$\big(\mathcal{D}^{\mathrm{p}}(A)_S, \Qoppa^{\geq m}_M\big) \longrightarrow \big(\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\geq m}_M\big) \longrightarrow \big(\mathcal{D}^{\mathrm{p}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]}\big)$$

and a Poincaré-Verdier sequence

$$\big(\mathcal{D}^{\mathrm{c}}(A)_S, \Qoppa^{\geq m}_M\big) \longrightarrow \big(\mathcal{D}^{\mathrm{c}}(A), \Qoppa^{\geq m}_M\big) \longrightarrow \big(\mathcal{D}^{\mathrm{im(c)}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]}\big),$$

where the subscript S in the source denotes the full subcategory of complexes whose homology is S-torsion.

This example will serve as the main input to obtain localisation sequences of Grothendieck-Witt spectra in §4.4.

Proof. Note only that $A[S^{-1}]$ is flat thus a derived localisation on account of the Ore condition, as the construction of $A[S^{-1}]$ as one-sided fractions displays it as a filtered colimit of free A^{op}-modules of rank 1, so that Proposition 1.4.8 applies. ☐

The Ore condition is in fact often necessary to achieve flatness of the localisation: In [Tei03, Main Theorem], Teichner shows that if S is the set of elements that become invertible modulo a two-sided ideal I, then flatness of $A[S^{-1}]$ as a right A-module is equivalent to S being left Ore.

Let us finally consider examples involving the tensor and cotensor constructions as considered in §[I].6.3 and [I].6.4; several of the results will be required for our analysis of the hermitian Q-construction in the next section.

1.4.10. **Proposition.** *Let $p \colon \mathcal{I} \to \mathcal{J}$ be a functor of ∞-categories which exhibits $\mathcal{J}$ as a localisation of $\mathcal{I}$ and let $\mathcal{C}$ be a stable ∞-category. Then the following hold:*

i) The induced functor $\mathcal{C}_{\mathcal{I}} \to \mathcal{C}_{\mathcal{J}}$ is a Verdier projection,

ii) the induced functor $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ is a Verdier inclusion,

iii) if $\Qoppa$ is a Poincaré structure on $\mathcal{C}$, such that $(\mathcal{C}_{\mathcal{J}}, \Qoppa_{\mathcal{J}})$ is Poincaré and the kernel of $\mathcal{C}_{\mathcal{I}} \to \mathcal{C}_{\mathcal{J}}$ is closed under the duality of $\Qoppa_{\mathcal{I}}$ then

$$(\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}}) \xrightarrow{p_*} (\mathcal{C}_{\mathcal{J}}, \Qoppa_{\mathcal{J}})$$

is a Poincaré-Verdier projection, and

iv) if $\Qoppa$ is a Poincaré structure on $\mathcal{C}$, such that $(\mathcal{C}^{\mathcal{I}}, \Qoppa^{\mathcal{I}})$ is Poincaré and $\mathcal{C}^{\mathcal{J}}$ is closed under the duality in $\mathcal{C}^{\mathcal{I}}$, then

$$(\mathcal{C}^{\mathcal{J}}, \Qoppa^{\mathcal{J}}) \xrightarrow{p^*} (\mathcal{C}^{\mathcal{I}}, \Qoppa^{\mathcal{I}})$$

is a Poincaré-Verdier inclusion.

Denoting by W the set of maps in $\mathcal{I}$ taken to equivalences in $\mathcal{J}$ and by $\overline{W}$ the collection of maps in $\mathcal{C}_{\mathcal{J}}$ arising as images of arrows (id_x, α) under the canonical functor $\mathcal{C}\times\mathcal{I} \to \mathcal{C}_{\mathcal{J}}$ with $x \in \mathcal{C}$ and α in W, then $\mathcal{C}_{\mathcal{I}} \to \mathcal{C}_{\mathcal{J}}$ is in fact a localisation at $\overline{W}$ and the kernel appearing in iii) is the smallest stable subcategory of $\mathcal{C}_{\mathcal{I}}$, that is closed under retracts and contains the fibres of maps in $\overline{W}$.

Proof. The universal properties of $\mathcal{C}_{\mathcal{I}}$ and $\mathcal{C}_{\mathcal{J}}$ as tensors imply that for every stable ∞-category $\mathcal{D}$, we have a square with invertible horizontal arrows

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_{\mathcal{J}}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}^{l}(\mathcal{C}\times\mathcal{J}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{J}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \arrow[d] \\
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_{\mathcal{I}}, \mathcal{D}) \arrow[r] & \mathrm{Fun}^{l}(\mathcal{C}\times\mathcal{I}, \mathcal{D}) \arrow[r] & \mathrm{Fun}(\mathcal{I}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})),
\end{tikzcd}$$

where $\mathrm{Fun}^{l}(\mathcal{C}\times\mathcal{I}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}\times\mathcal{I}, \mathcal{D})$ spanned by those functors which are exact in the $\mathcal{C}$ entry, and similarly for $\mathrm{Fun}^{l}(\mathcal{C}\times\mathcal{J}, \mathcal{D})$. The assumption that $\mathcal{I} \to \mathcal{J}$ exhibits $\mathcal{J}$ as the localisation of $\mathcal{I}$ at W now implies that the right most vertical arrow is fully faithful, with essential image spanned by those diagrams $\mathcal{I} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ which send the arrows in W to equivalences. It then follows that the left most vertical arrow is fully faithful with essential image spanned by those exact functors $\mathcal{C}_{\mathcal{I}} \to \mathcal{D}$ which send the arrows in $\overline{W}$ to equivalences. By Proposition A.1.6 this establishes i) and the addendum.

For the second statement, we recall $\mathcal{C}^{\mathcal{I}} \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{C})$, so the fact that $\mathcal{I} \to \mathcal{J}$ is a localisation implies that $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ is fully faithful, and the characterisation of the image as those functors that invert W shows that the image is closed under retract. ii) now follows from Proposition A.1.9.

Now, since $(\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}})$ is assumed to be Poincaré and the kernel $\mathcal{E} \subseteq \mathcal{C}_{\mathcal{I}}$ closed under the associated duality in statement iii), it follows that $(\mathcal{E}, (\Qoppa_{\mathcal{I}})|_{\mathcal{E}})$ is Poincaré as well. In light of Proposition 1.1.4, to show that the map in iii) is a Poincaré-Verdier projection it thus suffices to show that extended by its kernel it is a cofibre sequence in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$, since the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{h}}_{\infty}$ detects colimits. For this, by the universal properties of $(\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}})$ and $(\mathcal{C}_{\mathcal{J}}, \Qoppa_{\mathcal{J}})$ as tensors in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$, we obtain for every hermitian ∞-category $(\mathcal{D}, \Phi)$ a diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J}, \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi))) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}_{\mathcal{J}}, \Qoppa_{\mathcal{J}}), (\mathcal{D}, \Phi)) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_{\mathcal{J}}, \mathcal{D}) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Fun}(\mathcal{J}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \arrow[d] \\
\mathrm{Fun}(\mathcal{I}, \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi))) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}}), (\mathcal{D}, \Phi)) \arrow[r] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_{\mathcal{I}}, \mathcal{D}) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}(\mathcal{I}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})).
\end{tikzcd}$$

Since $\mathcal{I} \to \mathcal{J}$ is a localisation at W, the left most diagonal arrow is fully faithful with essential image consisting of those diagrams $\mathcal{I} \to \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi))$ which send the arrows in W to equivalences, and similarly for the rightmost arrow. The conservativity of $\mathrm{Fun}^{\mathrm{h}}((\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ implies that the outer square (and hence the middle one) are cartesian and since $\mathcal{C}_{\mathcal{I}} \to \mathcal{C}_{\mathcal{J}}$ is a Verdier projection,

$$\mathrm{Fun}^{\mathrm{h}}((\mathcal{C}_{\mathcal{J}}, \Qoppa_{\mathcal{J}}), (\mathcal{D}, \Phi)) \to \mathrm{Fun}^{\mathrm{h}}((\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}}), (\mathcal{D}, \Phi))$$

is fully faithful with essential image those hermitian functors $(\mathcal{C}_{\mathcal{I}}, \Qoppa_{\mathcal{I}}) \to (\mathcal{D}, \Phi)$ which invert $\overline{W}$, i.e. which vanish on the kernel of $\mathcal{C}_{\mathcal{I}} \to \mathcal{C}_{\mathcal{J}}$. This shows iii).

For iv), we have to check that $\Qoppa^{\mathcal{J}} \simeq p^*\Qoppa^{\mathcal{J}}$, but this follows from the definition of the hermitian structure on cotensors as a limit, together with the fact that localisations are final. □

For simplicity, we shall restrict attention to the case of finite posets in the remainder of this section, where we recall that our convention for interpreting a poset as a category is that $i \leq j$ means a morphism from i to j. Given a finite poset $\mathcal{J}$, a full subposet $\mathcal{I} \subseteq \mathcal{J}$ is said to be a upwards closed if $i, j \in \mathcal{J}$ are such that $i \in \mathcal{I}$ and $i \leq j$ then $j \in \mathcal{I}$. In particular, if $r\colon \mathcal{I} \hookrightarrow \mathcal{J}$ is upwards closed then for every $i \in \mathcal{I}$ the functor $\mathcal{I}_{i/} \to \mathcal{J}_{r(i)/}$ is an isomorphism and hence r satisfies the condition of Proposition [I].6.3.18. Given a Poincaré ∞-category

$(\mathcal{C},\Qoppa)$, we then have that the functor $r^*\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ commutes with the respective (possibly non-perfect) dualities. Thus, in the case where both $(\mathcal{C}^{\mathcal{I}},\Qoppa^{\mathcal{I}})$ and $(\mathcal{C}^{\mathcal{J}},\Qoppa^{\mathcal{J}})$ are Poincaré the hermitian functor

$$(r^*,\eta)\colon (\mathcal{C}^{\mathcal{J}},\Qoppa^{\mathcal{J}}) \to (\mathcal{C}^{\mathcal{I}},\Qoppa^{\mathcal{I}}) \tag{11}$$

is Poincaré as well.

1.4.11. **Proposition.** *Let $r\colon \mathcal{I} \hookrightarrow \mathcal{J}$ be an upwards closed inclusion between finite posets, and let $(\mathcal{C},\Qoppa)$ be a Poincaré ∞-category such that the hermitian ∞-categories $(\mathcal{C}^{\mathcal{I}},\Qoppa^{\mathcal{I}})$ and $(\mathcal{C}^{\mathcal{J}},\Qoppa^{\mathcal{J}})$ are Poincaré. Then the Poincaré functor* (11) *is a split Poincaré-Verdier projection.*

Proof. A fully faithful left adjoint to r is given by the exact functor $r_!\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}}$ performing left Kan extension. In fact, since r is upwards closed this left Kan extension admits a very explicit formula: for a diagram $\varphi\colon \mathcal{I} \to \mathcal{C}$ the value of $r_!\varphi$ is given by

$$r_!\varphi(j) = \begin{cases} \varphi(j) & j \in \mathcal{I} \\ 0 & j \notin \mathcal{I} \end{cases}$$

Since $\Qoppa(0) \simeq 0$, the spectrum valued diagram $j \mapsto \Qoppa(r_!\varphi(j))$ is, for a similar reason, a right Kan extension of its restriction to $\mathcal{I}^{\mathrm{op}}$, and so the natural map

$$\Qoppa^{\mathcal{J}}(r_!\varphi) \simeq \lim_{j\in\mathcal{J}^{\mathrm{op}}} \Qoppa(r_!\varphi(j)) \to \lim_{i\in\mathcal{I}^{\mathrm{op}}} \Qoppa(\varphi(i)) = \Qoppa^{\mathcal{I}}(\varphi)$$

is an equivalence. The Poincaré functor (11) is then Poincaré-Verdier projection by Corollary 1.2.3 ii). ☐

In the situation of Proposition 1.4.11, one may also consider instead the hermitian ∞-categories $(\mathcal{C}_{\mathcal{I}},\Qoppa_{\mathcal{I}})$ and $(\mathcal{C}_{\mathcal{J}},\Qoppa_{\mathcal{J}})$ obtained by applying the tensor construction instead of the cotensor construction. When $\mathcal{I}$ is a finite poset, the underlying ∞-category $\mathcal{C}_{\mathcal{I}}$ identifies with $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{C})$ by Lemma [I].6.5.6, and $\Qoppa_{\mathcal{I}}$ sends such a diagram $\varphi\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ to $\mathrm{colim}_{i\in\mathcal{I}}\Qoppa(\varphi(i))$; of course the same holds for $\mathcal{J}$. In the case where $(\mathcal{C}_{\mathcal{I}},\Qoppa_{\mathcal{I}})$ and $(\mathcal{C}_{\mathcal{J}},\Qoppa_{\mathcal{J}})$ are both Poincaré and $r\colon \mathcal{I} \hookrightarrow \mathcal{J}$ is an upwards closed inclusion, the induced hermitian functor

$$(r_*,\eta)\colon (\mathcal{C}_{\mathcal{I}},\Qoppa_{\mathcal{I}}) \to (\mathcal{C}_{\mathcal{J}},\Qoppa_{\mathcal{J}}) \tag{12}$$

refining the right Kan extension functor r_* is Poincaré by Proposition [I].6.5.13; we shall use the symbol r_* even though the right Kan extension is taken along the functor $r^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{J}^{\mathrm{op}}$.

1.4.12. **Proposition.** *Let $r\colon \mathcal{I} \to \mathcal{J}$ be an upwards closed inclusion of finite posets, and let $(\mathcal{C},\Qoppa)$ be a Poincaré ∞-category such that the hermitian ∞-categories $(\mathcal{C}_{\mathcal{I}},\Qoppa_{\mathcal{I}})$ and $(\mathcal{C}_{\mathcal{J}},\Qoppa_{\mathcal{J}})$ are Poincaré. Then the Poincaré functor* (12) *is a split Poincaré-Verdier inclusion.*

Proof. We first note that the right Kan extension r_* is fully faithful (since r is) and admits a left adjoint given by restriction. To finish the proof, it will suffice by Corollary 1.2.3 i) to show that for every diagram $\varphi\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$, the composite map

$$\operatorname*{colim}_{i\in\mathcal{I}} \Qoppa(\varphi(i)) \to \operatorname*{colim}_{i\in\mathcal{I}} \Qoppa(r^*r_*\varphi(i)) \to \operatorname*{colim}_{j\in\mathcal{J}} \Qoppa(r_*\varphi(j))$$

is an equivalence. Here, the first map is an equivalence since r is fully faithful. To see that the second map is an equivalence, we argue as in the proof of Proposition 1.4.11 and observe that

$$r_*\varphi(j) = \begin{cases} \varphi(j) & j \in \mathcal{I} \\ 0 & j \notin \mathcal{I} \end{cases}$$

The spectrum valued diagram $j \mapsto \Qoppa(r_*\varphi(j))$ is then, for a similar reason, the left Kan extension of its restriction to $\mathcal{I}$, and so the second map above is an equivalence as well. ☐

We next consider Poincaré-Verdier projections involving the exceptional functoriality from Construction [I].6.5.14. To this end, let $\alpha\colon \mathcal{I} \to \mathcal{J}$ be a cofinal map between finite posets. In this case the restriction and right Kan extension maps

$$(\alpha^{\mathrm{op}})^*\colon \mathrm{Fun}(\mathcal{J}^{\mathrm{op}},\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{C}) \quad \text{and} \quad \alpha_*\colon \mathrm{Fun}(\mathcal{I},\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J},\mathcal{C})$$

acquire canonical hermitian structure upgrading them to functors

$$\alpha^*\colon (\mathcal{C},\Qoppa)_{\mathcal{J}} \longrightarrow (\mathcal{C},\Qoppa)_{\mathcal{I}} \quad \text{and} \quad \alpha_*\colon (\mathcal{C},\Qoppa)^{\mathcal{I}} \longrightarrow (\mathcal{C},\Qoppa)^{\mathcal{J}},$$

respectively.

1.4.13. **Proposition.** *Suppose that* $(\mathcal{C}, \Qoppa)$ *is a Poincaré* ∞*-category and* $\alpha : \mathcal{I} \hookrightarrow \mathcal{J}$ *is a cofinal and full inclusion of finite posets such that* $(\mathcal{C}, \Qoppa)_{\mathcal{I}}$ *and* $(\mathcal{C}, \Qoppa)_{\mathcal{J}}$ *are Poincaré. Then* $\alpha^* : (\mathcal{C}, \Qoppa)_{\mathcal{J}} \to (\mathcal{C}, \Qoppa)_{\mathcal{I}}$ *is a split Poincaré-Verdier projection.*

Proof. To prove the first claim, note that α^* admits fully faithful left and right adjoints given by left and right Kan extension. It follows directly from the explicit formula

$$[\mathrm{D}_{\mathcal{I}}(\varphi)](j) = \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{D}(\varphi(i))^{\hom_{\mathcal{I}}(i,j)}$$

of Proposition [I].6.5.8 that α^* preserves the dualities.

By Proposition 1.2.3, we are left to show that for $\varphi \in \mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{C})$, the natural map

$$\Qoppa_{\mathcal{I}}(\alpha_!^{\mathrm{op}}\varphi) \to \Qoppa_{\mathcal{J}}((\alpha^{\mathrm{op}})^*\alpha_!^{\mathrm{op}}\varphi) \to \Qoppa_{\mathcal{J}}(\varphi)$$

is an equivalence, where $\alpha_!$ denotes the left Kan extension functor. Indeed

$$\Qoppa_{\mathcal{I}}(\alpha_!^{\mathrm{op}}\varphi) \simeq \operatorname*{colim}_{i \in \mathcal{I}} \Qoppa(\alpha_!^{\mathrm{op}}\varphi(i)) \simeq \operatorname*{colim}_{j \in \mathcal{J}} \Qoppa(\alpha_!^{\mathrm{op}}\varphi(\alpha(j))) \cong \operatorname*{colim}_{j \in \mathcal{J}} \Qoppa(\varphi(j)),$$

where we have used that $\alpha : \mathcal{J} \to \mathcal{I}$ is cofinal and that $(\alpha^{\mathrm{op}})^*\alpha_!^{\mathrm{op}}\varphi \cong \varphi$ as a consequence of α being fully faithful. ☐

1.4.14. **Proposition.** *Suppose that* $(\mathcal{C}, \Qoppa)$ *is a Poincaré* ∞*-category and* $\alpha : \mathcal{I} \hookrightarrow \mathcal{J}$ *a localisation among finite posets such that* $(\mathcal{C}, \Qoppa)^{\mathcal{I}}$ *and* $(\mathcal{C}, \Qoppa)^{\mathcal{J}}$ *are Poincaré. Assume furthermore that the hermitian functor* α_* *is duality preserving. Then* $\alpha_* : (\mathcal{C}, \Qoppa)^{\mathcal{I}} \to (\mathcal{C}, \Qoppa)^{\mathcal{J}}$ *is a split Poincaré-Verdier projection.*

Note that by [Cis19, Proposition 7.1.10], localisations are cofinal, so that α_* is well-defined.

Proof. First up, restriction along α is left adjoint to α_* and fully faithful since α is a localisation. Thus, by 1.1.6, we are left to prove that the natural map

$$\Qoppa^{\mathcal{J}}(\varphi) \longrightarrow \Qoppa^{\mathcal{I}}(\varphi \circ \alpha)$$

is an equivalence. Indeed,

$$\Qoppa^{\mathcal{J}}(\varphi) \simeq \lim_{j \in \mathcal{J}^{\mathrm{op}}} \Qoppa(\varphi(j)) \simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}} \Qoppa(\varphi\alpha(i)) \simeq \Qoppa^{\mathcal{I}}(\varphi \circ \alpha)$$

since α^{op} is final by [Cis19, Proposition 7.1.10]. ☐

Finally, let us also record:

1.4.15. **Proposition.** *Let us consider a (split) Poincaré-Verdier sequence* $(\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa') \to (\mathcal{C}'', \Qoppa'')$ *and a finite poset* $\mathcal{I}$ *such that* $(-)_{\mathcal{I}} : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ *preserves Poincaré* ∞*-categories. Then the induced sequences*

$$(\mathcal{C}, \Qoppa)^{\mathcal{I}} \longrightarrow (\mathcal{C}', \Qoppa')^{\mathcal{I}} \longrightarrow (\mathcal{C}'', \Qoppa'')^{\mathcal{I}}$$

and

$$(\mathcal{C}, \Qoppa)_{\mathcal{I}} \longrightarrow (\mathcal{C}', \Qoppa')_{\mathcal{I}} \longrightarrow (\mathcal{C}'', \Qoppa'')_{\mathcal{I}}$$

are (split) Poincaré-Verdier sequence.

Note that by [I].6.5.12, the functor $(-)^{\mathcal{I}} : \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ preserves $\mathrm{Cat}_\infty^{\mathrm{p}}$ provided $(-)_{\mathcal{I}}$ does, which in turn is equivalent to $(\mathcal{S}p^\omega, \Qoppa^{\mathrm{u}})_{\mathcal{I}}$ being Poincaré.

Proof. Let us treat the tensoring, the argument for the cotensoring being entirely dual. As a left adjoint, the tensoring construction generally preserves colimits, and by [I].6.5.10 tensoring with a finite poset also preserves limits. This gives the part of the statement disregarding splittings. But for example from [I].6.5.8 we find that the operation $(-)_{\mathcal{I}} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ preserves adjoints, which implies the split case. ☐

1.5. Additive and localising functors.

In this section, we establish the basic notions of additive, Verdier-localising and Karoubi-localising functors. They are based on a mild generalisation of Poincaré-Verdier and Poincaré-Karoubi sequences in the form of certain bicartesian squares. Sending these particular bicartesian squares to bicartesian squares isolates the localisation properties enjoyed by Grothendieck-Witt theory axiomatically.

In the present paper, we focus almost exclusively on Verdier-localising (or even additive) functors. Together with the principal example of the Karoubi-Grothendieck-Witt functor, Karoubi-localising functors are studied thoroughly in Paper [IV] and we only briefly mention them here for completeness' sake.

1.5.1. **Definition.** A *(split) Verdier square* is a diagram

(13)
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{D}' \end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$ which is cartesian and whose vertical maps are (split) Verdier projections. We say that a square as in (13) is a *Karoubi square* if it becomes cartesian in $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ after applying completion and its vertical maps are Karoubi projections.

A *(split) Poincaré-Verdier square* is a diagram

(14)
$$\begin{tikzcd} (\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & (\mathcal{D},\Phi) \arrow[d] \\ (\mathcal{C}',\Qoppa') \arrow[r] & (\mathcal{D}',\Phi') \end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ which is cartesian and whose vertical maps are (split) Poincaré-Verdier projections. We say that a square as in (14) is a *Poincaré-Karoubi square* if it becomes cartesian after applying the idempotent completion functor of Proposition 1.3.3 and its vertical maps are Poincaré-Karoubi projections.

1.5.2. **Remarks.**

i) A (split) Poincaré-Verdier square with lower left corner $0 \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$ is exactly a (split) Poincaré-Verdier sequence. The same holds for Poincaré-Karoubi sequences.
ii) The classifying squares of Theorem 1.2.9 and Proposition A.2.12 give examples of split (Poincaré-)Verdier squares.
iii) Any Poincaré-Verdier square is also cocartesian in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$: Indeed, extend (14) to a commutative rectangle

(15)
$$\begin{tikzcd} (\mathcal{E},\Qoppa|_{\mathcal{E}}) \arrow[r] \arrow[d] & (\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & (\mathcal{D},\Phi) \arrow[d] \\ 0 \arrow[r] & (\mathcal{C}',\Qoppa') \arrow[r] & (\mathcal{D}',\Phi') \end{tikzcd}$$

in which both squares are cartesian and the vertical maps are Poincaré-Verdier projections. Then the external rectangle is cartesian by the pasting lemma, see [Lur09a, Lemma 4.4.2.1], hence cocartesian since the right vertical map is a Verdier projection. For the same reason, the left square is cocartesian and so the right square is cocartesian by the pasting lemma. Similarly, every Poincaré-Karoubi square becomes cocartesian in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ after applying idempotent completion.
iv) By Corollary 1.2.6 and Corollary A.2.7, the collection of split (Poincaré-)Verdier projections is closed under pullback. Therefore a cartesian square in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ is a split Poincaré-Verdier square if only its right vertical leg is a split Poincaré-Verdier projection. The same statement holds for general (Poincaré-)Verdier/Karoubi squares by A.1.11, A.3.10, 1.1.9 and 1.1.9.
v) Proposition 1.3.5 implies that every Poincaré-Verdier square is a Poincaré-Karoubi square. Conversely, a Poincaré-Karoubi square involving idempotent complete Poincaré ∞-categories is a Poincaré-Verdier square if and only if its vertical maps are essentially surjective.

The following are useful recognition criteria for (Poincaré)-Verdier squares:

1.5.3. **Lemma.** *Consider a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "p"] & \mathcal{C}' \arrow[d, "p'"] \\ \mathcal{D} \arrow[r, "j"] & \mathcal{D}' \end{tikzcd}$$

in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ *such that* p *and* p' *are (split) Verdier projections. Then the square is a Verdier square if and only if the induced map* $\ker(p) \to \ker(p')$ *is an equivalence and the square is* Pro/Ind-*adjointable.*

The same statement holds for a diagram

$$\begin{tikzcd} (\mathcal{C}, \Qoppa) \arrow[r, "i"] \arrow[d, "p"] & (\mathcal{C}', \Qoppa') \arrow[d, "p'"] \\ (\mathcal{D}, \Phi) \arrow[r, "j"] & (\mathcal{D}', \Phi') \end{tikzcd}$$

in $\mathrm{Cat}_\infty^{\mathrm{p}}$ *whose vertical maps are (split) Poincaré-Verdier projections, i.e. it is cartesian if and only if the induced map* $(\ker(p), \Qoppa) \to (\ker(p'), \Qoppa')$ *is an equivalence and the underlying diagram of stable* ∞*-categories is (*Pro/Ind-*)adjointable.*

Furthermore, for a diagram in $\mathrm{Cat}_\infty^{\mathrm{p}}$ *as above, left adjointability and right adjointability are equivalent.*

The adjointability condition is not automatic: Consider for example the shear map

$$\mathcal{C}^2 \to \mathcal{C}^2, \quad (c, c') \mapsto (c, c \oplus c')$$

as a self-map of the Verdier projection $\mathrm{pr}_1 : \mathcal{C}^2 \to \mathcal{C}$. It is, however, easily checked in practise, especially in the Poincaré setting: For example, for a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

of E_1-rings, Ind-adjointability of the square formed by the extension-of-scalars functors on compact modules is equivalent to the natural map $A' \otimes_A B \to B'$ being an equivalence. Under the assumption that the map $\ker(p) \to \ker(p')$ is an equivalence, Ind/Pro-adjointability is also easily checked equivalent to the condition that i induces equivalences

$$\mathrm{Hom}_{\mathcal{C}}(x, c) \to \mathrm{Hom}_{\mathcal{C}'}(i(x), i(c)) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{C}}(c, x) \to \mathrm{Hom}_{\mathcal{C}'}(i(c), i(x))$$

for all $x \in \ker(p)$ and $c \in \mathcal{C}$. In this guise, the non-hermitian part of Lemma 1.5.3 is directly verified by Krause in [Kra20, Lemma 3.9], whereas we will simply appeal to our classification of (Poincaré-)Verdier projections.

Proof. We explicitly checked that Verdier squares are Ind/Pro-adjointable as part of Proposition A.3.15; this contains the much simpler case of split Verdier squares. As mentioned, for the more interesting converse, we appeal to the classification results for Verdier sequences: In the split case, Proposition A.2.12 implies that both p and p' are pulled back from the same split Verdier projection, and thus from one another, since the classifying map of p is that of p' composed with the arrow $\mathcal{D} \to \mathcal{D}'$ by Proposition A.2.14. In the case of a Karoubi projection, the same argument can be made using Proposition A.3.14 and Proposition A.3.15 instead. For the case of general Verdier sequences, we then immediately obtain that

$$\begin{tikzcd} \mathcal{C}^\natural \arrow[r, "i"] \arrow[d, "p"] & (\mathcal{C}')^\natural \arrow[d, "p'"] \\ \mathcal{D}^\natural \arrow[r, "j"] & (\mathcal{D}')^\natural \end{tikzcd}$$

is a cartesian square and deduce that the map from $\mathcal{C}$ to the pullback in the original square is fully faithful. It remains to show that it is essentially surjective. This follows immediately from Thomason's classification of dense subcategories Theorem A.3.2, since Verdier projections induce exact sequences on K_0.

The Poincaré case follows by the exact same argument using Proposition 1.2.11 and Corollary 1.3.18 instead of Proposition A.2.14 and Proposition A.3.15.

Finally, to see that the two adjointability conditions are equivalent in the Poincaré case, simply note that $\mathrm{D}_{\mathfrak{Q}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ induces an equivalence

$$\mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathrm{Ind}(\mathcal{C})$$

and similarly for $\mathcal{C}'$, $\mathcal{D}$ and $\mathcal{D}'$. Since all functors in sight commute with the dualities, conjugation with the above equivalence exchanges left and right adjoints, which gives the claim. ☐

We now come to the main definition of this subsection.

1.5.4. **Definition.** Let $\mathcal{E}$ be an ∞-category which admits finite limits and $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ a functor. Recall that $\mathcal{F}$ is said to be *reduced* if $\mathcal{F}(0)$ is a terminal object in $\mathcal{E}$. We say that a reduced functor $\mathcal{F}$ is *additive*, *Verdier-localising* or *Karoubi-localising*, if it takes split Poincaré-Verdier squares, arbitrary Poincaré-Verdier squares or Poincaré-Karoubi squares to cartesian squares, respectively.

We shall denote the ∞-categories of these functors by

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}), \quad \mathrm{Fun}^{\mathrm{vloc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}), \quad \text{and} \quad \mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}),$$

respectively.

It follows from Remark 1.5.2 that there are inclusions

$$\mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}) \subseteq \mathrm{Fun}^{\mathrm{vloc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$$

as full subcategories. We note that additive, Verdier-localising and Karoubi-localising invariants are closed in $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ under limits (which are computed pointwise), such as taking loops. Colimits on the other hand are generally not computed pointwise (unless $\mathcal{E}$ is stable), and we shall see in the next section that the Q-construction implements suspension in the ∞-category $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$, which is ultimately the reason for the universal property of Grothendieck-Witt theory.

Warning. Here we follow the convention of the fifth author and Tamme to divorce the preservation of filtered colimits from the preservation of certain fibre sequences and squares. As a result, the ∞-categories appearing in the end of Definition 1.5.4 are *not* locally small. The reader who is adverse to non-locally small ∞-categories is invited to restrict attention only to accessible additive/Verdier-localising/Karoubi-localising functors; this will not affect any of the statements in this paper, nor their proofs.

We also note that if one fixes a regular cardinal κ and restricts attention only to those additive/Verdier-localising/Karoubi-localising functors that preserve κ-filtered colimits, then the corresponding variants of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$, $\mathrm{Fun}^{\mathrm{vloc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ and $\mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ become presentable, and a reader who so prefers may fix at this moment once and for all a sufficiently large such κ. At any rate, the most interesting examples of such functors that appear in this paper, such as the Grothendieck-Witt, K- and L-theory spectra, even preserve ω-filtered colimits.

Any additive, Verdier-localising or Karoubi-localising functor sends split Poincaré-Verdier, Poincaré-Verdier or Poincaré-Karoubi sequences, respectively, to fibre sequences. If $\mathcal{E}$ is stable, the converse holds as well:

1.5.5. **Proposition.** *A reduced functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ with $\mathcal{E}$ stable is additive, Verdier-localising or Karoubi-localising if and only if it takes split Poincaré-Verdier, Poincaré-Verdier or all Poincaré-Karoubi sequences to exact sequences in $\mathcal{E}$.*

Proof. Apply $\mathcal{F}$ to the rectangle in Remark 1.5.2 and use the pasting lemma. ☐

For non-stable $\mathcal{E}$ we expect, however, that the condition of being additive or Verdier-localising is strictly stronger than sending split Poincaré-Verdier or Poincaré-Verdier sequences to fibre sequences, and similarly for the condition of being Karoubi-localising. We will need the stronger variant in §2.6 with target $\mathcal{S}$, when we discuss the additivity theorem for cobordism categories.

1.5.6. **Proposition.** *A functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is Karoubi-localising if and only if it is Verdier-localising and invariant under Karoubi equivalences.*

Proof. The "only if" part follows from Remark 1.5.2 and the fact that

$$\begin{tikzcd}
(\mathcal{C}, \Omega) \arrow[r] \arrow[d] & 0 \arrow[d] \\
(\mathcal{C}, \Omega)^\natural \arrow[r] & 0
\end{tikzcd}$$

forms a Poincaré-Karoubi square. The other direction follows from the fact that every Poincaré-Karoubi square is Karoubi equivalent to a Poincaré-Verdier square: Assume that $\mathcal{F}$ is Verdier-localising and sends Karoubi equivalences to equivalences. By definition of Karoubi squares it suffices to consider squares

$$\begin{tikzcd}
(\mathcal{C}, \Omega) \arrow[r] \arrow[d] & (\mathcal{D}, \Phi) \arrow[d] \\
(\mathcal{C}', \Omega') \arrow[r] & (\mathcal{D}', \Phi'),
\end{tikzcd} \tag{16}$$

all of whose corners are idempotent complete and whose vertical legs are Poincaré-Karoubi projections.Let then $\mathcal{A} \subseteq \mathcal{C}'$ and $\mathcal{B} \subseteq \mathcal{D}'$ be the essential images of the left and right vertical arrows, respectively, which are invariant under the respective dualities since these vertical arrows are Poincaré. Furthermore, their inclusions are Karoubi equivalences by Corollary 1.3.12. Since (16) is cartesian the full subcategory $\mathcal{A}$ coincides with the inverse image of $\mathcal{B} \subseteq \mathcal{D}'$ and the square

$$\begin{tikzcd}
(\mathcal{C}, \Omega) \arrow[r] \arrow[d] & (\mathcal{D}, \Phi) \arrow[d] \\
(\mathcal{A}, \Omega'|_{\mathcal{A}}) \arrow[r] & (\mathcal{B}, \Phi'|_{\mathcal{B}})
\end{tikzcd} \tag{17}$$

is again cartesian. Finally, it follows from Corollary 1.3.10 that the vertical maps in (17) are Poincaré-Verdier projections, which gives the claim. □

1.5.7. **Lemma.** *The categories*

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E}), \quad \mathrm{Fun}^{\mathrm{vloc}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E}) \quad \textit{and} \quad \mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E})$$

are semi-additive and the forgetful functor

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E})) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E})$$

and its localising analogues are equivalences.

Since the ∞-category $\mathrm{Cat}^{\mathrm{p}}_\infty$ is semi-additive but obviously not additive, see Proposition [I].6.1.7, the analogous statement for $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$ in place of $\mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E})$ is not true. Noting the unfortunate clash in the use of the word *additive* arising from this, we set:

1.5.8. **Definition.** An additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ is called *group-like* if the canonical lift of $\mathcal{F}$ to $\mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E})$ actually takes values in the full subcategory $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$.

Equivalently, this is the same as saying that for every Poincaré ∞-category $(\mathcal{C}, \Omega)$ it takes the shear map $(\mathcal{C}, \Omega) \times (\mathcal{C}, \Omega) \to (\mathcal{C}, \Omega) \times (\mathcal{C}, \Omega)$ (given at the level of objects by $(x, y) \mapsto (x, x \oplus y)$) to an equivalence in $\mathcal{E}$.

Proof of Lemma 1.5.7. Given the semi-additivity of $\mathrm{Cat}^{\mathrm{p}}_\infty$, the ∞-category of product preserving functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ is semi-additive by [GGN15, Corollary 2.4]. But products of additive, Verdier or Karoubi localising functors are again such, which implies the first statement. The second follows from [GGN15, Corollary 2.5 iii)]. □

1.5.9. **Remark.** If $\mathcal{E}$ is additive then any additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ is group-like, because both forgetful functors $\mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E}) \to \mathcal{E}$ and $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E}) \to \mathcal{E}$ are equivalences.

1.5.10. **Examples.** i) The functors Cr and Pn : $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ are Verdier-localising since, by virtue of being representable, they preserve all limits. They are not group-like.

ii) Lurie's version of L-theory, i.e. the functor $\mathrm{L}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ from [Lur11] is Verdier localising, see §4.4 below for a discussion.

iii) Many examples can be obtained from functors $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{E}$ satisfying the corresponding conditions for (non-Poincaré) stable ∞-categories: For example, the functors $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ and $\mathrm{K}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$, which associate to a Poincaré ∞-category the algebraic K-theory space or spectrum of its underlying stable ∞-category are Verdier-localising and group-like; this essentially follows from Waldhausen's additivity and fibration theorems, as implemented in the setting of stable ∞-categories by Blumberg-Gepner-Tabuada [BGT13], we will review the situation in §2.7. Similarly, the functor $\mathbb{K}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ which associates to a Poincaré ∞-category the nonconnective K-theory spectrum of its underlying stable ∞-category is Karoubi-localising by [BGT13].

iv) The functor $\mathrm{K}\circ(-)^\natural\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ (where $(-)^\natural$ is the idempotent completion functor of Proposition 1.3.3) is an example of an additive, but non-Verdier-localising functor. By contrast, the cofinality theorem implies that $\mathcal{K}\circ(-)^\natural\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ is Karoubi-localising. It is one of the main purposes of this paper series to show that these K-theoretic examples have hermitian analogues.

v) The functors $\mathrm{H}\,\mathrm{K}_i(-)\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ are further examples of additive functors that are not Verdier-localising and so is $\mathrm{H}(\mathrm{K}_0(-^\natural)/\,\mathrm{K}_0(-))$, where $\mathrm{H}\colon \mathcal{A}b \to \mathcal{S}p$ denotes the Eilenberg-Mac Lane functor. The analogous claims for Grothendieck-Witt and L-groups are, however, not correct: For example, it follows from the results of Section 4 that the functor $\mathrm{H}\,\mathrm{GW}_0\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ can only take all metabolic Poincaré-Verdier sequences to fibre sequences if the map $\mathrm{fgt}\colon \mathrm{GW}_0(\mathcal{C}, \Omega) \to \mathrm{K}_0(\mathcal{C})$ is always injective. This fails in many cases, e.g. for $(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Omega^{\mathrm{s}})$, where it is a map $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$.

Finally, we record the following simple consequence of the splitting lemma:

1.5.11. **Proposition.** *Let*

$$(\mathcal{C}, \Omega) \xrightarrow{i} (\mathcal{C}', \Omega') \xrightarrow{p} (\mathcal{C}'', \Omega'') \tag{18}$$

be a (split) Poincaré-Verdier sequence and let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathcal{E}$ *be a group-like (additive or) Verdier-localising functor. Assume that the Verdier projection* $(\mathcal{C}', \Omega') \longrightarrow (\mathcal{C}'', \Omega'')$ *admits a section* $s\colon (\mathcal{C}'', \Omega'') \longrightarrow (\mathcal{C}', \Omega')$ *in* $\mathrm{Cat}^{\mathrm{p}}_\infty$*. Then* i *and* s *together induce an equivalence*

$$\mathcal{F}(\mathcal{C}, \Omega) \oplus \mathcal{F}(\mathcal{C}'', \Omega'') \longrightarrow \mathcal{F}(\mathcal{C}', \Omega'). \tag{19}$$

If, in addition, the Poincaré functor i *admits a retraction* $r\colon (\mathcal{C}', \Omega') \longrightarrow (\mathcal{C}, \Omega)$ *in* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *then* p *and* r *together induce an equivalence*

$$\mathcal{F}(\mathcal{C}', \Omega') \longrightarrow \mathcal{F}(\mathcal{C}, \Omega) \oplus \mathcal{F}(\mathcal{C}'', \Omega''). \tag{20}$$

This equivalence is inverse to (19) *when* $r\circ s$ *is the zero Poincaré functor.*

Note that since $\mathrm{Cat}^{\mathrm{p}}_\infty$ is only semi-additive, but not additive, the middle term in a Poincaré-Verdier sequence admitting a Poincaré splitting as above, need not split as a direct sum before applying $\mathcal{F}$.

The proof of Proposition 1.5.11 relies on the following version of the classical splitting lemma [ML95, Proposition I.4.3] from homological algebra (it should be considered standard, but we were not able to locate a reference).

1.5.12. **Lemma.** *Let* $\mathcal{A}$ *be an additive* ∞*-category which admits fibres and cofibres and let*

$$x \xrightarrow{i} y \xrightarrow{r} x$$

be a retract diagram. Then the following statement hold:

i) The maps $i\colon x \longrightarrow y$ *and* $\mathrm{fib}(r) \longrightarrow y$ *induce an equivalence* $x \oplus \mathrm{fib}(r) \longrightarrow y$.

ii) The maps $r\colon y \longrightarrow x$ *and* $y \longrightarrow \mathrm{cof}(i)$ *induce an equivalence* $y \longrightarrow x \oplus \mathrm{cof}(i)$.

iii) The fibre sequence $\mathrm{fib}(r) \longrightarrow y \longrightarrow x$ *is also a cofibre sequence.*

iv) The cofibre sequence $x \longrightarrow y \longrightarrow \mathrm{cof}(i)$ *is also a fibre sequence.*

v) The composite map $\mathrm{fib}(r) \longrightarrow y \longrightarrow \mathrm{cof}(i)$ *is an equivalence.*

Proof. We first note that ii) and iv) follow from i) and iii), respectively, applied to the additive ∞-category $\mathcal{A}^{\mathrm{op}}$. To prove i), it is actually enough to argue at the level of the homotopy category. To see this, observe that for every z we have a fibre sequence of spaces

$$\mathrm{Map}_{\mathcal{A}}(z, \mathrm{fib}(r)) \longrightarrow \mathrm{Map}_{\mathcal{A}}(z, y) \longrightarrow \mathrm{Map}_{\mathcal{A}}(z, x).$$

Since r admits a section, the map $\pi_1 \operatorname{Map}_{\mathcal{E}}(z,y) \longrightarrow \pi_1 \operatorname{Map}_{\mathcal{E}}(z,x)$ is surjective and hence the long exact sequence in homotopy groups ends with a fibre sequence

$$\pi_0 \operatorname{Map}_{\mathcal{A}}(z, \operatorname{fib}(r)) \longrightarrow \pi_0 \operatorname{Map}_{\mathcal{A}}(z,y) \longrightarrow \pi_0 \operatorname{Map}_{\mathcal{A}}(z,x).$$

of sets. This means that $\operatorname{fib}(r)$ is also the fibre of r in the homotopy category $\operatorname{Ho}(\mathcal{A})$. Now, since products and coproducts descend to $\operatorname{Ho}(\mathcal{A})$, we have that $\operatorname{Ho}(\mathcal{A})$ is additive and the functor $\mathcal{A} \longrightarrow \operatorname{Ho}(\mathcal{A})$ preserves direct sums. Hence it suffices to show that i) holds for $\operatorname{Ho}(\mathcal{A})$, which is the classical splitting lemma (see, e.g., [Bor94, Proposition 1.8.7]); the splitting lemma is usually phrased for abelian categories only, but the proof from loc. cit. works verbatim in the additive case. Alternatively, it can be deduced from the abelian case by embedding $\operatorname{Ho}(\mathcal{A})$ into its abelian envelope.

Let us prove iii). Note that i) provides us in particular with a retraction $y \longrightarrow \operatorname{fib}(r)$ which vanishes when restricted to x. We may then consider the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \operatorname{fib}(r) \arrow[r] \arrow[d] & 0 \arrow[d] \\
x \arrow[r, "i"] \arrow[d] & y \arrow[r, "r"] \arrow[d] & x \arrow[d] \\
0 \arrow[r] & \operatorname{fib}(r) \arrow[r] & 0
\end{tikzcd}$$

in which the middle row and middle column are retract diagrams. By i), the top left square is cocartesian and hence by the pasting lemma the top right square is cocartesian as well. This gives iii).

To obtain v), use the pasting lemma to deduce that the bottom left square is cocartesian, which induces an equivalence $\operatorname{fib}(r) \longrightarrow \operatorname{cof}(i)$. But this map is the same as the one obtained from the composition $\operatorname{fib}(r) \longrightarrow y \longrightarrow \operatorname{cof}(i)$ because the $\operatorname{fib}(r) \longrightarrow y \longrightarrow \operatorname{fib}(r)$ is a retract diagram. ☐

Proof of Proposition 1.5.11. To obtain the first equivalence (19), we apply part i) of the splitting lemma 1.5.12 to the retract diagram

$$\mathcal{F}(\mathcal{C}'', \Qoppa'') \xrightarrow{s_*} \mathcal{F}(\mathcal{C}', \Qoppa') \xrightarrow{p_*} \mathcal{F}(\mathcal{C}'', \Qoppa'')$$

in the additive ∞-category $\operatorname{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$ and identify the fibre of p_* with $\mathcal{F}(\mathcal{C}, \Qoppa)$. By Part iii) of the same lemma it follows that the fibre sequence

$$\mathcal{F}(\mathcal{C}, \Qoppa) \xrightarrow{i_*} \mathcal{F}(\mathcal{C}', \Qoppa') \xrightarrow{p_*} \mathcal{F}(\mathcal{C}'', \Qoppa'')$$

is also a cofibre sequence in $\mathcal{A}$. The second equivalence (20) then follows from Part ii) of the splitting lemma applied the retract diagram

$$\mathcal{F}(\mathcal{C}, \Qoppa) \xrightarrow{i_*} \mathcal{F}(\mathcal{C}', \Qoppa') \xrightarrow{r_*} \mathcal{F}(\mathcal{C}, \Qoppa),$$

after identifying $\mathcal{F}(\mathcal{C}, \Qoppa)$ with the cofibre of i_* with $\mathcal{F}(\mathcal{C}'', \Qoppa'')$ using the above. To see the final statement, note that two equivalences are inverse to each other if and only if they are one-sided inverses. Composing in one direction, we get the functor

$$\mathcal{F}(\mathcal{C}, \Qoppa) \oplus \mathcal{F}(\mathcal{C}'', \Qoppa'') \to \mathcal{F}(\mathcal{C}, \Qoppa) \oplus \mathcal{F}(\mathcal{C}'', \Qoppa'') \tag{21}$$

whose "matrix components" are $\begin{pmatrix} \mathrm{id} & r_* s_* \\ 0 & \mathrm{id} \end{pmatrix}$, and so (21) is homotopic to the identity as soon as $r \circ s$ is the zero Poincaré functor. ☐

2. THE HERMITIAN Q-CONSTRUCTION AND ALGEBRAIC COBORDISM CATEGORIES

In this section, we introduce the main objects of study, namely the cobordism ∞-category constructed from a Poincaré ∞-category. To motivate our perspective, let $(\mathcal{C}, \Qoppa)$ be a Poincaré ∞-category and (x, q), (x', q') be two Poincaré objects in $\mathcal{C}$. A *cobordism* from (x, q) to (x', q') is a span of the form

$$x \xleftarrow{\alpha} w \xrightarrow{\beta} x'$$

together with a path $\eta\colon\ \alpha^*q \to \beta^*q'$ in the space $\Omega^\infty Ϙ(w)$ of hermitian structures on w, such that w satisfies the Poincaré-Lefschetz condition with respect to x and x', i.e. that the canonical map

$$\mathrm{fib}(w \to x) \simeq \mathrm{fib}(x' \to x \cup_w x') \to \mathrm{fib}(x' \to \mathrm{D}_Ϙ w) \simeq \Omega\mathrm{D}_Ϙ(\mathrm{fib}(w \to x')), \tag{22}$$

is an equivalence; here the middle map is induced by the map

$$w \to \mathrm{D}_Ϙ x \times_{\mathrm{D}_Ϙ w} \mathrm{D}_Ϙ x'$$

provided by η, and the condition above can also be phrased as asking this map to be an equivalence.

For example, if W is an oriented cobordism between two closed oriented d-manifolds M and N, we obtain a span of the form

$$C^*(M) \leftarrow C^*(W) \to C^*(N)$$

and the fundamental class $[W]$ determines a path relating the pullbacks of the two symmetric Poincaré structures q_M and q_N on $C^*(M)$ and $C^*(N)$, respectively. Lefschetz duality for manifolds with boundary precisely implies that this path exhibits the span as a cobordism between the Poincaré objects $(C^*(M), q_M)$ and $(C^*(N), q_N)$ of $(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), Ϙ^{s\,[-d]}_{\mathbb{Z}})$ in the sense above.

Now, cobordisms can be composed in a natural way, by first forming the corresponding composition at the level of spans and then at the level of the paths between hermitian structures. This will allow us to define an ∞-category $\mathrm{Cob}(\mathcal{C}, Ϙ)$ whose objects are the Poincaré objects of $(\mathcal{C}, Ϙ^{[1]})$ and whose morphisms are given by cobordisms; the choice in shifts here adheres to the usual convention from manifold theory that the category Cob_d have $(d-1)$-dimensional, closed, oriented, smooth manifolds as objects and d-dimensional cobordisms as morphisms.

To make this idea precise, we interpret a cobordism in $(\mathcal{C}, Ϙ)$ as a Poincaré object in the diagram ∞-category $(\mathrm{Fun}(P, \mathcal{C}), Ϙ^P)$, where P is the category $\bullet \leftarrow \bullet \to \bullet$, and $Ϙ^P$ is the Poincaré structure on the diagram ∞-category given by the limit of the values of $Ϙ$ on the diagram. This construction turns out to be the degree 1 part of a simplicial Poincaré ∞-category $\mathrm{Q}(\mathcal{C}, Ϙ)$, whose Poincaré objects in degree n may be interpreted as the datum of n composable tuples of cobordisms. Varying $(\mathcal{C}, Ϙ)$, this construction gives rise to a functor

$$\mathrm{Q}\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{sCat}^{\mathrm{p}}_\infty,$$

our implementation of the hermitian Q-construction; see §2.2. By considering the spaces of Poincaré objects of these diagram of ∞-categories, we will obtain a complete Segal space and then extract $\mathrm{Cob}(\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty$ as the associated ∞-category in §2.3.

Then, we develop the two main tools that will allow us to analyse this cobordism ∞-category and its homotopy type. First, we show how to describe the cobordism ∞-category using Ranicki's algebraic surgery techniques from [Ran80], adapted to the setting of Poincaré ∞-categories by Lurie in [Lur11]. Beside its uses in the present paper, this serves as a fundamental tool in [HS21] to compare our definition of Grothendieck-Witt theory with the classical L-, Witt and Grothendieck-Witt groups, and it is also used extensively in Paper [III]. The second topic, in §2.5 and §2.6, is the additivity theorem, which says that the functor

$$|\mathrm{Cob} - | = |\mathrm{Pn}\,\mathrm{Q}(-)|\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S};$$

is additive. This will be the basis for most of the structural results we prove about Grothendieck-Witt theory.

As far as the additivity theorem is concerned, the only property of the functor Pn that enters the proof, is that it is itself additive. In fact, we will show that the functor

$$|\mathcal{F}\,\mathrm{Q}(-)|\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$$

is additive whenever $\mathcal{F}\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ is additive. This added layer of generality will be used to establish the additivity of the spectral version of the Grothendieck-Witt functor, defined via iteration of the hermitian Q-construction, and also enters into the proof of its universal property.

Finally, in §2.7 we explain how our methods give rise to a new proof of the more classical additivity theorem for the algebraic K-theory of stable ∞-categories.

2.1. **Recollections on the Rezk nerve.** Before we get started, let us collect the necessary results regarding the relationship between ∞-categories and Rezk's complete Segal spaces, on which we rely in the analysis of the cobordism ∞-categories $\mathrm{Cob}(\mathcal{C}, \Omega)$. They were originally established in [Rez01, JT07], and suitably reformulated in [Lur09b, §1].

There is an adjoint pair of functors

$$\mathrm{asscat} \colon \mathrm{s}\mathcal{S} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Cat}_\infty \colon \mathrm{N}$$

with the Rezk nerve as right adjoint, given by

$$\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C}),$$

and left adjoint given by left Kan extending the cosimplicial ∞-category

$$\Delta^- \colon \Delta \longrightarrow \mathrm{Cat}_\infty$$

along the Yoneda embedding $\Delta \to \mathrm{s}\mathcal{S}$. By [Lur09b, Corollary 4.3.16], the nerve is fully faithful with essential image the complete Segal spaces $\mathrm{cs}\mathcal{S} \subseteq \mathrm{s}\mathcal{S}$, in particular making Cat_∞ a left Bousfield localisation of $\mathrm{s}\mathcal{S}$ (at what we shall refer to as the categorical equivalences). Consequently, there is also a left adjoint $\mathrm{comp} \colon \mathrm{s}\mathcal{S} \to \mathrm{cs}\mathcal{S}$ to the inclusion, often referred to as completion, and composing adjoints we find $\mathrm{asscat} \circ \mathrm{comp} = \mathrm{asscat}$. Recall that a simplicial space X is called a Segal space if the Segal maps

$$\mathrm{seg}_i \colon [1] \longrightarrow [n], \quad 0, 1 \longmapsto i-1, i$$

for $1 \leq i \leq n$ induce equivalences

$$(\mathrm{seg}_1, \dots, \mathrm{seg}_n) \colon X_n \longrightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for all $n \geq 1$. Completeness can be characterised in many ways; for us, the most convenient criterion is that a Segal space is complete (i.e. lies in the image of N) if and only if

$$\begin{tikzcd}
X_0 \arrow[r, "s"] \arrow[d, "\Delta"] & X_3 \arrow[d, "{(d_{02}, d_{13})}"] \\
X_0^2 \arrow[r, "{(s,s)}"] & X_1^2
\end{tikzcd}$$

is cartesian, where $d_{02} \colon [1] \to [3]$ is the unique injective map that misses 0 and 2 and similarly for d_{13}, see [Lur09b, Proposition 1.1.13] or [Rez10, Proposition 10.1].

The restriction of the nerve functor to $\mathcal{S} \subset \mathrm{Cat}_\infty$ is given by the inclusion of constant diagrams $\mathcal{S} \to \mathrm{s}\mathcal{S}$, and passing to adjoints again shows that $|\mathrm{asscat}\, X| \simeq |X|$ for every simplicial space X. In particular, $\pi_0|\mathrm{asscat}\, X|$ is always the coequaliser of the two boundary maps $\pi_0 X_1 \to \pi_0 X_0$.

Furthermore, Rezk showed in [Rez01, §14], see also [Lur09b, Proposition 1.2.27], that for any Segal space X the natural map $X \to \mathrm{comp}\, X$ induces equivalences from the fibres of $(d_1, d_0) \colon X_1 \to X_0 \times X_0$ to the same expression for $\mathrm{comp}\, X$. For $X = \mathrm{N}\, \mathcal{C}$ this fibre, say over (x, y), is given by $\mathrm{Hom}_{\mathcal{C}}(x, y)$. We therefore find that for any Segal space X and any $x, y \in X_0$, we have a canonical equivalence

$$\mathrm{Hom}_{\mathrm{asscat}\, X}(x, y) \simeq \mathrm{fib}_{(x,y)}(X_1 \to X_0 \times X_0).$$

Similarly, for every simplicial space X, the inclusion of 0-simplices induces a natural map

$$X_0 \to \iota(\mathrm{asscat}\, X)$$

on associated ∞-categories, which is a surjection on π_0 by [Lur09b, Remark 1.2.17]; here, ι denotes the groupoid core of an arbitrary ∞-category (we reserve the use of the symbol Cr for those occasions where we regard the groupoid core as an additive functor on stable or Poincaré ∞-categories). For X a Segal space $\iota(\mathrm{asscat}\, X) \simeq |X^\times|$, where $X_n^\times$ is the full subspace of X_n consisting of all those simplices whose edges become equivalences in $\mathrm{asscat}\, X$ by [Lur09b, Proposition 1.2.27]. In particular, the map $X_0 \to \iota(\mathrm{asscat}\, X)$ is an equivalence whenever X is complete. We will also have to use that the completion functor commutes with finite products, when restricted to Segal spaces. This also follows immediately from [Lur09b, Proposition 1.2.27] since Segal equivalences are evidently closed under finite products.

Finally, on the more abstract side, we note that by [Lur09a, Proposition 5.5.4.15], the categorical equivalences are the saturation of the spine inclusions, which encode the Segal condition, and the map

$$\Delta^3/\Delta^{0,2}, \Delta^{1,3} = \Delta^0 \cup_{\Delta}^{\{0,2\}} \Delta^3 \cup_{\Delta}^{\{1,3\}} \Delta^0 \longrightarrow \Delta^0.$$

In particular, any colimit preserving functor $s\mathcal{S} \to \mathcal{E}$ (with $\mathcal{E}$ cocomplete) factors (uniquely) through $\mathrm{asscat}\colon s\mathcal{S} \to \mathrm{Cat}_\infty$ if and only if it inverts these maps, i.e. if its restriction $\Delta^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ along the Yoneda embedding is a complete Segal object in $\mathcal{E}$.

2.2. **The hermitian Q-construction.** Let K be an ∞-category and $(\mathcal{C}, \Qoppa)$ a hermitian ∞-category.

2.2.1. **Definition.** Let $\mathrm{Q}_K(\mathcal{C}, \Qoppa)$ denote the following hermitian ∞-category: The underlying stable ∞-category is given as the full subcategory $\mathrm{Q}_K(\mathcal{C})$ of $\mathrm{Fun}(\mathrm{TwAr}(K), \mathcal{C})$ spanned by those functors F such that for every functor $[3] \to K$, say $i \to j \to k \to l \in K$, the square

$$\begin{tikzcd}
F(i \to l) \arrow[r] \arrow[d] & F(j \to l) \arrow[d] \\
F(i \to k) \arrow[r] & F(j \to k)
\end{tikzcd}$$

is bicartesian. The hermitian structure is given by restricting the quadratic functor

$$\Qoppa^{\mathrm{TwAr}(K)}(F) = \lim_{\mathrm{TwAr}(K)^{\mathrm{op}}} \Qoppa \circ F^{\mathrm{op}}$$

from Proposition [I].6.3.2.

When $K = \Delta^n$, we will shorten notation, and denote $\mathrm{Q}_K(\mathcal{C}, \Qoppa)$ by $\mathrm{Q}_n(\mathcal{C}, \Qoppa)$ and $\Qoppa^{\mathrm{TwAr}(\Delta^n)}$ by $\Qoppa_n$. Also, by definition, the hermitian ∞-category $(\mathrm{Fun}(\mathrm{TwAr}(K), \mathcal{C}), \Qoppa^{\mathrm{TwAr}(K)})$ is the cotensor $(\mathcal{C}, \Qoppa)^{\mathrm{TwAr}(K)}$, in the sense of §[I].6.3. It is usually not Poincaré, while $\mathrm{Q}_K(\mathcal{C}, \Qoppa)$ is, as we will see below.

2.2.2. **Remark.** By the pasting lemma for cartesian squares, we find that in order to establish the condition in Definition 2.2.1 for all $i \to j \to k \to l$, it suffices to check the case in which $j \to k$ is the identity.

2.2.3. **Examples.**

i) Since $\mathrm{TwAr}(\Delta^0) = \Delta^0$, we have $\mathrm{Q}_0(\mathcal{C}, \Qoppa) = (\mathcal{C}, \Qoppa)$. Similarly, we have $\mathrm{TwAr}(\Delta^1) = (0 \le 0) \leftarrow (0 \le 1) \to (1 \le 1)$, so $\mathrm{Q}_1(\mathcal{C})$ is simply the ∞-category of spans in $\mathcal{C}$, with no condition imposed. Using Proposition [I].6.3.2, the duality on $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ takes

$$\big(X \leftarrow Y \to Z\big) \quad \mapsto \quad \big(\mathrm{D}_\Qoppa X \longleftarrow \mathrm{D}_\Qoppa X \times_{\mathrm{D}_\Qoppa Y} \mathrm{D}_\Qoppa Z \longrightarrow \mathrm{D}_\Qoppa Z\big).$$

Following our explanation above, we interpret $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ as the ∞-*category of cobordisms* in $(\mathcal{C}, \Qoppa)$.

ii) $\mathrm{Q}_2(\mathcal{C})$ consists of those diagrams

$$\begin{tikzcd}
 & & F(0 \le 2) \arrow[dl] \arrow[dr] & & \\
 & F(0 \le 1) \arrow[dl] \arrow[dr] & & F(1 \le 2) \arrow[dl] \arrow[dr] & \\
F(0 \le 0) & & F(1 \le 1) & & F(2 \le 2)
\end{tikzcd}$$

in which the top square is bicartesian. It is therefore reasonable to think of $\mathrm{Q}_2(\mathcal{C}, \Qoppa)$ as the ∞-category of two composable cobordisms equipped with a chosen composite.

iii) By i), the functor

$$d_1\colon\ \mathrm{Q}_1(\mathcal{C}, \Qoppa) \to \mathrm{Q}_0(\mathcal{C}, \Qoppa) = (\mathcal{C}, \Qoppa), \quad (X \leftarrow Y \to Z) \mapsto X$$

is duality-preserving so that its kernel is closed under the duality of $\mathrm{Q}_1(\mathcal{C}, \Qoppa)$, and therefore a Poincaré ∞-category with the restricted Poincaré structure. In fact, there is a canonical equivalence of Poincaré ∞-categories

$$\ker(d_1) \simeq \mathrm{Met}(\mathcal{C}, \Qoppa)$$

that sends $0 \leftarrow w \to c$ to $w \to c$.

iv) We note that for the category $\mathcal{I}_n \subseteq \mathrm{TwAr}(\Delta^n)$ spanned by the pairs (i, j) with $j \le i + 1$ (the zig-zag along the bottom) the restriction functor

$$\mathrm{Q}_n(\mathcal{C}, \Qoppa) \to (\mathrm{Fun}(\mathcal{I}_n, \mathcal{C}), \Qoppa^{\mathcal{I}_n}) = (\mathcal{C}, \Qoppa)^{\mathcal{I}_n}$$

is an equivalence of hermitian ∞-categories: On underlying ∞-categories, it follows from [Lur09a, Proposition 4.3.2.15] that the right Kan extension functor $\mathrm{Fun}(\mathcal{I}_n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{C})$ is both fully faithful and a left inverse to restriction. For $X \in \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{C})$, it is then readily checked from the pointwise formulae [Lur09a, Lemma 4.3.2.13] that being in $\mathrm{Q}_n(\mathcal{C})$ is equivalent to being right Kan extended from $\mathcal{I}_n$. For the quadratic functor, it follows since the inclusion $\mathcal{I}_n^{\mathrm{op}} \subseteq \mathrm{TwAr}(\Delta^n)^{\mathrm{op}}$ is final. By Remark [I].6.5.18, the arising hermitian structure on the right Kan extension functor $\mathrm{Fun}(\mathcal{I}_n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{C})$ is an instance of the exceptional functoriality of Construction [I].6.5.14.

This description justifies us in thinking of $\mathrm{Q}_n(\mathcal{C}, \Omega)$ as the ∞-category of n composable cobordisms in $(\mathcal{C}, \Omega)$ also for larger n.

v) There is another description of the ∞-category underlying $\mathrm{Q}_n(\mathcal{C}, \Omega)$: Letting $\mathcal{J}_n \subseteq \mathrm{TwAr}(\Delta^n)$ denote the subset of those (i, j) with either $i = 0$ or $j = n$ (the arch along the top), the restriction functor

$$\mathrm{Q}_n(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}_n, \mathcal{C})$$

is also an equivalence: A functor $F\colon \mathrm{TwAr}(\Delta^n) \to \mathcal{C}$ is in $\mathrm{Q}_n(\mathcal{C})$ if and only if it is left Kan extended from $\mathcal{J}_n$. However, this equivalence does not translate the quadratic functor Ω_n into $\Omega^{\mathcal{J}_n}$, once $n \geq 2$. For example, for the element

$$X \xleftarrow{\mathrm{id}_X} X \leftarrow 0 \to Y \xrightarrow{\mathrm{id}_Y} Y$$

in $\mathrm{Fun}(\mathcal{J}_2, \mathcal{C})$ we find $\Omega^{\mathcal{J}_n}$ given by $\Omega(X) \oplus \Omega(Y)$, where as Ω_2 yields $\Omega(X \oplus Y)$, and these two terms differ by $\mathrm{B}_\Omega(X, Y)$. In fact, $(\mathrm{Fun}(\mathcal{J}_n, \mathcal{C}), \Omega^{\mathcal{J}_n})$ is usually not Poincaré, whereas we will next establish this for $\mathrm{Q}_n(\mathcal{C}, \Omega)$.

Denoting the ∞-category of ∞-categories by Cat_∞, we thus obtain a functor

$$\mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{h}}, \quad (K, \mathcal{C}, \Omega) \mapsto \mathrm{Q}_K(\mathcal{C}, \Omega),$$

from Proposition [I].6.3.11, since clearly induced maps preserve the cartesianness condition of Definition 2.2.1. Restricting along the inclusion $\Delta \subseteq \mathrm{Cat}_\infty$ and adjoining the construction above, we thus obtain a simplicial object $\mathrm{Q}(\mathcal{C}, \Omega) \in \mathrm{sCat}_\infty^{\mathrm{h}}$.

2.2.4. **Definition.** We call the functor $\mathrm{Q}\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{sCat}_\infty^{\mathrm{h}}$ just described the *hermitian* Q*-construction*.

We immediately note that the underlying ∞-category of $\mathrm{Q}_n(\mathcal{C}, \Omega)$ only depends on $\mathcal{C}$, and agrees with Barwick-Rognes' implementation $\mathrm{Q}_n(\mathcal{C})$ of the Q-construction, see [BR13, §3] upon restricting their set-up to stable ∞-categories.

The following is at the heart of the present section:

2.2.5. **Lemma.** *For every hermitian ∞-category* $(\mathcal{C}, \Omega)$ *the simplicial hermitian ∞-category* $\mathrm{Q}(\mathcal{C}, \Omega)$ *is a Segal object of* $\mathrm{Cat}_\infty^{\mathrm{h}}$. *Furthermore, it is complete in the sense that the diagram*

$$\begin{tikzcd}
\mathrm{Q}_0(\mathcal{C}, \Omega) \arrow[r, "s"] \arrow[d, "\Delta"] & \mathrm{Q}_3(\mathcal{C}, \Omega) \arrow[d, "(d_{02}, d_{13})"] \\
\mathrm{Q}_0(\mathcal{C}, \Omega)^2 \arrow[r, "(s,s)"] & \mathrm{Q}_1(\mathcal{C}, \Omega)^2
\end{tikzcd}$$

is cartesian in $\mathrm{Cat}_\infty^{\mathrm{h}}$, *with horizontal maps given by total degeneracies.*

At the level of underlying ∞-categories, this holds more generally for any of Barwick's adequate triples in place of the stable ∞-category $\mathcal{C}$, see [HHLN22, Lemma 2.17].

Proof. We will show that for every $0 \leq i \leq n$ the square

$$\begin{tikzcd}
\mathrm{Q}_n(\mathcal{C}, \Omega) \arrow[r] \arrow[d] & \mathrm{Q}_{[0,i]}(\mathcal{C}, \Omega) \arrow[d] \\
\mathrm{Q}_{[i,n]}(\mathcal{C}, \Omega) \arrow[r] & \mathrm{Q}_{[i,i]}(\mathcal{C}, \Omega)
\end{tikzcd}$$

is a pullback square of hermitian ∞-categories; the Segal condition then follows by iteration. The statement will follow readily from Example 2.2.3 iv). To this end, note that the inclusions $\mathrm{TwAr}(\Delta^i) \to \mathrm{TwAr}(\Delta^n)$ and

$\mathrm{TwAr}(\Delta^{\{i,\dots,n\}}) \to \mathrm{TwAr}(\Delta^n)$ take the subcategories $\mathcal{I}_i$ and $\mathcal{I}_{[i,\dots,n]}$ to $\mathcal{I}_n$, and in fact the induced diagram

$$\begin{tikzcd}
\mathcal{I}_{[i,i]} \arrow[r] \arrow[d] & \mathcal{I}_{[i,n]} \arrow[d] \\
\mathcal{I}_i \arrow[r] & \mathcal{I}_n
\end{tikzcd}$$

is readily checked to be cocartesian in Cat_∞, thus cartesian in $\mathrm{Cat}_\infty^{\mathrm{op}}$. But the functor

$$\mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{p}}, \quad I \mapsto (\mathrm{Fun}(I, \mathcal{C}), \Qoppa^I)$$

being a right adjoint preserves limits, whence we obtain the first claim.

To see that $\mathrm{Q}(\mathcal{C}, \Qoppa)$ is complete, recall that limits in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ may be computed in Cat_∞ (as limits in Cat_∞ of diagrams of stable ∞-categories and exact functors, are easily checked to be stable again), so the map $P \to \mathrm{Q}_3(\mathcal{C})$ from the pullback P of the diagram

$$\mathrm{Q}_0(\mathcal{C})^2 \to \mathrm{Q}_1(\mathcal{C})^2 \leftarrow \mathrm{Q}_3(\mathcal{C})$$

is fully faithful, since the degeneracy $\mathrm{Q}_0(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C})$ is, and fully faithful functors are stable under pullback. Its essential image is given by the diagrams consisting entirely of equivalences, as one can check directly using the defining property of the Q-construction, and these are precisely the constant diagrams, i.e. the totally degenerate ones, since $|\,\mathrm{TwAr}(\Delta^3)| \simeq *$.

The claim for the hermitian structure is immediate from Remark [I].6.1.3, since the diagram

$$\Qoppa_0(X)^2 \to \Qoppa_1(sX)^2 \leftarrow \Qoppa_3(sX),$$

whose pullback defines the hermitian structure on P, evaluates to

$$\Qoppa(X)^2 \xrightarrow{\mathrm{id}} \Qoppa(X)^2 \xleftarrow{\Delta} \Qoppa(X),$$

so has pullback $\Qoppa(X)$. □

2.2.6. **Lemma.** *For fixed* $(\mathcal{C}, \Qoppa) \in \mathrm{Cat}_\infty^{\mathrm{h}}$ *the functor* $\mathrm{Q}_{\_}(\mathcal{C}, \Qoppa) \colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ *preserves limits.*

Of course the functor $\mathrm{Q}_K(-) \colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ also preserves limits, essentially by construction.

Proof. On underlying ∞-categories, this is [HHLN22, Proposition 2.20]. We repeat the argument with hermitian structures tagging along. The preservation of limits can be recast as $\mathrm{Q}(\mathcal{C}, \Qoppa) \colon \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ being a complete Segal object (which we showed above), and the full functor $\mathrm{Q}_{\_}(\mathcal{C}, \Qoppa) \colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ being right Kan extended from its restriction to Δ^{op}; see [HHLN22, Lemma 2.21]. By the pointwise formula this means that the natural map $\mathrm{Q}_J(\mathcal{C}, \Qoppa) \to \lim_{n \in (\Delta/J)^{\mathrm{op}}} \mathrm{Q}_n(\mathcal{C}, \Qoppa)$ has to be an equivalence for every $J \in \mathrm{Cat}_\infty$.

The diagram $\Delta/J \to \Delta \to \mathrm{Cat}_\infty$ is a typical example of a functor $I \colon K \to \mathrm{Cat}_\infty$ whose colimit is preserved by the nerve functor. Such colimits are also preserved by the functor $\mathrm{TwAr} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$, by a direct calculation at the level of Rezk nerves, and so are the subcategories making up the Q-construction. Therefore,

$$\lim_{k \in K} \mathrm{Q}_{I_k}(\mathcal{C}, \Qoppa) \quad \text{and} \quad \mathrm{Q}_{\mathrm{colim}_{k \in K} I_k}(\mathcal{C}, \Qoppa)$$

are the same hermitian subcategory of

$$\lim_{k \in K} (\mathcal{C}, \Qoppa)^{\mathrm{TwAr}(I_k)} \simeq (\mathcal{C}, \Qoppa)^{\mathrm{colim}_{k \in K} \mathrm{TwAr}(I_k)} \simeq (\mathcal{C}, \Qoppa)^{\mathrm{TwAr}(\mathrm{colim}_{k \in K} I_k)},$$

and thus $\mathrm{Q}_{\_}(\mathcal{C}, \Qoppa)$ commutes with limits over diagrams that are compatible with the Rezk nerve, which is more than we need. □

We next show the following pair of statements, whose proofs share some notation we will not need again:

2.2.7. **Lemma.** *The functor* $\mathrm{Q} \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ *restricts to a functor* $\mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{p}}$. *In particular,* $\mathrm{Q}(\mathcal{C}, \Qoppa)$ *is a complete Segal object of* $\mathrm{Cat}_\infty^{\mathrm{p}}$, *whenever* $(\mathcal{C}, \Qoppa)$ *is Poincaré.*

2.2.8. **Lemma.** *For* $(\mathcal{C}, \Qoppa)$ *a Poincaré ∞-category, all face maps in* $\mathrm{Q}(\mathcal{C}, \Qoppa)$*, and more generally all maps induced by injections in* Δ*, are split Poincaré-Verdier projections.*

Proof of Lemmata 2.2.7& 2.2.8. There are two good approaches to the statements about $\mathrm{Q}(\mathcal{C}, \Omega)$. Either, one directly attacks them using the machinery developed in §[I].6.6, or one reduces the statement to explicit checks for small values of n using the Segal condition. At the cost of being less elementary, we will here use the former route as it leads to shorter proofs.

That the ∞-categories $\mathrm{Q}_n(\mathcal{C}, \Omega)$ are Poincaré follows immediately from Proposition [I].6.6.1 and Examples 2.2.3, since $\mathcal{I}_n$ is the poset of faces for the triangulation of the interval using $n+1$ vertices. It also follows from [I].6.6.1 that Poincaré functors $(\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ induce Poincaré functors $\mathrm{Q}_n(\mathcal{C}, \Omega) \to \mathrm{Q}_n(\mathcal{C}', \Omega')$.

To see that the induced hermitian functors $\alpha^* : \mathrm{Q}_n(\mathcal{C}, \Omega) \to \mathrm{Q}_m(\mathcal{C}, \Omega)$ for $\alpha : \Delta^m \to \Delta^n$ preserve the dualities, we distinguish two cases, namely the inner face maps on the one hand, and the outer face maps and degeneracies on the other. Since every morphism in Δ can be written as a composition of such, this will suffice for the claim.

The latter maps all take the subset $\mathcal{I}_m \subseteq \mathrm{TwAr}(\Delta^m)$ into $\mathcal{I}_n$, and the restriction is induced by a map of the simplicial complexes giving rise to $\mathcal{I}_m$ and $\mathcal{I}_n$. Thus, Proposition [I].6.6.2 gives the claim. The interior faces do not preserves the subsets $\mathcal{I}_m$, however. Instead, we claim that they are instances of the exceptional functoriality of Construction [I].6.5.14 associated to a refinement among triangulations. Namely, one readily checks that $d_i : \mathrm{TwAr}(\Delta^n) \to \mathrm{TwAr}(\Delta^{n+1})$ admits a right adjoint $r_i : \mathrm{TwAr}(\Delta^{n+1}) \to \mathrm{TwAr}(\Delta^n)$ explicitly given by

$$(k \leq l) \longmapsto \begin{cases} (k \leq l) & l < i \text{ or } k < l = i \\ (k-1 \leq l) & k = l = i \\ (k \leq l-1) & k < i < l \\ (k-1 \leq l-1) & i \leq k < l \text{ or } i < k = l. \end{cases}$$

As a right adjoint, r_i is cofinal, so by Example [I].6.5.15, the pullback functor

$$(d_i)^* : (\mathcal{C}, \Omega)^{\mathrm{TwAr}(\Delta^{n+1})} \longrightarrow (\mathcal{C}, \Omega)^{\mathrm{TwAr}(\Delta^n)}$$

agrees with the exceptional functoriality along r_i. From the explicit formula, it is clear that r_i takes $\mathcal{I}_{n+1}$ into $\mathcal{I}_n$, so we find a diagram

$$\begin{tikzcd}
(\mathcal{C}, \Omega)^{\mathcal{I}_{n+1}} \arrow[r, "(r_i)_*"] \arrow[d] & (\mathcal{C}, \Omega)^{\mathcal{I}_n} \arrow[d] \\
(\mathcal{C}, \Omega)^{\mathrm{TwAr}(\Delta^{n+1})} \arrow[r, "(r_i)_*"] & (\mathcal{C}, \Omega)^{\mathrm{TwAr}(\Delta^n)}
\end{tikzcd}$$

where vertical maps are the exceptional functorialities associated to the inclusions $\mathcal{I}_n \subseteq \mathrm{TwAr}(\Delta^n)$ which are also cofinal (the diagram commutes since exceptional functorialities compose by Remark [I].6.5.17). But the vertical maps are equivalences onto $\mathrm{Q}_n(\mathcal{C}, \Omega)$ by Example 2.2.3 iv). The claim now follows from Proposition [I].6.6.2, since the restriction of r_i to $\mathcal{I}_{n+1} \to \mathcal{I}_n$ comes from the refinement of triangulation of the interval that adds a new i-th vertex.

This shows that Q restricts to a functor $\Delta^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$, and in particular, it follows from Lemma 2.2.5 above, that $\mathrm{Q}(\mathcal{C}, \Omega)$ is a complete Segal object of $\mathrm{Cat}^{\mathrm{p}}_\infty$ for every $(\mathcal{C}, \Omega) \in \mathrm{Cat}^{\mathrm{p}}_\infty$, because limits in $\mathrm{Cat}^{\mathrm{p}}_\infty$ are computed in $\mathrm{Cat}^{\mathrm{h}}_\infty$. Since generally $K = \mathrm{colim}_{n \in \Delta/K} \Delta^n$ in Cat_∞, we find

$$\mathrm{Q}_K(\mathcal{C}, \Omega) = \lim_{n \in \Delta/K} \mathrm{Q}_n(\mathcal{C}, \Omega)$$

in $\mathrm{Cat}^{\mathrm{h}}_\infty$ from Lemma 2.2.6. But the right hand side lies in $\mathrm{Cat}^{\mathrm{p}}_\infty$ again because the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ preserves limits. This finishes the proof of Lemma 2.2.7.

We finally establish Lemma 2.2.8. We only need to consider face maps, since split Poincaré-Verdier projections are stable under composition by the characterisation in Corollary 1.1.6. For the inner faces, this is immediate from Proposition 1.4.14, since $r_i : \mathcal{I}_{n+1} \to \mathcal{I}_n$ is evidently a localisation at the edges $(i-1 \leq i) \to (i \leq i)$ and $(i \leq i+1) \to (i \leq i)$. For the outer faces, it is an instance of Proposition 1.4.11. □

2.2.9. **Remark.** If $(\mathcal{C}, \Omega)$ is a commutative monoid in $\mathrm{Cat}^{\mathrm{p}}_\infty$ with respect to the symmetric monoidal structure constructed in §[I].5.2, then each $\mathrm{Q}_n(\mathcal{C}, \Omega)$ canonically promotes to a commutative monoid as well; however these structures are *not* compatible with the simplicial structure.

2.3. **The cobordism ∞-category of a Poincaré ∞-category.** We now proceed to extract the cobordism ∞-category from the hermitian Q-construction. As mentioned in the introduction, it is useful to do this in the generality of an arbitrary additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$, see Definition 1.5.4, but the reader is encouraged to envision $\mathcal{F} = \mathrm{Pn}$ throughout.

2.3.1. **Proposition.** *Let* $(\mathcal{C}, \Qoppa)$ *be a Poincaré ∞-category and* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *an additive functor. Then* $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ *is a Segal space and if, furthermore,* $\mathcal{F}$ *preserves arbitrary pullbacks, it is complete.*

When $\mathcal{F}$ is the functor $\mathrm{Cr}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$, completeness was established in [BR13, 3.4 Proposition] by different means. For arbitrary additive $\mathcal{F}$, the Segal space $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ is in general not complete. For example, if $\mathcal{F}$ is group-like, then $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ is complete if and only if $\mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \simeq 0$, see Remark 3.2.18.

Proof. For the first part we need to show that

$$\begin{tikzcd}
\mathcal{F}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathcal{F}\,\mathrm{Q}_{[0,i]}(\mathcal{C}, \Qoppa) \arrow[d] \\
\mathcal{F}\,\mathrm{Q}_{[i,n]}(\mathcal{C}, \Qoppa) \arrow[r] & \mathcal{F}\,\mathrm{Q}_{[i,i]}(\mathcal{C}, \Qoppa)
\end{tikzcd}$$

is cartesian for every $0 \leq i \leq n$. But before applying $\mathcal{F}$ the square is a Poincaré-Verdier square by Lemmas 2.2.8 and 2.2.5, and by assumption $\mathcal{F}$ preserves the cartesianness of such squares.

The assertion on completeness is immediate from the final part of Lemma 2.2.5. ☐

2.3.2. **Definition.** Let $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)$ denote the ∞-category associated to the Segal space $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})$. We shall write $\mathrm{Cob}(\mathcal{C}, \Qoppa)$ for $\mathrm{Cob}^{\mathrm{Pn}}(\mathcal{C}, \Qoppa)$ and call it the *cobordism ∞-category* of $(\mathcal{C}, \Qoppa)$. Furthermore, we set $\mathrm{Cob}^{\partial}(\mathcal{C}, \Qoppa) = \mathrm{Cob}(\mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}))$, the *cobordism ∞-category with boundaries*.

We shall refer to $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)$ as the $\mathcal{F}$-based cobordism ∞-category and hope the two possible superscripts ($\mathcal{F}$ and ∂) will not lead to confusion. By the functoriality of the Q-construction and the previous discussion, the construction of these ∞-categories assemble into a functor

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}) \times \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}_{\infty}.$$

An entirely analogous definition can be made for additive functors $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{S}$ (i.e. reduced and sending split Verdier squares to cartesian squares), resulting in an ∞-category $\mathrm{asscat}(\mathcal{F}\,\mathrm{Q}(\mathcal{C})) = \mathrm{Span}^{\mathcal{F}}(\mathcal{C})$, with $\mathcal{F} = \mathrm{Cr}$ giving rise to the usual span ∞-category $\mathrm{Span}(\mathcal{C})$ considered for example in [BR13].

2.3.3. **Example.**

i) Straight from the definition we have $\mathrm{Cob}^{\mathrm{Cr}}(\mathcal{C}, \Qoppa) \simeq \mathrm{Span}(\mathcal{C})$ for every Poincaré ∞-category $(\mathcal{C}, \Qoppa)$.

ii) Similarly, one obtains an equivalence

$$\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Span}^{\mathcal{F} \circ \mathrm{Hyp}}(\mathcal{C})$$

by commuting the hyperbolic and Q-constructions: From the natural equivalences of Remarks [I].6.4.6 and [I].7.2.24, we find

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}, \Qoppa), \mathrm{Q}_n\,\mathrm{Hyp}(\mathcal{C})) &\simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}, \Qoppa), \mathrm{Hyp}(\mathcal{C})^{\mathcal{J}_n}) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}, \Qoppa)_{\mathcal{J}_n}, \mathrm{Hyp}(\mathcal{C})) \\
&\simeq \mathrm{Hyp}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}_{\mathcal{J}_n}, \mathcal{C})) \\
&\simeq \mathrm{Hyp}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}^{\mathcal{J}_n})) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}, \Qoppa), \mathrm{Hyp}\,\mathrm{Q}_n(\mathcal{C})).
\end{aligned}$$

so the natural map $\mathrm{Q}\,\mathrm{Hyp}(\mathcal{C}) \Rightarrow \mathrm{Hyp}\,\mathrm{Q}(\mathcal{C})$ in $\mathrm{sCat}^{\mathrm{p}}_{\infty}$ is an equivalence.

iii) In particular, $\mathrm{Pn}\,\mathrm{Hyp}(\mathcal{C}) \simeq \mathrm{Cr}(\mathcal{C})$ gives

$$\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Span}(\mathcal{C})$$

for every stable ∞-category, see Proposition [I].2.2.5.

iv) There are canonical equivalences

$$\mathrm{Cob}(\mathcal{C}, Ϙ^{\mathrm{s}}) \simeq \mathrm{Span}(\mathcal{C})^{\mathrm{hC}_2} :$$

By Remark [I].2.2.8, a Poincaré structure on an ∞-category $\mathcal{D}$ induces a natural C_2-action on $\iota\mathcal{D}$. In particular, $Ϙ^{\mathrm{s}}$ induces a C_2-action on the simplicial space $\mathrm{Cr}\,\mathrm{Q}\,\mathcal{C}$ and therefore a C_2-action on the associated ∞-category $\mathrm{Span}(\mathcal{C})$. By Proposition [I].6.2.2, the Poincaré structure $(Ϙ^{\mathrm{s}})^{\mathrm{TwAr}[n]}$ is symmetric so that by Proposition [I].2.2.11 $\mathrm{Pn}\,\mathrm{Q}_n(\mathcal{C}, Ϙ^{\mathrm{s}}) \simeq \mathrm{Cr}\,\mathrm{Q}_n(\mathcal{C})^{\mathrm{hC}_2}$. As $\mathrm{Cr}\,\mathrm{Q}\,\mathcal{C}$ is a complete Segal space, this implies the claim.

v) There is a canonical equivalence

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \simeq \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)^{\mathrm{op}}$$

natural in the Poincaré ∞-category $(\mathcal{C}, Ϙ)$, since $\mathrm{Q}(\mathcal{C}, Ϙ)$ is naturally identified with $\mathrm{Q}(\mathcal{C}, Ϙ)^{\mathrm{op}}$ (the reversal of the simplicial object) via the canonical (and in fact unique) identification $\mathrm{TwAr}(\Delta^n) \cong \mathrm{TwAr}((\Delta^n)^{\mathrm{op}})$ of cosimplicial objects.

We now collect a few basic properties of such cobordism ∞-categories. Note that the inclusion of 0-simplices of $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ^{[1]})$ gives a natural map

$$\mathcal{F}(\mathcal{C}, Ϙ^{[1]}) \longrightarrow \iota\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$$

that is surjective on π_0. Informally, for $\mathcal{F} = \mathrm{Pn}$, this map takes any Poincaré object to itself and an equivalence $f : x \to x'$ to the cobordism $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{f} x'$. Proposition 2.3.1 implies:

2.3.4. **Corollary.** *The natural map*

$$\mathcal{F}(\mathcal{C}, Ϙ^{[1]}) \to \iota\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$$

is an equivalence, whenever $\mathcal{F}$ preserves pullbacks. In particular, a Poincaré cobordism

$$(x, q) \leftarrow (w, p) \to (x', q')$$

considered as a morphism in $\mathrm{Cob}(\mathcal{C}, Ϙ)$ *is invertible if and only if both underlying maps $w \to x$ and $w \to x'$ are equivalences in* $\mathcal{C}$.

2.3.5. **Remark.** In the geometric cobordism category Cob_d, one can perform a similar analysis: If a morphism W in Cob_d is invertible, then it is an h-cobordism and the converse is true if $d \neq 4$, the inverse of W being given by the h-cobordism with Whitehead torsion $-\tau(W) \in \mathrm{Wh}(\pi_1(\partial_0 W))$.

Furthermore, the homotopy type of $\iota\mathrm{Cob}_d$ is closely related to the classifying space for h-cobordisms [RS21].

There is a similar simple way for producing diagrams in $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$, via the canonical composite with $K \in \mathrm{Cat}_\infty$ and $(\mathcal{C}, Ϙ) \in \mathrm{Cat}^{\mathrm{p}}_\infty$:

$$\mathcal{F}\,\mathrm{Q}_K(\mathcal{C}, Ϙ) \longrightarrow \mathrm{Hom}_{\mathrm{s}\mathcal{S}}(\mathrm{N}\,K, \mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ)) \longrightarrow \mathrm{Hom}_{\mathrm{s}\mathcal{S}}(\mathrm{N}\,K, \mathrm{comp}(\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ))) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(K, \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ));$$

the first map arises by regarding $X \mapsto \mathcal{F}\,\mathrm{Q}_{\mathrm{asscat}\,X}(\mathcal{C}, Ϙ)$ as a functor $\mathrm{s}\mathcal{S}^{\mathrm{op}} \to \mathcal{S}$ and observing that $X \mapsto \mathrm{Hom}_{\mathrm{s}\mathcal{S}}(X, \mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ))$ is the right Kan extension of its restriction along the Yoneda embedding $\Delta^{\mathrm{op}} \to \mathrm{s}\mathcal{S}^{\mathrm{op}}$ (since it preserves limits and this characterises Kan extended functors in the present situation by (the opposite of) [Lur09a, Lemma 5.1.5.5]). Restricted to Δ^{op}, the two functors agree (by definition) so the adjunction unit (of restriction and lKan extension) provides the desired map upon precomposition with N.

The second map simply applies completion, and the third is an instance of the full faithfulness of N.

2.3.6. **Proposition.** *If $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ preserves arbitrary limits, the above map gives an equivalence*

$$\mathcal{F}\,\mathrm{Q}_K(\mathcal{C}, Ϙ) \longrightarrow \mathrm{Hom}_{\mathrm{Cat}_\infty}(K, \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ))$$

for every $K \in \mathrm{Cat}^{\mathrm{p}}_\infty$ and $(\mathcal{C}, Ϙ) \in \mathrm{Cat}^{\mathrm{p}}_\infty$.

Proof. All maps in the construction above are equivalences in this case, the first by Lemma 2.2.6, the second by Lemma 2.2.7 (and the third since N is fully faithful). □

Since the association $(\mathcal{C}, Ϙ) \mapsto \mathrm{Q}_n(\mathcal{C}, Ϙ)$ preserves finite products, as does completion of Segal spaces, it follows that the functor $\mathrm{Cob}^{\mathcal{F}} : \mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathrm{Cat}_\infty$ preserves finite products. Since $\mathrm{Cat}^{\mathrm{p}}_\infty$ is semi-additive (see Proposition [I].6.1.7), the ∞-categories $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ acquire natural symmetric monoidal structures induced by the direct sum operation in $\mathcal{C}$ by [GGN15, Corollary 2.5]. In particular, $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ is naturally a commutative monoid; explicitly when $\mathcal{F} = \mathrm{Pn}$, $\pi_0|\mathrm{Cob}(\mathcal{C}, Ϙ)|$ is the monoid of cobordism classes of Poincaré objects in $(\mathcal{C}, Ϙ)$ under orthogonal sum. Now, $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ is in fact a group by the following result:

2.3.7. **Proposition.** *The Poincaré functor* $(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{Ϙ}) : (\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$ *induces the inversion map on* $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ *for every additive* $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *and every Poincaré* ∞*-category* $(\mathcal{C}, Ϙ)$*. In particular,* $|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ *is always an* E_∞*-group in its canonical* E_∞*-structure.*

2.3.8. **Remark.** Let us warn the reader, that the Poincaré functor $(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{Ϙ})$ does not generally induce the inversion on the entirety of $|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$, the difference between the two maps merely vanishes on π_0. We will give a formula for the inversion map at the space level in Corollary 3.1.8 below.

For the proof, we need a construction which will reappear later:

2.3.9. **Construction.** Consider the hermitian functor $\mathrm{bcyl} : (\mathcal{C}, Ϙ) \to \mathrm{Q}_1(\mathcal{C}, Ϙ)$, representing a *bent cylinder*, which consists of the functor

$$X \mapsto [X \oplus X \xleftarrow{\Delta_X} X \to 0]$$

and the map of quadratic functors induced by the diagram

$$\begin{tikzcd}
ϘX \arrow[rrrd, bend left, "{(\mathrm{id},-\mathrm{id},0)}"] \arrow[rd, dashed] \arrow[rdd, bend right] & & & \\
& Ϙ_1(\mathrm{bcyl}X) \arrow[r] \arrow[d] & Ϙ(X \oplus X) \arrow[r] \arrow[d, "\Delta^*"] & ϘX \oplus ϘX \oplus \mathrm{B}_Ϙ(X, X) \arrow[ld, "\mathrm{pr}_1+\mathrm{pr}_2"] \\
& * \arrow[r] & ϘX &
\end{tikzcd}$$

whose left hand square is cartesian by definition of $Ϙ_1$, and whose right most horizontal map is an equivalence by definition of $\mathrm{B}_Ϙ$. The construction is readily checked to give a Poincaré functor by unwinding definitions.

Informally, the bent cylinder provides a nullbordism of the sum of any Poincaré object with its reversed hermitian form.

Proof of Proposition 2.3.7. Recall from the discussion of Segal spaces that $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ is the coequaliser of the two boundary maps $\pi_0\mathcal{F}\,\mathrm{Q}_1(\mathcal{C}, Ϙ^{[1]}) \to \pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[1]})$. By construction, the element $\mathrm{bcyl}_* x \in \pi_0\mathcal{F}\,\mathrm{Q}_1(\mathcal{C}, Ϙ^{[1]})$ then witnesses

$$0 = x + (\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{Ϙ})_* x \in \pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[1]})$$

for every $x \in \pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[1]})$. The claim follows. □

2.3.10. **Corollary.** *For any additive functor* $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$*, the natural map* $\pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[1]}) \to \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$ *fits into a cocartesian square*

$$\begin{tikzcd}
\pi_0\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ^{[1]})) \arrow[r, "\mathrm{met}"] \arrow[d] & \pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[1]}) \arrow[d] \\
0 \arrow[r] & \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|
\end{tikzcd}$$

of commutative monoids.

Recall that the cokernel of $f : A \to B$ in the category of commutative monoids is given by quotienting the smallest congruence relation on B that has $f(y) \sim 0$ for all $y \in A$; one easily checks that explicitly $x \sim x'$ iff there exist $y, y' \in A$ with $x + f(y) = x' + f(y')$.

In particular, applying 2.3.10 to $\mathcal{F} = \mathrm{Pn}$, we obtain an isomorphism

$$\pi_0|\mathrm{Cob}(\mathcal{C}, Ϙ)| \cong \mathrm{L}_{-1}(\mathcal{C}, Ϙ)$$

with the L-groups from §[I].2.4. We will further explain the relation in §4.4 below.

Proof. It suffices to show that

$$\pi_0\mathcal{F}(\mathrm{Met}(\mathcal{C},\Omega^{[1]}))\xrightarrow{\mathrm{met}}\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})\to\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)|\to 0$$

is an exact sequence of commutative monoids: For then, we obtain a homomorphism

$$\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})/\mathrm{im}(\mathrm{met})\to\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)|,$$

which is surjective with vanishing kernel, and whose target is a group by Proposition 2.3.7. One easily checks in such a situation that inverses in the target lift to inverses in the source, which is therefore a group. Consequently, the vanishing of the kernel implies injectivity of the homomorphism as desired.

Now, recall that $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)|$ is the coequaliser of the two boundary maps $d_0,d_1\colon\pi_0\mathcal{F}\,\mathrm{Q}_1(\mathcal{C},\Omega^{[1]})\to\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})$. Thus the second map is indeed surjective. For exactness at $\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})$, we note that the image of (d_0,d_1) is an equivalence relation in $\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})$: It is clearly reflexive and transitive, and symmetry follows from the evident automorphism of $\mathrm{Q}_1(\mathcal{C},\Omega)$ swapping source and target. Thus $x\in\pi_0\mathcal{F}(\mathcal{C},\Omega^{[1]})$ vanishes in $\pi_0|\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Omega^{[1]})|$ if and only if there exists a $w\in\pi_0\mathcal{F}\,\mathrm{Q}_1(\mathcal{C},\Omega^{[1]})$ with $d_0w=x$ and $d_1w=0$. But since

$$\mathcal{F}(\mathrm{Met}(\mathcal{C},\Omega^{[1]}))\to\mathcal{F}\,\mathrm{Q}_1(\mathcal{C},\Omega^{[1]})\to\mathcal{F}(\mathcal{C},\Omega^{[1]})$$

is a fibre sequence by Lemma 2.2.8, this is equivalent to w lifting to $\pi_0\mathcal{F}(\mathrm{Met}(\mathcal{C},\Omega^{[1]}))$. □

As the maps

$$\mathrm{met}\colon\ \mathrm{Met}(\mathrm{Met}(\mathcal{C},\Omega))\to\mathrm{Met}(\mathcal{C},\Omega)\quad\text{and}\quad\mathrm{met}\colon\ \mathrm{Met}(\mathrm{Hyp}(\mathcal{C}))\to\mathrm{Hyp}(\mathcal{C})$$

are split by Remark [I].7.3.23 and Corollary [I].2.4.9, we obtain:

2.3.11. **Corollary.** *For any Poincaré ∞-category* $(\mathcal{C},\Omega)$*, any small stable ∞-category* $\mathcal{D}$ *and any additive functor* $\mathcal{F}\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty\to\mathcal{S}$*, the ∞-categories* $\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C},\Omega))$ *and* $\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}(\mathcal{D}))$ *are connected.*

Let us have a closer look at these two cobordism ∞-categories. We recorded in Example 2.3.3 that the forgetful functor $\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C}))\to\mathrm{Span}(\mathcal{C})$ is an equivalence, so in particular we find:

2.3.12. **Observation.** *For every small stable ∞-category* $\mathcal{C}$ *there is a canonical equivalence*

$$|\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C}))|\simeq\Omega^{\infty-1}\,\mathrm{K}(\mathcal{C}).$$

Here, $\mathrm{K}(\mathcal{C})$ denotes the connective algebraic K-theory spectrum of $\mathcal{C}$, defined for instance through the iterated Q-construction for stable ∞-categories. In the case of $\mathrm{Met}(\mathcal{C})$, we have:

2.3.13. **Proposition.** *There is a natural equivalence of ∞-categories*

$$\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Omega^{[1]}))\to\mathrm{Span}(\mathrm{He}(\mathcal{C},\Omega)).$$

Furthermore, the forgetful functor $\mathrm{Span}(\mathrm{He}(\mathcal{C},\Omega))\to\mathrm{Span}(\mathcal{C})$ *induces an equivalence on realisations. Thus,*

$$|\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Omega))|\simeq\Omega^{\infty-1}\,\mathrm{K}(\mathcal{C}).$$

The resulting equivalence

$$|\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Omega))|\simeq|\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C}))|$$

in fact holds more generally for the $\mathcal{F}$-based cobordism ∞-categories as a formal consequence $|\mathrm{Cob}^{\mathcal{F}}-|$ being additive and group-like, see Corollary 3.1.5.

Proof. Commuting diagram categories, we find

$$\mathrm{Q}(\mathrm{Met}(\mathcal{C},\Omega))\simeq\mathrm{Met}\,\mathrm{Q}(\mathcal{C},\Omega)$$

so that Proposition [I].2.4.6 implies

$$\mathrm{Pn}\,\mathrm{Q}(\mathrm{Met}(\mathcal{C},\Omega^{[1]}))\simeq\mathrm{Fm}\,\mathrm{Q}(\mathcal{C},\Omega).$$

But, without a non-degeneracy condition, hermitian objects in a diagram category are just diagrams of hermitian objects; see Corollary [I].6.3.15. So the right hand side is equivalent to $\iota\,\mathrm{Q}(\mathrm{He}(\mathcal{C},\Omega))$. Passing to associated ∞-categories gives the first claim.

For the second claim, we show that

$$\pi : \mathrm{Span}(\mathrm{He}(\mathcal{C}, \Qoppa)) \longrightarrow \mathrm{Span}(\mathcal{C})$$

is cofinal and appeal to [Lur09a, Corollary 4.1.1.12]. By [Lur09a, Theorem 4.1.3.1], it suffices to show that for every $x \in \mathrm{Span}(\mathcal{C})$, the comma ∞-category $\mathrm{Span}(\mathrm{He}(\mathcal{C}))_{x/}$ is contractible. Since $\iota\,\mathrm{Span}(\mathcal{C}) \simeq \iota\mathcal{C}$ (which is immediate from our discussion of Segal spaces), we may naturally interpret x as an object of $\mathcal{C}$ and hence consider the comparison map

$$(\mathrm{He}(\mathcal{C}, \Qoppa)_{/x})^{\mathrm{op}} \simeq (\mathrm{He}(\mathcal{C}, \Qoppa)^{\mathrm{op}})_{x/} \longrightarrow \mathrm{Span}(\mathrm{He}(\mathcal{C}, \Qoppa))_{x/} \tag{23}$$

induced by the following functor $\mathrm{He}(\mathcal{C}, \Qoppa)^{\mathrm{op}} \to \mathrm{Span}(\mathrm{He}(\mathcal{C}, \Qoppa))$: It is given by the identity on objects and takes a morphism $f : x' \to x''$ to the span $x'' \xleftarrow{f} x' \xrightarrow{\mathrm{id}} x'$; more formally the target functor $\mathrm{TwAr}(\Delta^n) \to (\Delta^n)^{\mathrm{op}}$ gives a natural transformation of complete Segal spaces

$$\iota\,\mathrm{Fun}(\Delta, -^{\mathrm{op}}) \simeq \iota\,\mathrm{Fun}(\Delta^{\mathrm{op}}, -) \to \iota\,\mathrm{Q}(-),$$

which has the desired behaviour on associated ∞-categories. Now, the functor (23) admits a right adjoint: Using the fibre sequence relating mapping spaces in comma categories with those in the original ∞-category one readily checks that $w \to x$ is right adjoint to $x' \leftarrow w \to x$, and thus Yoneda's lemma assembles this assignment into a right adjoint functor. We conclude that (23) induces an equivalence on realisations. But the ∞-category $\mathrm{He}(\mathcal{C}, \Qoppa)_{/x}$ has an initial object (the zero object of $\mathcal{C}$ with the trivial hermitian structure) and is hence contractible. □

2.4. Algebraic surgery.

In this subsection, we translate Ranicki's algebraic surgery to our set-up. This provides a useful way of producing cobordisms, that we will heavily exploit in §[III].1 of Paper [III] and [HS21]. It also gives a description of comma categories of $\mathrm{Cob}(\mathcal{C}, \Qoppa)$. We will approach these statements by translating them into assertions about certain Segal spaces derived from the Q-construction, and for the present paper it is, in fact, the analysis thereof that plays the largest role. We follow the basic description of algebraic surgery given by Lurie in [Lur11, Lecture 11].

Let $(\mathcal{C}, \Qoppa)$ be a Poincaré ∞-category, and (X, q) be a Poincaré object therein. A *surgery datum* on (X, q) consists of a map $f : T \to X$ and a nullhomotopy of $f^*q \in \Omega^\infty \Qoppa(T)$. In other words, this is the extension of (X, q) to a hermitian (but not necessarily Poincaré) nullbordism, i.e. to an object of $\mathrm{He}(\mathrm{Met}(\mathcal{C}, \Qoppa))$. Surgery data organise into a space, and, more generally, into an ∞-category:

2.4.1. **Definition.** The ∞*-category of surgery data in* $(\mathcal{C}, \Qoppa)$ is given by

$$\mathrm{Surg}(\mathcal{C}, \Qoppa) = \mathrm{Pn}(\mathcal{C}, \Qoppa) \times_{\mathrm{He}(\mathcal{C},\Qoppa)} \mathrm{He}(\mathrm{Met}(\mathcal{C}, \Qoppa)),$$

where the right hand map in the pullback is induced by $\mathrm{met} : \ \mathrm{Met}(\mathcal{C}, \Qoppa) \to (\mathcal{C}, \Qoppa)$. The fibre of $\mathrm{Surg}(\mathcal{C}, \Qoppa)$ over some $(X, q) \in \mathrm{Pn}(\mathcal{C}, \Qoppa)$ is called the *category of surgery data on* (X, q) and denoted by $\mathrm{Surg}_{(X,q)}(\mathcal{C}, \Qoppa)$.

We shall refer to the groupoid cores of these ∞-categories as the *spaces of surgery data*.

2.4.2. **Remark.** In geometric topology, a surgery datum on a closed oriented d-dimensional manifold M is an embedding $\coprod_{i\in I} S^k \to M$ with trivialised normal bundle (and I finite). The induced map on singular chains inherits the structure of an algebraic surgery datum in $(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Qoppa^{\mathrm{s}[-d]})$ (the Poincaré form on $\mathrm{C}_*(M)$ arises via its identification with $\mathrm{C}^*(M; \mathbb{Z})$ through Poincaré duality), for example by feeding the trace of the geometric surgery datum into the surgery equivalence of Proposition 2.4.3 below.

Let us warn the reader that our presentation of algebraic surgery does not follow the overall convention of creating Poincaré chain complexes from manifolds via their cochains; that convention would require us to describe an algebraic surgery datum in a more cumbersome, though equivalent, fashion via the map $X \to S = \mathrm{D}_\Qoppa T$, together with a null-homotopy of the form after pull-back along $\mathrm{D}_\Qoppa S \longrightarrow \mathrm{D}_\Qoppa X$.

Like in the geometric setting, surgery data can be used to produce cobordisms: Given a surgery datum $(f : T \to X, h : f^*q \simeq 0)$, the composition

$$T \xrightarrow{f} X \xrightarrow{q_\sharp} \mathrm{D}_\Qoppa X \xrightarrow{\mathrm{D}_\Qoppa f} \mathrm{D}_\Qoppa T$$

is identified with $(f^*q)_\sharp$ and therefore null via h. Therefore one can form the following diagram

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\chi(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathrm{D}_{\mathfrak{Q}}T \arrow[d, equal] \\
X_f \arrow[r] & X/T \arrow[r] & \mathrm{D}_{\mathfrak{Q}}T
\end{tikzcd}$$

with exact rows and columns: Here, $\chi(f)$ is the fibre of the composition $X \simeq \mathrm{D}_{\mathfrak{Q}}X \xrightarrow{\mathrm{D}_{\mathfrak{Q}}f} \mathrm{D}_{\mathfrak{Q}}T$ and X_f is defined to be the cofibre of $T \to \chi(f)$. When the map f is clear from context, we shall also abusively refer to X_f simply as X_T.

The resulting span $[X \leftarrow \chi(f) \to X_f] \in \mathrm{Q}_1(\mathcal{C})$ is then the underlying object of the desired cobordism:

2.4.3. **Proposition** (Surgery equivalence)**.** *The association χ promotes to an equivalence*

$$\chi : \iota\mathrm{Surg}(\mathcal{C}, \mathfrak{Q}) \longrightarrow \mathrm{Pn}\,\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q}),$$

such that the diagram

$$\begin{tikzcd}
\iota\mathrm{Surg}(\mathcal{C}, \mathfrak{Q}) \arrow[rr, "\chi"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{Pn}\,\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q}) \arrow[dl, "d_1"] \\
& \mathrm{Pn}(\mathcal{C}, \mathfrak{Q}). &
\end{tikzcd}$$

commutes, naturally in the Poincaré ∞-category $(\mathcal{C}, \mathfrak{Q})$.

The image of a surgery datum under this equivalence is called the *trace* of the surgery. By the commutativity of the diagram above, the trace of a surgery on (X, q) starts at (X, q), and the other end of the trace, that is X_f, is called the *result of surgery*. As already done here, we will use $\chi(f)$ for both the trace and its total object.

Proof. We identify $\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q})$ with the full subcategory of $\mathrm{Met}(\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]}))$ on those objects whose "boundary of the boundary" is zero, i.e., with the fibre of

$$\mathrm{Met}(\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]})) \xrightarrow{\mathrm{met}} \mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]}) \xrightarrow{\mathrm{met}} (\mathcal{C}, \mathfrak{Q}^{[1]}),$$

via

$$\begin{tikzcd}
 & Y \arrow[r] \arrow[d] & X \arrow[d] \\
(X \leftarrow Y \to Z) \arrow[r, mapsto] & Z \arrow[r] & 0.
\end{tikzcd}$$

One readily checks that this yields an equivalence

$$\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q}) \simeq (\mathcal{C}, \mathfrak{Q}) \times_{\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]})} \mathrm{Met}\,\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]}))$$

in $\mathrm{Cat}^{\mathrm{p}}_\infty$, where the maps in the pull-back are given by taking boundaries on the right, and including objects with boundary zero on the left. We obtain an equivalence

$$\begin{aligned}
\mathrm{Pn}(\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q})) &\simeq \mathrm{Pn}(\mathcal{C}, \mathfrak{Q}) \times_{\mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]}))} \mathrm{Pn}(\mathrm{Met}\,\mathrm{Met}(\mathcal{C}, \mathfrak{Q}^{[1]})) \\
&\simeq \mathrm{Pn}(\mathcal{C}, \mathfrak{Q}) \times_{\mathrm{Fm}(\mathcal{C}, \mathfrak{Q})} \mathrm{Fm}(\mathrm{Met}(\mathcal{C}, \mathfrak{Q})) \\
&= \iota\mathrm{Surg}(\mathcal{C}, \mathfrak{Q})
\end{aligned}$$

as desired from the algebraic Thom isomorphism (see Corollary [I].2.4.6). □

2.4.4. **Remark.** From the proof, one also obtains the following explicit description of the inverse equivalence on objects. Given a Poincaré cobordism with underlying object

$$X \leftarrow W \to Y,$$

its associated surgery datum has as underlying object the canonical map

$$\mathrm{fib}(W \to Y) \to X.$$

The form on $\mathrm{fib}(W \to Y)$ is the pull-back of the form on W to the fibre, which comes with a canonical nullhomotopy, since this pulls back further from Y.

Now, by construction of $\mathrm{Cob}(\mathcal{C}, \Qoppa)$ there is a cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cob}(\mathcal{C},\Qoppa)}(X,Y) \arrow[r] \arrow[d] & \mathrm{Pn}\,\mathrm{Q}_1(\mathcal{C}, \Qoppa^{[1]}) \arrow[d, "{(d_1,d_0)}"] \\
\Delta^0 \arrow[r, "{(X,Y)}"'] & \mathrm{Pn}(\mathcal{C}, \Qoppa^{[1]})^2,
\end{tikzcd}$$

so from a surgery datum T on $(X,q) \in \mathrm{Pn}(\mathcal{C}, \Qoppa^{[1]})$, we obtain the element $\chi_T \in \mathrm{Hom}_{\mathrm{Cob}(\mathcal{C},\Qoppa)}(X, X_T)$. As mentioned, we will make extensive use of this construction in §[III].1. Due to the inherently asymmetrical nature of the surgery process, it is, however, not particularly convenient to describe the spaces $\mathrm{Hom}_{\mathrm{Cob}(\mathcal{C},\Qoppa)}(X,Y)$ themselves (with prescribed Y) in terms of surgery data on (X,q). The entire process does, however, generalise very well to describe the comma ∞-categories $\mathrm{Cob}(\mathcal{C}, \Qoppa)_{X/}$ and more generally the comma ∞-category of $\iota\mathrm{Cob}(\mathcal{C}, \Qoppa)$ over $\mathrm{Cob}(\mathcal{C}, \Qoppa)$. Let us denote the latter ∞-category by $\mathrm{dec}(\mathrm{Cob}(\mathcal{C}, \Qoppa))$, so that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{dec}(\mathrm{Cob}(\mathcal{C},\Qoppa)) \arrow[r] \arrow[d] & \mathrm{Ar}(\mathrm{Cob}(\mathcal{C},\Qoppa)) \arrow[d, "s"] \\
\mathrm{Pn}(\mathcal{C}, \Qoppa^{[1]}) \arrow[r] & \mathrm{Cob}(\mathcal{C},\Qoppa)
\end{tikzcd}$$

with s the source map. The terminology dec is issued from the word *décalage*, see Lemma 2.4.7 below.

2.4.5. **Theorem.** *The surgery process results in an equivalence* χ

$$\begin{tikzcd}
\mathrm{Surg}(\mathcal{C}, \Qoppa^{[1]}) \arrow[rr, "\chi"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{dec}(\mathrm{Cob}(\mathcal{C},\Qoppa)) \arrow[dl, "s"] \\
& \mathrm{Pn}(\mathcal{C}, \Qoppa^{[1]}) &
\end{tikzcd}$$

natural in the Poincaré ∞-category $(\mathcal{C}, \Qoppa)$. *In particular, there result equivalences*

$$\mathrm{Surg}_X(\mathcal{C}, \Qoppa^{[1]}) \simeq \mathrm{Cob}(\mathcal{C}, \Qoppa)_{X/}$$

for all $X \in \mathrm{Pn}(\mathcal{C}, \Qoppa^{[1]})$.

2.4.6. **Remark.** We will not exploit this description of $\mathrm{Cob}(\mathcal{C}, \Qoppa)_{X/}$ in the present paper as we are forced to consider $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)$ for arbitrary additive $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ in the sequel and do not know a similarly nice description in that generality (see Remark 2.4.12 below for a discussion of this point). The description features very prominently in [HS21].

The proof of Theorem 2.4.5 occupies the remainder of this section. The construction of the equivalence will proceed by first translating the assertion to the language of Segal spaces, see Proposition 2.4.10 below. To this end, let us first recall the following well-known construction of comma categories in Segal spaces (for which we could not find a reference). We denote by $\mathrm{dec} \colon s\mathcal{S} \to s\mathcal{S}$ the shift or décalage functor induced by the endofunctor $[0] * - \colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, and similarly for simplicial objects in other ∞-categories. Recall also that we set $\mathrm{dec}(\mathcal{C}) \simeq \iota(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Ar}(\mathcal{C})$ for any ∞-category $\mathcal{C}$, where the pullback is formed using the source functor $\mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$.

2.4.7. **Lemma.** *There are canonical equivalences*

$$\mathrm{N}(\mathrm{dec}(\mathcal{C})) \simeq \mathrm{dec}\,\mathrm{N}(\mathcal{C})$$

natural in the ∞-category $\mathcal{C}$, under which the nerve of the source and target functors

$$\mathrm{dec}(\mathcal{C}) \to \iota\mathcal{C} \quad \textit{and} \quad \mathrm{dec}(\mathcal{C}) \longrightarrow \mathcal{C}$$

correspond to the maps

$$\mathrm{N}_{1+n}(\mathcal{C}) \longrightarrow \mathrm{N}_0(\mathcal{C})$$
$$\mathrm{N}_{1+n}(\mathcal{C}) \longrightarrow \mathrm{N}_n(\mathcal{C})$$

induced by $+1\colon [n] \to [1+n]$ *and the inclusion* $[0] \to [1+n]$*, respectively. In particular, there result equivalences*

$$\mathrm{N}(\mathcal{C}_{X/}) \simeq \mathrm{fib}(t\colon \mathrm{dec}(\mathrm{N}(\mathcal{C})) \to \mathrm{N}_0(\mathcal{C})),$$

naturally in n and $(\mathcal{C}, X)$*, where the fibre is taken over* $X \in \iota\mathcal{C} = \mathrm{N}_0(\mathcal{C})$*.*

Proof. Note that the statement is entirely analogous to the comparison of thin and fat slices in the theory of quasi-categories and the proof is conceptually similar as well. Unwinding the definitions the claim is equivalent to there being a cocartesian square

$$\begin{tikzcd}
\Delta^n \arrow[r, "d_1"] \arrow[d] & \Delta^n \times \Delta^1 \arrow[d] \\
\Delta^0 \arrow[r, "d_0"] & \Delta^{1+n}
\end{tikzcd}$$

in Cat_∞, that is natural in n; the contraction of Δ^{1+n} onto its first vertex produces the map $\Delta^n \times \Delta^1 \to \Delta^{1+n}$ appearing on the right. Explicitly, it is given by

$$(k, 0) \longmapsto 0 \quad \text{and} \quad (k, 1) \longmapsto k + 1.$$

That this diagram is cocartesian can be deduced from [Lur09a, Proposition 4.2.1.2], together with the fact that homotopy cartesian diagrams in Joyal's model structure on simplicial sets give cocartesian diagrams in Cat_∞.

But one can also give an internal argument: The contraction admits an explicit degreewise right inverse (which is not natural in n) as follows: Simply include Δ^{n+1} into $\Delta^n \times \Delta^1$ by sending 0 to $(0, 0)$ and k to $(k-1, 1)$ for all $0 < k \le n + 1$. Then the composition

$$\Delta^{n+1} \longrightarrow \Delta^n \times \Delta^1 \longrightarrow \Delta^{n+1}$$

is the identity, and conversely the composition

$$\Delta^n \times \Delta^1 \longrightarrow \Delta^{n+1} \longrightarrow \Delta^n \times \Delta^1$$

comes with a unique natural transformation $\Delta^n \times \Delta^1 \times \Delta^1 \to \Delta^n \times \Delta^1$ to the identity. Given now an ∞-category $\mathcal{E}$ against which to test the cocartesianness of the square above, this transformation preserves $\Delta^n \times \{0\}$ so adjoins to a transformation

$$\Delta^1 \times \mathrm{Fun}(\Delta^n \times \Delta^1, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^n, \mathcal{E})} \mathcal{E} \longrightarrow \mathrm{Fun}(\Delta^n \times \Delta^1, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^n, \mathcal{E})} \mathcal{E}$$

from the composition in question to the identity. This is readily checked to be a pointwise equivalence. □

It follows conversely that for a complete Segal space $\mathcal{C} \in \mathrm{cS}\mathcal{S}$ and $X \in \mathcal{C}_0$ we find

$$\mathrm{asscat}(\mathcal{C})_{X/} \simeq \mathrm{asscat}(\mathrm{fib}(s\colon \mathrm{dec}(\mathcal{C}) \longrightarrow \mathcal{C}_0))$$

where the fibres are taken over X. It is also easy to see that the right hand side is not affected by completion, so this formula is valid for all Segal spaces $\mathcal{C}$.

In particular, the ∞-category $\mathrm{Cob}(\mathcal{C}, \Omega)_{0/}$ is modelled by the following Segal object:

2.4.8. **Definition.** Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category. We define the simplicial object $\mathrm{Null}(\mathcal{C}, \Omega)$ in $\mathrm{Cat}^{\mathrm{p}}_\infty$ as the fibre of the simplicial map $\mathrm{dec}(\mathrm{Q}(\mathcal{C}, \Omega)) \to \mathrm{Q}_0(\mathcal{C}, \Omega) = (\mathcal{C}, \Omega)$.

Explicitly, $\mathrm{Null}_n(\mathcal{C}, \Omega)$ consists of those diagrams $\varphi\colon \mathrm{TwAr}[1+n] \to \mathcal{C}$ in $\mathrm{Q}_{1+n}(\mathcal{C}, \Omega)$ such that $\varphi(0 \le 0) = 0$, with the Poincaré structure restricted from $\mathrm{Q}_{1+n}(\mathcal{C}, \Omega)$. In particular, $\mathrm{Null}_0(\mathcal{C}, \Omega) \cong \mathrm{Met}(\mathcal{C}, \Omega)$. In fact, the Poincaré ∞-category $\mathrm{Null}_n(\mathcal{C}, \Omega)$ is metabolic in the sense of Definition [I].7.3.10: Let $\mathcal{L}_n^- \subseteq \mathrm{Null}_n(\mathcal{C}, \Omega)$ be the full subcategory spanned by those diagrams $\varphi\colon \mathrm{TwAr}[1+n]^{\mathrm{op}} \longrightarrow \mathcal{C}$ with $\varphi(0 \le i) \simeq 0$ for all $i \in [1+n]$. Then, since φ is left Kan extended from $\mathcal{J}_{1+n} \subseteq \mathrm{TwAr}(\Delta^{1+n})$ by Examples 2.2.3, the restriction to the subposet of $\mathrm{TwAr}(\Delta^{1+n})$ spanned by all $(j \le 1+n)$, $j \neq 0$ gives an equivalence

$$p_n\colon \mathcal{L}_n^- \to \mathrm{Fun}(\Delta^n, \mathcal{C})$$

and furthermore one readily checks that the restriction of the hermitian structure of $\mathrm{Null}_n(\mathcal{C}, \Omega)$ corresponds precisely to Ω^{Δ^n} under p_n.

2.4.9. **Proposition.** *We have a natural equivalence*

$$\mathrm{Pn}(\mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]}))\simeq \mathrm{Fm}(\mathrm{Fun}(\Delta^n,\mathcal{C}),\Qoppa^{\Delta^n}).$$

for Poincaré ∞-categories $(\mathcal{C},\Qoppa)$.

Proof. We will show more generally that the full subcategory $\mathcal{L}_n^+\subseteq \mathrm{Null}_n(\mathcal{C})$ formed by the duals of the objects in $\mathcal{L}_n^-$ is a Lagrangian in the sense of Definition [I].7.3.10; so that p_n induces an equivalence

$$\mathrm{Pair}\left(\mathrm{Fun}(\Delta^n,\mathcal{C}),\Qoppa^{\Delta^n}\right)\to \mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]})$$

by the recognition principle for pairing ∞-categories, Proposition [I].7.3.11, from which the claim follows from the generalised algebraic Thom isomorphism, Proposition [I].7.3.5.

To see this, we observe that $\mathcal{L}_n^+$ consists of all those $\varphi:\ \mathrm{TwAr}[1+n]^{\mathrm{op}}\longrightarrow\mathcal{C}$ that are left Kan-extended from the subposet $\boldsymbol{B}_n$ of $\mathrm{TwAr}(\Delta^{1+n})$ spanned by all $(0\le j)$, or equivalently for which $\varphi(0\le j)\longrightarrow\varphi(i\le j)$ is an equivalence for $i\le j\in[1+n]$ (in addition to $\varphi(0\le 0)=0$). The second description immediately implies that the restriction of $\Qoppa_{1+n}$ indeed vanishes, while the first exhibits left Kan extension from $\boldsymbol{B}_n$ as a right adjoint $\boldsymbol{R}$ to the inclusion $\mathcal{L}_n^+\subseteq\mathrm{Null}_n(\mathcal{C},\Qoppa)$. Since also, by definition, $\mathcal{L}_n^-=\ker(\boldsymbol{R})$, the subcategory $\mathcal{L}_n^+$ is indeed a Lagrangian. ☐

Now under the equivalences of Lemma 2.4.7, Theorem 2.4.5 translates to the following generalisation of Proposition 2.4.3:

2.4.10. **Proposition.** *The algebraic surgery construction canonically extends to a cartesian diagram*

$$\begin{tikzcd}
\mathrm{Pn}\,\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa) \arrow[r, "d_0"] \arrow[d] & \mathrm{Pn}(\mathcal{C},\Qoppa) \arrow[d, "\mathrm{const}"] \\
\iota\,\mathrm{Fun}(\Delta^n,\mathrm{He}(\mathrm{Met}(\mathcal{C},\Qoppa))) \arrow[r, "\mathrm{met}"] & \iota\,\mathrm{Fun}(\Delta^n,\mathrm{He}(\mathcal{C},\Qoppa))
\end{tikzcd}$$

of functors $\mathrm{Cat}^{\mathrm{p}}_\infty\times\Delta^{\mathrm{op}}\to\mathcal{S}$.

Proof. Identify the Poincaré ∞-category $\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)$ with the full Poincaré subcategory of $\mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})$ on all objects whose boundary in $\mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]})$ is of the form

$$\begin{tikzcd}
 & & b \arrow[dl] \arrow[dr] & & \dots & & \\
 & b \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & \dots & \\
0 & & 0 & & 0 & & \dots,
\end{tikzcd}$$

that is with the fibre of the composition

$$\mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})\xrightarrow{\mathrm{met}}\mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]})\xrightarrow{d_0}\mathrm{Q}_n(\mathcal{C},\Qoppa^{[1]});\tag{24}$$

this is achieved by the equivalences

$$\begin{aligned}
\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)&\simeq \mathrm{Q}_1(\mathcal{C},\Qoppa)\times_{(\mathcal{C},\Qoppa)}\mathrm{Q}_n(\mathcal{C},\Qoppa)\\
&\simeq \mathrm{Met}\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})\times_{\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})}\mathrm{Q}_n(\mathcal{C},\Qoppa)\\
&\simeq \mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})\times_{\mathrm{Q}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})}\mathrm{Q}_n(\mathcal{C},\Qoppa),
\end{aligned}$$

where the third identification is obtained from the pullback

$$\begin{tikzcd}
\mathrm{Met}(\mathcal{C},\Qoppa) \arrow[r, "\mathrm{met}"] \arrow[d] & (\mathcal{C},\Qoppa) \arrow[d] \\
\mathrm{Null}_n(\mathcal{C},\Qoppa) \arrow[r, "d_0"] & \mathrm{Q}_n(\mathcal{C},\Qoppa)
\end{tikzcd}$$

(straight from the Segal condition Lemma 2.2.5) by pasting pullbacks. Since the right hand map in the last description is fully faithful, this embeds $\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)$ fully faithfully into $\mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})$, and it is clear that the essential image is as desired. But invoking the displayed pullback again, we find that the fibre of the right hand maps in (24) is equivalent to

$$\mathrm{fib}(\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})\to(\mathcal{C},\Qoppa^{[1]}))\simeq(\mathcal{C},\Qoppa),$$

the latter by the metabolic fibre sequence of Example 1.2.5. In total, we obtain an equivalence

$$\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)\simeq(\mathcal{C},\Qoppa)\times_{\mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]})}\mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})$$

in $\mathrm{Cat}^{\mathrm{p}}_\infty$. But the functor Pn preserves limits, so

$$\begin{aligned}\mathrm{Pn}\,\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)&\simeq\mathrm{Pn}\mathrm{Null}_n\,\mathrm{Met}(\mathcal{C},\Qoppa^{[1]})\times_{\mathrm{Pn}\mathrm{Null}_n(\mathcal{C},\Qoppa^{[1]})}\mathrm{Pn}(\mathcal{C},\Qoppa)\\&\simeq\mathrm{Fm}\,\mathrm{Fun}(\Delta^n,\mathrm{Met}(\mathcal{C},\Qoppa))\times_{\mathrm{Fm}\,\mathrm{Fun}(\Delta^n,(\mathcal{C},\Qoppa))}\mathrm{Pn}(\mathcal{C},\Qoppa)\\&\simeq\iota\,\mathrm{Fun}(\Delta^n,\mathrm{He}\,\mathrm{Met}(\mathcal{C},\Qoppa))\times_{\iota\,\mathrm{Fun}(\Delta^n,\mathrm{He}(\mathcal{C},\Qoppa))}\mathrm{Pn}(\mathcal{C},\Qoppa),\end{aligned}$$

the second equivalence following from Proposition 2.4.9 and the third from Corollary [I].6.3.15. □

2.4.11. **Remark.** Unwinding the algebraic Thom construction used in the proof above, we can extract the following description of the functor

$$\mathrm{dec}(\mathrm{Cob}(\mathcal{C},\Qoppa))\to\mathcal{C}$$

that takes a cobordism to its underlying surgery datum (without any form data): Per construction, it is induced by the composite

$$\mathrm{Pn}\,\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)\longrightarrow\mathrm{Fun}(\Delta^n,\mathrm{He}(\mathrm{Met}(\mathcal{C},\Qoppa)))\xrightarrow{\mathrm{fgt}}\mathrm{Fun}(\Delta^n,\mathrm{Ar}(\mathcal{C}))\xrightarrow{\mathrm{s}}\mathrm{Fun}(\Delta^n,\mathcal{C})$$

where the first map is the left hand vertical one in 2.4.10, the middle one forgets the hermitian form, and the right one takes the source of an arrow. It factors, naturally in $n\in\Delta$, as the composite of the forgetful map

$$\mathrm{Pn}\,\mathrm{Q}_{1+n}(\mathcal{C},\Qoppa)\longrightarrow\mathrm{Cr}\,\mathrm{Q}_{1+n}(\mathcal{C})$$

followed by the map which takes a diagram $F\colon\ \mathrm{TwAr}(\Delta^{1+n})\to\mathcal{C}$, forms the fibre G of the counit $t^*t_*F\to F$ of the adjunction

$$t^*\colon\ \mathrm{Fun}((\Delta^{1+n})^{\mathrm{op}},\mathcal{C})\longleftrightarrow\mathrm{Fun}(\mathrm{TwAr}(\Delta^{1+n}),\mathcal{C})\ :t_*$$

and then takes its preimage under the fully faithful functor

$$s^*\colon\ \mathrm{Fun}(\Delta^{1+n},\mathcal{C})\longrightarrow\mathrm{Fun}(\mathrm{TwAr}(\Delta^{1+n}),\mathcal{C});$$

and forgets the initial vertex; here $(s,t)\colon\ \mathrm{TwAr}(\Delta^{1+n})\to\Delta^{1+n}\times(\Delta^{1+n})^{\mathrm{op}}$ takes source and target; s^* is fully faithful since s admits $k\mapsto(k\le n)$ as a fully faithful left adjoint.

To prove this description, observe that the map $p_n\colon\ \mathcal{L}^-_n\to\mathrm{Fun}(\Delta^n,\mathcal{C})$ inducing the equivalence in 2.4.9 is inverse to the restriction of s^* to diagrams $\Delta^{1+n}\to\mathcal{C}$ that vanish at 0, and that the Lagrangian $\mathcal{L}^+_n$ occurring in the proof is precisely the image of the fully faithful functor t^*.

Unwinding further the statement means that F is taken to the functor informally described by

$$k\longmapsto\mathrm{fib}(F(0\le k)\to F(k\le k))$$

on objects and with the morphism induced by some $k\to l$ obtained by inverting the left hand map in

$$\mathrm{fib}(F(0\le k)\to F(k\le k))\longleftarrow\mathrm{fib}(F(0\le l)\to F(k\le l))\longrightarrow\mathrm{fib}(F(0\le l)\to F(l\le l)),$$

which is an equivalence by the definition of the Q-construction.

Proof of 2.4.5. Since the inclusion of constant diagrams induces an equivalence

$$\mathrm{Pn}(\mathcal{C},\Qoppa)\longrightarrow\iota\,\mathrm{Fun}(\Delta^n,\mathrm{Pn}(\mathcal{C},\mathrm{Q}))$$

by the contractibility of Δ^n, Proposition 2.4.10 can be restated as an equivalence

$$\mathrm{dec}(\mathrm{Pn}\,\mathrm{Q}(\mathcal{C},\Qoppa))\simeq\mathrm{N}(\mathrm{Surg}(\mathcal{C},\Qoppa)).$$

The claim thus follows from Lemma 2.4.7. □

2.4.12. **Remark.** Finally, let us explain the reason for sticking to the functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ in this section: For an arbitrary additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$, one can produce a functor $\mathrm{c}\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}_{\infty}$ by setting

$$\mathrm{c}\mathcal{F}(\mathcal{C}, \Qoppa) = \mathrm{asscat}\,\mathcal{F}(\mathrm{Null}(\mathcal{C}, \Qoppa^{[1]})).$$

For example, $\mathrm{cPn} = \mathrm{He}$ by Proposition 2.4.9. One can then set

$$\mathrm{Surg}^{\mathcal{F}}(\mathcal{C}, \Qoppa) = \mathcal{F}(\mathcal{C}, \mathrm{Q}) \times_{\mathrm{c}\mathcal{F}(\mathcal{C},\Qoppa)} \mathrm{c}\mathcal{F}\,(\mathrm{Met}(\mathcal{C}, \Qoppa))$$

and attempt to obtain generalisations of Proposition 2.4.3 and Theorem 2.4.5 for arbitrary additive $\mathcal{F}$. The crucial (and in fact necessary) ingredient for these statement is, however, that the canonical map

$$\mathcal{F}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \to \iota(\mathrm{c}\mathcal{F}(\mathcal{C}, \Qoppa^{[-1]}))$$

is an equivalence, which unwinds exactly to the completeness of the Segal space $\mathcal{F}(\mathrm{Null}(\mathcal{C}, \Qoppa^{[1]}))$. As already mentioned after Proposition 2.3.1, this generally fails unless $\mathcal{F}$ preserves arbitrary pullbacks.

This does, however, also hold for $\mathcal{F} = \mathrm{Cr} = \mathrm{Pn} \circ \mathrm{Hyp}$, where one finds $\mathrm{cCr} = \mathrm{Span}(\mathcal{C})_{0/}$, which identifies with $\mathrm{TwAr}(\mathcal{C})$, via

$$(0 \leftarrow X \to Y) \quad \longmapsto (\mathrm{fib}(X \to Y) \to X),$$

a non-hermitian analogue of the algebraic Thom construction; one can verify this directly, or indeed use the equivalence $\mathrm{He}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{TwAr}(\mathcal{C})$ from Section [I].2.2 to derive it as a special case of the surgery equivalence 2.4.5. The analogue of the full surgery equivalence $\mathrm{Surg}(\mathcal{C}, \Qoppa^{[1]}) \simeq \mathrm{decCob}(\mathcal{C}, \Qoppa)$ is then a cartesian square

$$\begin{tikzcd}
\mathrm{dec}(\mathrm{Span}(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Cr}\,\mathcal{C} \arrow[d, "\mathrm{const}"] \\
\mathrm{TwAr}(\mathrm{Ar}(\mathcal{C})) \arrow[r, "\mathrm{t}"] & \mathrm{TwAr}(\mathcal{C})
\end{tikzcd}$$

with the top horizontal map given by

$$(X \leftarrow Y \to Z) \quad \longmapsto \quad X$$

and the left vertical one by

$$\begin{tikzcd}
 & \mathrm{fib}(Y \to Z) \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\
(X \leftarrow Y \to Z) \arrow[r, phantom, "\longmapsto"] & Y \arrow[r] & X.
\end{tikzcd}$$

We leave the details to the reader.

2.5. **The additivity theorem.**

As we will see, the decisive step towards understanding the homotopy type of the cobordism ∞-categories $\mathrm{Cob}(\mathcal{C}, \Qoppa)$ consists in analysing their behaviour under split Poincaré-Verdier sequences. To this end, we show:

2.5.1. **Theorem** (Additivity)**.** *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ be additive. Then the functor $|\mathrm{Cob}^{\mathcal{F}}|$ is also additive. In particular, a split Poincaré-Verdier sequence*

$$(\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa') \to (\mathcal{C}'', \Qoppa'')$$

induces a fibre sequence

$$|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \to |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', \Qoppa')| \to |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}'', \Qoppa'')|$$

of E_{∞}*-groups.*

We heavily exploit the result in §3 below. In particular, we use it to compute $\pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$, produce deloopings of $|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$ via the iterated Q-construction, and it serves as the basis for Grothendieck-Witt theory in §4. It also contains Waldhausen's additivity theorem for K-theory as a special case, as we will detail in §2.7 below.

On the other hand, Theorem 2.5.1 yields an algebraic analogue of Genauer's fibre sequence from geometric topology. To explain this analogy recall that there exists a fibre sequence

$$|\mathrm{Cob}_{d+1}| \to |\mathrm{Cob}^{\partial}_{d+1}| \to |\mathrm{Cob}_{d}|$$

relating cobordism categories of manifolds of different dimension (with the middle term allowing objects to have boundary). As mentioned in the introduction, this was originally proven by identifying the sequence term by term with the infinite loop spaces of certain Thom spectra, together with a direct verification that these Thom spectra form a fibre sequence; see [Gen12, Proposition 6.2] and the main result of [GTMW09].

Applying Theorem 2.5.1 for $\mathcal{F} = \mathrm{Pn}$ to the metabolic Poincaré-Verdier sequence

$$(\mathcal{C}, Ϙ^{[-1]}) \to \mathrm{Met}(\mathcal{C}, Ϙ^{[-1]}) \to (\mathcal{C}, Ϙ)$$

from Example 1.2.5, we obtain the following algebraic analogue of the Genauer fibre sequence:

2.5.2. **Corollary.** *For every Poincaré* ∞*-category* $(\mathcal{C}, Ϙ)$*, there is a fibre sequence*

$$|\mathrm{Cob}(\mathcal{C}, Ϙ^{[-1]})| \to |\mathrm{Cob}^{\partial}(\mathcal{C}, Ϙ^{[-1]})| \to |\mathrm{Cob}(\mathcal{C}, Ϙ)|$$

of E_∞*-groups.*

Even more, our proof of the Additivity theorem will follow the strategy developed in [Ste21] by the ninth author in his approach to Genauer's fibre sequence. It is based on a recognition criterion for realisation fibrations, whose assumption we verify with the following result:

2.5.3. **Theorem.** *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *be additive and* $(p,\eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ *a split Poincaré-Verdier projection. Then the induced map*

$$(p,\eta)_*\colon \mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ) \longrightarrow \mathcal{F}\,\mathrm{Q}(\mathcal{C}', Ϙ')$$

is a bicartesian fibration of Segal spaces and in particular

$$(p,\eta)_*\colon \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ')$$

a bicartesian fibration of ∞*-categories.*

We refer to [Ste21, section 2] for the definition of (co-)cartesian fibrations between possibly incomplete Segal spaces. The proof of Theorem 2.5.3 will show that an edge in $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ)$ is $\mathcal{F}\,\mathrm{Q}(p)$-cocartesian if and only if it lies in the image of $\mathcal{F}\,\mathrm{Q}(\mathcal{E}, Ϙ_1)$ where $\mathcal{E} \subseteq \mathrm{Q}_1(\mathcal{C})$ is the subcategory spanned by those diagrams $x \leftarrow w \rightarrow y$ with left hand map p-cartesian and right hand map p-cocartesian; the roles are reversed for $\mathrm{Q}(p)$-cartesian edges.

2.5.4. **Remark.**
- i) A similar result in the context of ∞-categories of spans was given by Barwick as part of his unfurling construction in [Bar17, Theorem 12.2], but see [HHLN22, Remark 3.3] for a small correction. While the main motivation for that construction is also K-theoretic in nature, its use does not seem at all related to additivity in Barwick's work. Our proof, furthermore, proceeds rather differently than Barwick's combinatorial approach.
- ii) Neither Theorem 2.5.1 nor Theorem 2.5.3 remain true upon assuming $\mathcal{F}$ Verdier-localising and the input Poincaré-Verdier, but not necessarily split. For example, with $\mathcal{F} = \mathcal{K} \circ (-)^\natural$, which is Karoubi-localising and group-like, Corollary 2.3.10 and Theorem 3.3.4 below in combination show that $|\mathrm{Cob}^{\mathcal{K}\circ(-)^\natural}| \simeq \mathrm{B}\,\mathcal{K}\circ(-)^\natural$ is the connected delooping, which is famously not (Poincaré-)Verdier-localising, since Verdier projections need not induce surjections on $\mathrm{K}_0 \circ (-)^\natural$.

Proof of the Additivity theorem, assuming Theorem 2.5.3. Suppose given a Poincaré-Verdier square

$$\begin{tikzcd}
(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] & (\mathcal{D}, \Phi) \arrow[d] \\
(\mathcal{C}', Ϙ') \arrow[r] & (\mathcal{D}', \Phi').
\end{tikzcd}$$

Since $\mathcal{F}\,\mathrm{Q}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{sCat}^{\mathrm{p}}_\infty$ is additive, we find an associated cartesian square of Segal spaces, and we claim that also

$$\begin{tikzcd}
\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}, \Phi) \arrow[d] \\
\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ') \arrow[r] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}', \Phi')
\end{tikzcd}$$

is cartesian. Since completion of Segal spaces preserves mapping spaces, the functor from the top left corner to the pullback is fully faithful on general grounds, but might a priori fail to be essentially surjective. However, if one of the maps involved is an isofibration we claim this is true (and (co)cartesian fibrations are easily checked to be isofibrations); here we call a map $C \to B$ of Segal spaces an isofibration if

$$C_1^\times \longrightarrow B_1^\times \times_{B_0} C_0$$

surjective on π_0, where the target is formed using either d_0 or d_1. Employing the formula $\iota\,\mathrm{asscat}(-) \simeq |(-)^\times|$ for Segal spaces, we have to check that

$$\pi_0|C^\times \times_{B^\times} A^\times| \longrightarrow \pi_0\left(|C^\times| \times_{|B^\times|} |A^\times|\right)$$

is surjective, but by the theorem of Seifert and van Kampen, any component in the target contains an element formed by some $c \in C_0$, $a \in A_0$, a path w in B_0 ending at the image of a, and a zig-zag of edges in $B^\times$ starting at the image of c, and ending at $w(0)$. Per assumption the zig-zag can be lifted to $C^\times$, and the endpoint of such a lift, together with a and w defines a preimage.

With the square above established as cartesian, we next argue that it remains cartesian after realisation. Writing $\mathrm{Un}(G)$ for Lurie's cocartesian unstraightening of a functor $G\colon \mathcal{E} \to \mathrm{Cat}_\infty$, it is generally true that if

$$\mathcal{E} \xrightarrow{G} \mathrm{Cat}_\infty \xrightarrow{|-|} \mathcal{S}$$

factors over $|\mathcal{E}|$, the diagram

$$\begin{tikzcd}
{|\mathrm{Un}(GH)|} \arrow[r] \arrow[d] & {|\mathrm{Un}(G)|} \arrow[d] \\
{|\mathcal{F}|} \arrow[r] & {|\mathcal{E}|}
\end{tikzcd}$$

is cartesian for any $H\colon \mathcal{F} \to \mathcal{E}$: Simply observe, for example via [Lur09a, Corollary 3.3.4.3], that the upper terms can also be regarded as the unstraightenings of $|G|\colon |\mathcal{E}| \to \mathcal{S}$ and its precomposition with $|H|$, whence this follows from unstraightening translating compositions to pullbacks (by the adjoint of [Lur09a, Proposition 3.2.1.4]). Taking for G the cocartesian straightening of $\mathrm{Cob}^{\mathcal{F}}(\mathcal{D},\Phi) \to \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}',\Phi')$ and for H the functor $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega) \to \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}',\Phi')$ then gives the claim, since the cocartesian straightening of a bicartesian fibration takes values in $\mathrm{Cat}_\infty^{\mathrm{L}}$, so satisfies the assumption on G, as adjoint functors realise to equivalences.

Alternatively, one can consider the square

$$\begin{tikzcd}
\mathcal{F}\mathrm{Q}(\mathcal{C},\Omega) \arrow[r] \arrow[d] & \mathcal{F}\mathrm{Q}(\mathcal{D},\Phi) \arrow[d] \\
\mathcal{F}\mathrm{Q}(\mathcal{C}',\Omega') \arrow[r] & \mathcal{F}\mathrm{Q}(\mathcal{D}',\Phi').
\end{tikzcd}$$

and apply [Ste21, Theorem 2.11], which verifies directly that bicartesian fibrations are realisation fibrations also among incomplete Segal spaces. Incidentally, that result also gives another proof that completion commutes with pullbacks with one leg an isofibration. □

2.6. **Fibrations between cobordism categories.** The present section is devoted to the proof of Theorem 2.5.3. We remind the reader of the notation $(\mathcal{C},\Omega)^{\mathcal{J}} = (\mathrm{Fun}(\mathcal{J},\mathcal{C}),\Omega^{\mathcal{J}})$ for the cotensoring of a hermitian ∞-category $(\mathcal{C},\Omega)$ with an ∞-category $\mathcal{J}$.

The strategy of proof is as follows: After recording that a split Verdier projection (of stable ∞-categories) is a bicartesian fibration, we improve on this by showing that the maps

$$p_*\colon (\mathcal{C},\Omega)^{\Delta^n} \to (\mathcal{C}',\Omega')^{\Delta^n}$$

behave like a bicartesian fibration between Segal objects in $\mathrm{Cat}_\infty^{\mathrm{h}}$; we will not give a formal definition of this term, but instead formulate the relevant statements directly in Lemmas 2.6.3 and 2.6.4. We then use this to show that the map

$$\mathrm{Q}(p)\colon \mathrm{Q}(\mathcal{C},\Omega) \to \mathrm{Q}(\mathcal{C}',\Omega')$$

also behaves like such a bicartesian fibration; the cocartesian part is formulated in Lemmas 2.6.7 and 2.6.8, and the cartesian one follows by invariance of the Q-construction under taking opposites. From there we will deduce the theorem by observing that any additive functor $\mathcal{F}$ can be used as a 'cut-off' to obtain a bicartesian fibration $\mathcal{F}\mathrm{Q}(\mathcal{C}, \Omega) \to \mathcal{F}\mathrm{Q}(\mathcal{C}', \Omega')$ of Segal objects in $\mathcal{S}$, which implies the result.

To get started we need:

2.6.1. **Lemma.** *Let $p\colon \mathcal{C} \to \mathcal{C}'$ be a functor with left adjoint g. Then:*

i) A morphism $\alpha\colon x \to y$ in $\mathcal{C}$ is p-cocartesian if and only if the square

$$\begin{tikzcd} gp(x) \arrow[r, "gp(\alpha)"] \arrow[d, "c_x"'] & gp(y) \arrow[d, "c_y"] \\ x \arrow[r, "\alpha"] & y, \end{tikzcd}$$

obtained by applying the counit transformation to α, is a pushout square.

ii) If $\mathcal{C}$ admits pushouts which p preserves and g is fully faithful, then p is a cocartesian fibration.

Proof. The first statement is immediate from the mapping space criterion for cocartesian morphisms [Lur09a, Proposition 2.4.4.3]. For the second one readily checks that for $c \in \mathcal{C}$ and a map $p(c) \to d$ in $\mathcal{C}'$ the edge $c \to c \cup_{gp(c)} g(d)$ is a p-cocartesian lift; here the pushout is formed using the counit $gp(c) \to c$ of the adjunction. □

Applying as well the previous lemma to the opposite ∞-category we find:

2.6.2. **Corollary.** *Any split Verdier projection $p\colon \mathcal{C} \to \mathcal{C}'$ of stable ∞-categories is a bicartesian fibration.*

In fact, the converse also holds for exact functors $\mathcal{C} \to \mathcal{C}'$ by Remark A.2.9 below. Now, denote by

$$\mathrm{Cart}(p), \mathrm{Cocart}(p) \subseteq \mathrm{Ar}(\mathcal{C})$$

the full subcategories on p-cartesian, resp. p-cocartesian morphisms. These are stable subcategories as a consequence of Lemma 2.6.1, and the hermitian structure Ω^{Δ^1} endows $\mathrm{Cart}(\mathcal{C})$ and $\mathrm{Cocart}(\mathcal{C})$ with the structure of hermitian ∞-categories (we warn the reader that $\Omega^{\Delta^1}(x \to y) \simeq \Omega(y)$ is distinct from the Poincaré structure Ω_{ar} from §[I].2.4). Finally, we denote by s and $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ source and target functor, respectively.

2.6.3. **Lemma.** *Let $p\colon (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ be a split Poincaré-Verdier projection. Then the diagrams*

$$\begin{tikzcd} (\mathrm{Cart}(p), \Omega^{\Delta^1}) \arrow[r, "t"] \arrow[d, "p"] & (\mathcal{C}, \Omega) \arrow[d, "p"] \\ (\mathcal{C}', \Omega')^{\Delta^1} \arrow[r, "t"] & (\mathcal{C}', \Omega') \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{Cocart}(p), \Omega^{\Delta^1}) \arrow[r, "s"] \arrow[d, "p"] & (\mathcal{C}, \Omega) \arrow[d, "p"] \\ (\mathcal{C}', \Omega')^{\Delta^1} \arrow[r, "s"] & (\mathcal{C}', \Omega') \end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{h}}_\infty$ are cartesian.

2.6.4. **Lemma.** *Let $p\colon (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ be a split Poincaré-Verdier projection. Then the square*

$$\begin{tikzcd} (\mathrm{Cart}(\mathcal{C}), \Omega^{\Delta^1}) \times_{(\mathcal{C},\Omega)^{\Delta^1}} (\mathcal{C}, \Omega)^{\Delta^2} \arrow[r, "(\mathrm{id},d_1)"] \arrow[d, "p"] & (\mathrm{Cart}(\mathcal{C}), \Omega^{\Delta^1}) \times_{(\mathcal{C},\Omega)} (\mathcal{C}, \Omega)^{\Delta^1} \arrow[d, "p"] \\ (\mathcal{C}', \Omega')^{\Delta^2} \arrow[r, "(d_0,d_1)"] & (\mathcal{C}', \Omega')^{\Delta^1} \times_{(\mathcal{C}',\Omega')} (\mathcal{C}', \Omega')^{\Delta^1}, \end{tikzcd}$$

where the pullback in the top left corner is formed using $d_0\colon \Delta^1 \to \Delta^2$ and those on the right using the target functor is cartesian in $\mathrm{Cat}^{\mathrm{h}}_\infty$. Similarly,

$$\begin{tikzcd} (\mathrm{Cocart}(\mathcal{C}), \Omega^{\Delta^1}) \times_{(\mathcal{C},\Omega)^{\Delta^1}} (\mathcal{C}, \Omega)^{\Delta^2} \arrow[r, "(\mathrm{id},d_1)"] \arrow[d, "p"] & (\mathrm{Cocart}(\mathcal{C}), \Omega^{\Delta^1}) \times_{(\mathcal{C},\Omega)} (\mathcal{C}, \Omega)^{\Delta^1} \arrow[d, "p"] \\ (\mathcal{C}', \Omega')^{\Delta^2} \arrow[r, "(d_2,d_1)"] & (\mathcal{C}', \Omega')^{\Delta^1} \times_{(\mathcal{C}',\Omega')} (\mathcal{C}', \Omega')^{\Delta^1}. \end{tikzcd}$$

with top left corner formed using $d_2\colon \Delta^1 \to \Delta^2$ *and right hand using the source functor, is cartesian in* $\mathrm{Cat}^{\mathrm{h}}_\infty$.

Proof of Lemma 2.6.3. One readily checks straight from the definitions and the mapping space criterion for cartesian edges [Lur09a, Proposition 2.4.4.3] that the map $\mathrm{Cart}(p) \to \mathrm{Ar}(\mathcal{C}') \times_{\mathcal{C}'} \mathcal{C}$ is essentially surjective and fully faithful for any cartesian fibration p. Now, apply Corollary 2.6.2 to obtain the statement at the level of underlying ∞-categories.

To see that this map is an equivalence $\mathrm{Cat}^{\mathrm{h}}_\infty$, note first that by the discussion in §[I].6.1 it is enough to show that for a cartesian morphism $f\colon \Delta^1 \to \mathcal{C}$ the square

$$\begin{tikzcd}
\lim(\Omega\circ f^{\mathrm{op}}) \arrow[r] \arrow[d] & \Omega(f(1)) \arrow[d] \\
\lim(\Omega'\circ(pf)^{\mathrm{op}}) \arrow[r] & \Omega'(pf(1))
\end{tikzcd}$$

is a pullback of spectra. But this is clear since the horizontal maps are equivalences, as 1 is initial in $(\Delta^1)^{\mathrm{op}}$.

Now we deal with the second square. That the underlying square of ∞-categories is cartesian is again easy (or indeed follows from the cartesian case applied to p^{op}). For the hermitian structure, we need to show that

$$\begin{tikzcd}
\Omega(f(1)) \arrow[r] \arrow[d] & \Omega(f(0)) \arrow[d] \\
\Omega'(pf(1)) \arrow[r] & \Omega'(pf(0)).
\end{tikzcd} \tag{25}$$

is a pullback for every p-cocartesian morphism f.

To see this, recall from Lemma 2.6.1 that $f(1) \simeq f(0) \cup_{lpf(0)} lpf(1)$, where l is the left adjoint to p. Furthermore, the canonical map $\Omega\circ l \to \Omega'$ is an equivalence, since p is a split Poincaré-Verdier projection; see Corollary 1.2.3. Thus, the square in question is equivalent to

$$\begin{tikzcd}
\Omega(f(0) \cup_{lpf(0)} lpf(1)) \arrow[r] \arrow[d] & \Omega(f(0)) \arrow[d] \\
\Omega(lpf(1)) \arrow[r] & \Omega(lpf(0)).
\end{tikzcd}$$

By Lemma [I].1.1.19 it is therefore enough to show that

$$\mathrm{B}_{\Omega}(\mathrm{cof}(lpf), \mathrm{cof}(c)) \simeq 0,$$

where $c\colon lpf(0) \to f(0)$ is the counit of the adjunction We compute

$$\begin{aligned}
\mathrm{B}_{\Omega}(\mathrm{cof}(lpf), \mathrm{cof}(c)) &\simeq \mathrm{Hom}_{\mathcal{C}}(l\,\mathrm{cof}(pf), \mathrm{D}_{\Omega}\,\mathrm{cof}(c)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'}(\mathrm{cof}(pf), p\mathrm{D}_{\Omega}\,\mathrm{cof}(c)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'}(\mathrm{cof}(pf), \mathrm{D}_{\Omega'}\,\mathrm{cof}(pc))
\end{aligned}$$

but pc is an equivalence so this term vanishes as desired. □

For the proof of Lemma 2.6.4, we use the following observation; compare [Lur09a, Corollary 2.4.2.5]:

2.6.5. **Observation.** *Let* $p\colon \mathcal{C} \to \mathcal{C}'$ *be a cartesian fibration, and* $\mathcal{C}_0 \subseteq \mathcal{C}$ *be a full subcategory that contains all p-cartesian morphisms whose target lies in* $\mathcal{C}_0$. *Then the restricted functor* $p\colon \mathcal{C}_0 \to \mathcal{C}'$ *is also a cartesian fibration.*

Proof of Lemma 2.6.4. We again start by showing that the upper square is a pullback of ∞-categories. We first claim that both vertical maps are cartesian fibrations. By [Lur09a, 3.1.2.1] the functors $p_*\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C}')$ are again cartesian fibrations, with cartesian edges detected pointwise. Applying this with $K = \Delta^2$ and Λ^2_2, the claim easily follows from Observation 2.6.5 and the cancellability of cartesian edges [Lur09a, Proposition 2.4.1.7]. The pointwise nature of cartesian edges also implies that the top horizontal map preserves cartesian edges, so to check that the underlying diagram is cartesian in Cat_∞ it suffices to check that the induced map on vertical fibres are equivalences by [Lur09a, Corollary 2.4.4.4].

But here again, one readily checks that the induced functors are fully faithful and essentially surjective straight from the mapping space criterion for cartesian edges [Lur09a, Proposition 2.4.4.3] together with the description of spaces of natural transformation as iterated pullbacks arising from [GHN17, Proposition 5.1].

This concludes the proof that the underlying diagram of the top square is a pullback in Cat_∞, and the argument for the bottom one is entirely analogous. To make the left square a pullback in $\mathrm{Cat}^{\mathrm{h}}_\infty$, we need to show that for each $f : \Delta^2 \to \mathcal{C}$, the square of spectra

$$\begin{tikzcd}
\lim_{(\Delta^2)^{\mathrm{op}}} \Qoppa \circ f^{\mathrm{op}} \arrow[r] \arrow[d] & \lim_{(\Lambda^2_2)^{\mathrm{op}}} \Qoppa \circ f^{\mathrm{op}} \arrow[d] \\
\lim_{(\Delta^2)^{\mathrm{op}}} \Qoppa' \circ p f^{\mathrm{op}} \arrow[r] & \lim_{(\Lambda^2_2)^{\mathrm{op}}} \Qoppa' \circ p f^{\mathrm{op}}
\end{tikzcd}$$

is a pullback. But this is clear since 2 is terminal in both Δ^2 and Λ^2_2, so the inclusion $(\Lambda^2_2)^{\mathrm{op}} \subset (\Delta^2)^{\mathrm{op}}$ is final and the horizontal maps are equivalences.

To see that the second square is a pullback in $\mathrm{Cat}^{\mathrm{h}}_\infty$, we have to show that the following square is a pull-back:

$$\begin{tikzcd}
\lim_{(\Delta^2)^{\mathrm{op}}} \Qoppa \circ f^{\mathrm{op}} \arrow[r] \arrow[d] & \lim_{(\Lambda^2_0)^{\mathrm{op}}} \Qoppa \circ f^{\mathrm{op}} \arrow[d] \\
\lim_{(\Delta^2)^{\mathrm{op}}} \Qoppa' \circ p f^{\mathrm{op}} \arrow[r] & \lim_{(\Lambda^2_0)^{\mathrm{op}}} \Qoppa' \circ p f^{\mathrm{op}}
\end{tikzcd}$$

Since 2 is terminal in Δ^2, this reads

$$\begin{tikzcd}
\Qoppa(f(2)) \arrow[r] \arrow[d] & \Qoppa(f(1)) \times_{\Qoppa(f(0))} \Qoppa(f(2)) \arrow[d] \\
\Qoppa'(pf(2)) \arrow[r] & \Qoppa'(pf(1)) \times_{\Qoppa'(pf(0))} \Qoppa'(pf(2)).
\end{tikzcd}$$

But either by decoding the statement of Lemma 2.6.3 or more directly from (25), we find

$$\Qoppa(f(1)) \simeq \Qoppa'(pf(1)) \times_{\Qoppa'(pf(0))} \Qoppa(f(0)),$$

since $f(0) \to f(1)$ is p-cocartesian by assumption. □

Let $\mathcal{E} \subset \mathrm{Q}_1(\mathcal{C})$ denote the full subcategory on objects of the form $c \leftarrow w \rightarrow d$ where the left arrow is p-cartesian, and the right arrow is p-cocartesian. This is a stable subcategory which inherits a hermitian structure from $\mathrm{Q}_1(\mathcal{C})$.

2.6.6. **Lemma.** $\mathcal{E} \subset \mathrm{Q}_1(\mathcal{C})$ *is closed under the duality* $\mathrm{D}_{\Qoppa_1}$.

Therefore, $(\mathcal{E}, \Qoppa_1)$ is a Poincaré ∞-category and the inclusion functor $\mathcal{E} \to \mathrm{Q}_1(\mathcal{C})$ canonically refines to a Poincaré functor.

Proof. Let $c \xleftarrow{f} w \xrightarrow{g} d$ be an object of $\mathcal{E}$, so that f is p-cartesian and g is p-cocartesian. The dual arrow is obtained by first completing the diagram to a pushout square; then applying $\mathrm{D}_\Qoppa$ termwise, and deleting the value at the terminal object of the square, see Proposition [I].6.3.2. The claim now follows from the fact that p-(co-)cartesian morphisms are stable under (co-)base change, and that the dualities interchange p-cartesian with p-cocartesian morphisms since the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "\mathrm{D}_\Qoppa"] \arrow[d, "p^{\mathrm{op}}"] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{C}'^{\mathrm{op}} \arrow[r, "\mathrm{D}_{\Qoppa'}"] & \mathcal{C}'
\end{tikzcd}$$

commutes as p is Poincaré. □

We are now ready to state the main technical results of this section, namely that $\mathrm{Q}(p)\colon \mathrm{Q}(\mathcal{C},\Qoppa) \to \mathrm{Q}(\mathcal{C}',\Qoppa')$ behaves like a cocartesian fibration of Segal objects in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$, with cocartesian lifts given by $(\mathcal{E},\Qoppa_1) \subset \mathrm{Q}_1(\mathcal{C},\Qoppa)$. Since the Q-construction is invariant under taking the opposite simplicial object, it follows that it also behaves like a cartesian fibration, see Example 2.3.3.

2.6.7. **Lemma.** *The diagram*

$$\begin{tikzcd}
(\mathcal{E},\Qoppa_1) \arrow[r, "d_1"] \arrow[d, "p"] & (\mathcal{C},\Qoppa) \arrow[d, "p"] \\
\mathrm{Q}_1(\mathcal{C}',\Qoppa') \arrow[r, "d_1"] & (\mathcal{C}',\Qoppa')
\end{tikzcd}$$

is a split Poincaré-Verdier square.

2.6.8. **Lemma.** *The diagram*

$$\begin{tikzcd}
(\mathcal{E},\Qoppa_1) \times_{\mathrm{Q}_1(\mathcal{C},\Qoppa)} \mathrm{Q}_2(\mathcal{C},\Qoppa) \arrow[r, "(\mathrm{id},d_1)"] \arrow[d, "p"] & (\mathcal{E},\Qoppa_1) \times_{(\mathcal{C},\Qoppa)} \mathrm{Q}_1(\mathcal{C},\Qoppa) \arrow[d, "p"] \\
\mathrm{Q}_2(\mathcal{C}',\Qoppa') \arrow[r, "(d_2,d_1)"] & \mathrm{Q}_1(\mathcal{C}',\Qoppa') \times_{(\mathcal{C}',\Qoppa')} \mathrm{Q}_1(\mathcal{C}',\Qoppa'),
\end{tikzcd}$$

where the upper left pullback is formed using $d_2\colon \mathrm{Q}_2(\mathcal{C},\Qoppa) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ and the right hand ones using d_1, is a split Poincaré-Verdier square.

In particular, both diagrams are cartesian in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$.

Proof of Lemma 2.6.7. We factor the square in question as

$$\begin{tikzcd}
(\mathcal{E},\Qoppa_1) \arrow[r] \arrow[d, "p"] & (\mathrm{Cart}(\mathcal{C}),\Qoppa^{\Delta^1}) \arrow[r, "t"] \arrow[d, "p"] & (\mathcal{C},\Qoppa) \arrow[d, "p"] \\
\mathrm{Q}_1(\mathcal{C}',\Qoppa') \arrow[r] & (\mathcal{C}',\Qoppa')^{\Delta^1} \arrow[r, "t"] & (\mathcal{C}',\Qoppa').
\end{tikzcd}$$

Here the left horizontal maps are given by including Δ^1 into TwAr Δ^1 as the morphism $(0 \leq 1) \to (0 \leq 0)$. The right square is a pullback by Lemma 2.6.3. Now

$$\mathrm{Q}_1(\mathcal{C},\Qoppa) \simeq (\mathcal{C},\Qoppa)^{\Lambda^2_0} \simeq (\mathcal{C},\Qoppa)^{\Delta^1} \times_{(\mathcal{C},\Qoppa)} (\mathcal{C},\Qoppa)^{\Delta^1}$$

using the source and target arrows for the pullback, and this equivalence restricts to an equivalence

$$\mathcal{E} \simeq \mathrm{Cart}(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Cocart}(\mathcal{C})$$

by construction. So, the left square is obtained by pullback from the right hand square of Lemma 2.6.3 and therefore cartesian as well (in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$, and hence in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$).

Since p is a split Poincaré-Verdier projection by assumption this implies the claim by Corollary 1.2.6. □

Proof of Lemma 2.6.8. The ∞-category in the upper left corner is equivalent (as a hermitian ∞-category) to the full subcategory of $\mathrm{Q}_2(\mathcal{C})$ on those diagrams $F\colon \mathrm{TwAr}\,\Delta^2 \to \mathcal{C}$,

$$\begin{tikzcd}[column sep=small]
 & & F(0 \leq 2) \arrow[dl] \arrow[dr, "\text{(III)}"'] & & \\
 & F(0 \leq 1) \arrow[dl, "\text{(I)}"'] \arrow[dr, "\text{(II)}"] & & F(1 \leq 2) \arrow[dl] \arrow[dr] & \\
F(0 \leq 0) & & F(1 \leq 1) & & F(2 \leq 2)
\end{tikzcd} \tag{26}$$

such that (i) the map labelled by (I) is p-cartesian, (ii) the map labelled by (II) is p-cocartesian, and (iii) the middle square is bicartesian. In view of Lemma 2.6.1, one easily checks by pasting squares that condition (iii) is equivalent to the following two conditions: (iii') the map labelled by (III) is p-cocartesian, and (iii”) the image of the middle square in $\mathcal{C}'$ is cocartesian. In other words, if we denote by $(\mathcal{C}, \Qoppa)_p^{\mathrm{TwAr}(\Delta^2)} \subset (\mathcal{C}, \Qoppa)^{\mathrm{TwAr}(\Delta^2)}$ the full subcategory on diagrams satisfying (i), (ii), and (iii’), then the diagram

(27)
$$\begin{tikzcd}
(\mathcal{E}, \Qoppa_1) \times_{\mathrm{Q}_1(\mathcal{C},\Qoppa)} \mathrm{Q}_2(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d, "p"] & (\mathcal{C}, \Qoppa)_p^{\mathrm{TwAr}(\Delta^2)} \arrow[d, "p"] \\
\mathrm{Q}_2(\mathcal{C}', \Qoppa') \arrow[r] & (\mathcal{C}', \Qoppa')^{\mathrm{TwAr}(\Delta^2)}
\end{tikzcd}$$

is a pullback in $\mathrm{Cat}^{\mathrm{h}}_\infty$, since it is one in Cat_∞ and the hermitian structures on the left are the restrictions of those on the right.

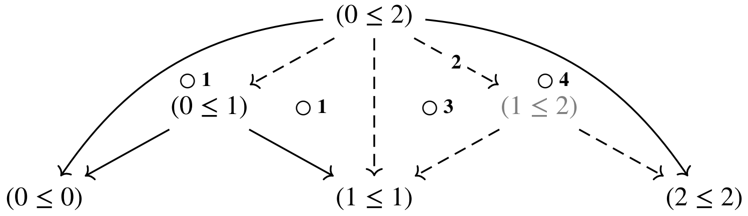
Now consider the following filtration

$$I_0 \to I_1 \to \ldots I_4 = \mathrm{TwAr}(\Delta^2)$$

through (non-full) subposets, starting with

$$I_0 = d_2(\mathrm{TwAr}\,\Delta^1) \cup d_1(\mathrm{TwAr}\,\Delta^1).$$

The remaining I_i are obtained by adding relations in the order indicated in the following picture, in which circles indicate 2-cells:



Now, one readily checks that each $I_i \to I_{i+1}$ is obtained from an outer horn inclusion by cobase change (namely using Λ^2_2, Λ^1_0 and then Λ^2_0 twice) in Cat_∞: This either follows from a simple direct argument by writing the posets involved as iterated pushouts of simplices, or from the corresponding statement at the level of simplicial sets using that homotopy pushouts in the Joyal model structure model pushouts in Cat_∞.

For $i \in \{0, \ldots, 4\}$, let $(\mathcal{C}, \Qoppa)_p^{I_i} \subset (\mathcal{C}, \Qoppa)^{I_i}$ denote the full subcategory on functors that satisfy whichever of condition (i), (ii), and (iii’) apply. Then for $i = 0$ the map $(\mathcal{C}, \Qoppa)_p^{I_i} \to (\mathcal{C}', \Qoppa')^{I_i}$ induced by p is equivalent to that in the right hand column of the statement of the Lemma, and for $i = 4$ it is the right hand map in (27).

We then claim that the diagram

(28)
$$\begin{tikzcd}
(\mathcal{C}, \Qoppa)_p^{I_i} \arrow[r] \arrow[d, "p"] & (\mathcal{C}, \Qoppa)_p^{I_{i-1}} \arrow[d, "p"] \\
(\mathcal{C}', \Qoppa')^{I_i} \arrow[r] & (\mathcal{C}', \Qoppa')^{I_{i-1}},
\end{tikzcd}$$

with horizontal maps given by restriction, is a pullback in $\mathrm{Cat}^{\mathrm{h}}_\infty$ for $i = 1, 2, 3, 4$. This establishes the lemma by pasting pullbacks.

Indeed, I_2 is obtained from I_1 by filling the 1-horn $\Lambda^1_0 \subset \Delta^1$ with a cocartesian edge, so that the restriction map $(\mathcal{C}, \Qoppa)^{I_2} \to (\mathcal{C}, \Qoppa)^{I_1}$ is pulled back from the restriction map $s\colon (\mathcal{C}, \Qoppa)^{\Delta^1} \to (\mathcal{C}, \Qoppa)$. It follows that the diagram in question is obtained from the second diagram of Lemma 2.6.3 by base changes, and therefore is a pullback.

Similarly, we see that the diagrams for $i = 1, 3, 4$ are obtained by base-changes from the diagrams of Lemma 2.6.4 and therefore pullbacks.

We are left to show that the right vertical map in the statement of the lemma, namely

$$(\mathcal{E}, \Qoppa_1) \times_{(\mathcal{C},\Qoppa)} \mathrm{Q}_1(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}_1(\mathcal{C}', \Qoppa') \times_{(\mathcal{C}',\Qoppa')} \mathrm{Q}_1(\mathcal{C}', \Qoppa'),$$

is a split Poincaré-Verdier projection. But Lemma 2.6.7 identifies this map as a base change of $\mathrm{Q}_1(p)\colon\ \mathrm{Q}_1(\mathcal{C}, Ϙ) \to \mathrm{Q}_1(\mathcal{C}', Ϙ')$, which is a Poincaré-Verdier projection by Proposition 1.4.15. The claim thus follows from Corollary 1.2.6. □

Proof of Theorem 2.5.3. Applying $\mathcal{F}$ to the squares of Lemmas 2.6.7 and 2.6.8, and using additivity, we deduce that the following squares are also pullbacks:

$$\begin{tikzcd}
\mathcal{F}(\mathcal{E}, Ϙ_1) \arrow[r, "d_1"] \arrow[d, "p"] & \mathcal{F}(\mathcal{C}, Ϙ) \arrow[d, "p"] & \mathcal{F}(\mathcal{E}, Ϙ_1) \times_{\mathcal{F}(\mathrm{Q}_1(\mathcal{C}, Ϙ))} \mathcal{F}(\mathrm{Q}_2(\mathcal{C}, Ϙ)) \arrow[r, "(\mathrm{id}, d_1)"] \arrow[d, "p"] & \mathcal{F}(\mathcal{E}, Ϙ_1) \times_{\mathcal{F}(\mathcal{C}, Ϙ)} \mathcal{F}(\mathrm{Q}_1(\mathcal{C}, Ϙ)) \arrow[d, "p"] \\
\mathcal{F}(\mathrm{Q}_1(\mathcal{C}', Ϙ')) \arrow[r, "d_1"] & \mathcal{F}(\mathcal{C}', Ϙ') & \mathcal{F}(\mathrm{Q}_2(\mathcal{C}', Ϙ')) \arrow[r, "(d_2, d_1)"] & \mathcal{F}(\mathrm{Q}_1(\mathcal{C}', Ϙ')) \times_{\mathcal{F}(\mathcal{C}', Ϙ')} \mathcal{F}(\mathrm{Q}_1(\mathcal{C}', Ϙ'));
\end{tikzcd}$$

here, the pullback in the right hand square is formed using d_2 on the left and d_1 on the right. Now, the right hand square tells us that the image of $\pi_0\mathcal{F}(\mathcal{E}, Ϙ_1) \to \pi_0\mathcal{F}(\mathrm{Q}_1(\mathcal{C}, Ϙ))$ consists of $\mathcal{F}\,\mathrm{Q}(p)$-cocartesian arrows, whence the left hand square provides sufficiently many $\mathcal{F}\,\mathrm{Q}(p)$-cocartesian lifts to make $\mathcal{F}\,\mathrm{Q}(p)\colon\ \mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ) \to \mathcal{F}\,\mathrm{Q}(\mathcal{C}', Ϙ')$ into a cocartesian fibration of Segal spaces: To see the former claim map the right square to

$$\begin{tikzcd}
\mathcal{F}(\mathcal{E}, Ϙ_1) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{E}, Ϙ_1) \arrow[d] \\
\mathcal{F}\,\mathrm{Q}_1(\mathcal{C}', Ϙ') \arrow[r] & \mathcal{F}\,\mathrm{Q}_1(\mathcal{C}', Ϙ')
\end{tikzcd}$$

in the evident fashion and take fibres of a given point $\hat{f} \in \mathcal{F}(\mathcal{E}, Ϙ_1)$ and its images. The resulting fibre square is precisely the necessary square making its image $f \in \mathcal{F}(\mathrm{Q}_1(\mathcal{C}, Ϙ))$ a $\mathcal{F}\,\mathrm{Q}(p)$-cocartesian morphism, see [Ste21, Definition 2.6]. Mapping instead to the square

$$\begin{tikzcd}
\mathcal{F}(\mathcal{E}, Ϙ_1) \times \mathcal{F}(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{E}, Ϙ_1) \times \mathcal{F}(\mathcal{C}, Ϙ) \arrow[d] \\
\mathcal{F}\,\mathrm{Q}_1(\mathcal{C}', Ϙ') \times \mathcal{F}(\mathcal{C}', Ϙ') \arrow[r] & \mathcal{F}\,\mathrm{Q}_1(\mathcal{C}', Ϙ') \times \mathcal{F}(\mathcal{C}', Ϙ')
\end{tikzcd}$$

by additionally extracting the last vertex and passing to fibres over $(f, t) \in \mathcal{F}(\mathcal{E}, Ϙ_1) \times \mathcal{F}(\mathcal{C}, Ϙ)$, we obtain the cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)}(t(f), x) \arrow[r, "-\circ f"] \arrow[d] & \mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)}(s(f), t) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ')}(t(pf), px) \arrow[r, "-\circ pf"] & \mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ')}(s(pf), px)
\end{tikzcd}$$

via the equivalence

$$\mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)}(c, d) \simeq \mathrm{fib}_{(c,d)}\big(\mathcal{F}(\mathrm{Q}_1(\mathcal{C}, Ϙ)) \xrightarrow{(d_1, d_0)} \mathcal{F}(\mathcal{C}, Ϙ)\big).$$

Thus, any $f \in \mathcal{F}(\mathcal{E}, Ϙ_1)$ also defines a $\mathrm{Cob}^{\mathcal{F}}(p)$-cocartesian morphism making $\mathrm{Cob}^{\mathcal{F}}(p)$ a cocartesian fibration as well.

Since $\mathrm{Q}(\mathcal{C})$ is naturally identified with $\mathrm{Q}(\mathcal{C})^{\mathrm{op}}$ through the canonical identification $\mathrm{TwAr}(\Delta^n) \cong \mathrm{TwAr}((\Delta^n)^{\mathrm{op}})$, see Example 2.3.3, we conclude that both $\mathcal{F}\,\mathrm{Q}(p)$ and $\mathrm{Cob}^{\mathcal{F}}(p)$ are also cartesian fibrations. □

2.7. **Additivity in K-Theory.**

The arguments presented in the previous section work verbatim upon dropping hermitian structures and working with additive functors $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$. In the present section, we briefly record the statements that are obtained this way.

Let us first formally set terminology obviously analogous to that of Definition 1.5.4.

2.7.1. **Definition.** Let $\mathcal{E}$ be an ∞-category with finite limits and $\mathcal{F}\,:\,\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{E}$ a reduced functor. We say that $\mathcal{F}$ is *additive*, *Verdier-localising* or *Karoubi-localising* if it sends split Verdier squares, arbitrary Verdier squares or Karoubi squares to cartesian squares, respectively.

Part of the following result also appears in [BR13], though in incommensurable generality.

2.7.2. **Proposition.** *For a stable ∞-category $\mathcal{C}$, the simplicial ∞-category $\mathrm{Q}(\mathcal{C})$ is a complete Segal object in* Cat_∞*, whose boundary maps are split Verdier projections. For an additive functor* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$*, the simplicial space $\mathcal{F}\,\mathrm{Q}(\mathcal{C})$ is a Segal space, which is complete if $\mathcal{F}$ preserves pullbacks.*

Proof. This first two statements are obtained during the proofs of Lemmas 2.2.5 (see also [HHLN22, Lemma 2.17]) and 2.2.8. The latter two statements are proven just as Proposition 2.3.1. □

In particular, we can extract an ∞-category $\mathrm{Span}^{\mathcal{F}}(\mathcal{C})$ from $\mathcal{F}\,\mathrm{Q}(\mathcal{C})$ by means of the functor asscat, and it inherits a symmetric monoidal structure since $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is semi-additive and $\mathcal{C} \mapsto \mathrm{Span}^{\mathcal{F}}(\mathcal{C})$ preserves finite products. The proof of Corollary 2.3.10 gives the statement that

$$\begin{tikzcd}
\pi_0\mathcal{F}(\mathrm{Ar}(\mathcal{C})) \arrow[r, "t"] \arrow[d] & \pi_0\mathcal{F}(\mathcal{C}) \arrow[d] \\
0 \arrow[r] & \pi_0|\,\mathrm{Span}^{\mathcal{F}}(\mathcal{C})|
\end{tikzcd}$$

is a pushout. In the non-hermitian situation, the top horizontal map is, however, surjective: It is split for example by the exact functor $x \mapsto (0 \to x)$. We obtain:

2.7.3. **Proposition.** *The ∞-category $\mathrm{Span}^{\mathcal{F}}(\mathcal{C})$ is connected for any stable $\mathcal{C}$ and additive* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$.

In particular, $|\,\mathrm{Span}^{\mathcal{F}}(\mathcal{C})|$ is always an E_∞-group. Our notion of additive functor is geared to permit the following strong version of Waldhausen's additivity theorem:

2.7.4. **Theorem** (Additivity)**.** *If* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$ *is additive, then so is* $|\,\mathrm{Span}^{\mathcal{F}} -| \simeq |\mathcal{F}\,\mathrm{Q}-|$.

Just as the hermitian version 2.5.1, this theorem is deduced from the following statement and the fact that bicartesian fibrations are realisation fibrations.

2.7.5. **Theorem.** *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$ *be additive and* $p\colon \mathcal{C} \to \mathcal{C}'$ *a split Verdier projection, then*

$$\mathcal{F}\,\mathrm{Q}(p)\colon \mathcal{F}\,\mathrm{Q}(\mathcal{C}) \to \mathcal{F}\,\mathrm{Q}(\mathcal{C}')$$

is a bicartesian fibration of Segal spaces and thus a realisation fibration.

The proof of Theorem 2.5.3 in §2.6, in particular, verifies Theorem 2.7.5 upon dropping all mention of Poincaré structures (which in fact made up the bulk of the work). For the reader averse to stripping away the hermitian structure themselves, here is a digest.

Proof sketch. The starting point is the observation that a split Verdier projection is itself a bicartesian fibration together with a result of Barwick's which implies that an exact bicartesian fibration $p\colon \mathcal{C} \to \mathcal{C}'$ among stable ∞-categories induces another bicartesian fibration $\mathrm{Span}(p)\colon \mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{C}')$, with a morphism $X \leftarrow Y \to Z$ being $\mathrm{Span}(p)$-cocartesian if and only if $Y \to X$ is p-cartesian and $Y \to Z$ is p-cocartesian (and reversed roles for $\mathrm{Span}(p)$-cartesian edges), see [Bar17, Theorem 12.2] or [HHLN22, Theorem 3.1 and Remark 3.4]. This already implies the result for $\mathcal{F} = \mathrm{Cr}$. To obtain it for general additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$, we categorify: Let $\mathcal{E} \subseteq \mathrm{Q}_1(\mathcal{C})$ denote the full subcategory spanned by the $\mathrm{Span}(p)$-cocartesian edges of $\mathrm{Span}(\mathcal{C})$. Then one checks that for a split Verdier projection p the squares

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "d_1"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
\mathrm{Q}_1(\mathcal{C}') \arrow[r, "d_1"] & \mathcal{C}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E} \times_{\mathrm{Q}_1(\mathcal{C})} \mathrm{Q}_2(\mathcal{C}) \arrow[r, "{(\mathrm{id},d_1)}"] \arrow[d, "p"] & \mathcal{E} \times_{(\mathcal{C},\mathcal{Q})} \mathrm{Q}_1(\mathcal{C}) \arrow[d, "p"] \\
\mathrm{Q}_2(\mathcal{C}') \arrow[r, "{(d_2,d_1)}"] & \mathrm{Q}_1(\mathcal{C}') \times_{\mathcal{C}'} \mathrm{Q}_1(\mathcal{C}')
\end{tikzcd}$$

are split Verdier squares, where the upper left pullback on the right is formed using $d_2\colon \mathrm{Q}_2(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C})$. Applying $\mathcal{F}$ these become pullbacks by assumption on $\mathcal{F}$, and after unwinding definitions, the right square says that all edges of $\mathcal{F}\,\mathrm{Q}(\mathcal{C})$ that lie in the image of $\mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathrm{Q}_1(\mathcal{C}))$ are $\mathcal{F}\,\mathrm{Q}(p)$-cocartesian, and the left square says that there is a sufficient supply of these, so $\mathcal{F}\,\mathrm{Q}(p)$ is a cocartesian fibration. Exchanging the legs of $\mathcal{E}$ shows that $\mathcal{F}\,\mathrm{Q}(p)$ is also cartesian. □

Waldhausen's additivity theorem now follows, by inserting the analogue of the metabolic sequence, i.e. the split Verdier sequence

$$\mathcal{C} \to \mathrm{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C}$$

into the additivity theorem (whence our terminology), and noting that either adjoint of t give rise to splittings of the resulting fibre sequence

$$\mathcal{F}(\mathcal{C}) \to \mathcal{F}(\mathrm{Ar}(\mathcal{C})) \xrightarrow{t} \mathcal{F}(\mathcal{C}).$$

This runs contrary to the situation of the metabolic sequence, where the adjoints of $\mathrm{met}\colon \mathrm{Met}(\mathcal{C}, \text{Ϙ}) \to (\mathcal{C}, \text{Ϙ})$ are not compatible with the Poincaré structures. The splitting lemma 1.5.12 in $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{S})$ then gives the equivalence

$$\mathcal{F}(\mathcal{C})^2 \simeq \mathcal{F}(\mathcal{C}^2) \simeq \mathcal{F}(\mathrm{Ar}(\mathcal{C})).$$

Applying this to $\mathcal{K} = \Omega|\operatorname{Span} -|$, which is additive by 2.7.4 above, we find $\mathcal{K}(\mathcal{C})^2 \simeq \mathcal{K}(\mathcal{C}^2) \simeq \mathcal{K}(\mathrm{Ar}(\mathcal{C}))$ as desired.

In summary, the metabolic fibre sequence is not just an algebraic analogue of Genauer's fibre sequence regarding geometric cobordism categories, but also of Waldhausen's additivity, the connection between which was first realised by the ninth author in [Ste21]. The idea to systematically relate cobordism categories of manifolds to algebraic K-theory by viewing spans as a formal type of cobordism was originally put forward in joint work of his with Raptis, see [RS17, RS19, RS20].

2.7.6. **Remark.** i) We repeat the caveat that it is not generally true, that $|\mathcal{F}\,\mathrm{Q}-|$ is Verdier-localising whenever $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}$ is, a counterexample being $\mathcal{K} \circ (-)^{\mathrm{idem}}$.

ii) In the set-up of stable ∞-categories, any group-like additive functor $\mathcal{F}$ in fact takes all right split and all left split Verdier-sequences (i.e. those where the projection admits only one adjoint) to fibre sequences by unpacking an old argument of Waldhausen's: Observe that as a consequence of additivity, there is an equivalence $\mathcal{F}(G_2) \simeq \mathcal{F}(G_1) + \mathcal{F}(G_3)\colon \mathcal{F}(\mathcal{C}) \to \mathcal{F}(\mathcal{C}')$ for any cofibre sequence $G_1 \Rightarrow G_2 \Rightarrow G_3$ of exact functors $\mathcal{C} \to \mathcal{C}'$ (since both source and cofibre of $G_1 \Rightarrow G_2$ and $G_1 \Rightarrow G_1 \oplus G_3\colon \mathcal{C} \to \mathrm{Ar}(\mathcal{C}')$ agree). But for a, say, left split Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{D} \arrow[l, bend right=50, "g"'] \arrow[r, "p"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{E}, \arrow[l, bend right=50, "q"'] \end{tikzcd}$$

we find a fibre sequence $fg \Rightarrow \mathrm{id}_{\mathcal{D}} \Rightarrow qp$ from A.2.5, which gives $\mathrm{id}_{\mathcal{F}(\mathcal{D})} \simeq \mathcal{F}(f)\mathcal{F}(g) + \mathcal{F}(q)\mathcal{F}(p)$, and thus that the two maps

$$(\mathcal{F}(g), \mathcal{F}(p))\colon\ \mathcal{F}(\mathcal{D}) \longleftrightarrow \mathcal{F}(\mathcal{C}) \times \mathcal{F}(\mathcal{E})\ :\mathcal{F}(f) + \mathcal{F}(q)$$

are inverse equivalences, which in particular means that $\mathcal{F}(\mathcal{C}) \to \mathcal{F}(\mathcal{D}) \to \mathcal{F}(\mathcal{E})$ is a fibre sequence.

In the hermitian case, we will replace this simple argument with the more elaborate isotropic decomposition principle, see Proposition 3.1.7 and Theorem 3.2.10.

iii) Let us also mention that the proof of Theorem 3.3.4 below also carries over without change to the setting of stable ∞-categories. As a consequence one obtains:

a) $\mathcal{K} \simeq \Omega|\operatorname{Span}(-)|\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}$ is the initial group-like additive functor under Cr, compare 3.3.6,
b) the iterated Q-construction defines a positive Ω-spectrum $\mathbb{S}\mathrm{pan}(\mathcal{C})$, with $\Omega^\infty \mathbb{S}\mathrm{pan}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{C})$, compare 3.4.5,
c) whose spectrification $\mathrm{K}(\mathcal{C})$ gives the initial additive functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p$ under $\mathbb{S}[\mathrm{Cr}]$, and in fact that K is the suspension spectrum of Cr in the stabilisation of ∞-category of additive functors $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}$, compare 3.4.6 and 3.4.9.

This gives a simple and uniform approach to these fundamental results, which to the best of our knowledge have not been treated together in the literature.

Finally, as we will have to make use of this result in the next section, let us also record the computation of $\pi_0\,\mathrm{K}(\mathcal{C}) = \pi_1|\operatorname{Span}(\mathcal{C})|$ in the generality of an arbitrary additive $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}$: The natural equivalence

$\mathrm{Hom}_{\mathrm{Span}^{\mathcal{F}}(\mathcal{C})}(0,0) \simeq \mathcal{F}(\mathcal{C})$ provides maps

$$\begin{tikzcd}
\pi_0\mathcal{F}(\mathcal{C}^2) \arrow[d] & \pi_0\mathcal{F}(\mathrm{Ar}\,\mathcal{C}) \arrow[l, "(\mathrm{s},\mathrm{cof})"'] \arrow[r, "\mathrm{t}"] \arrow[d] & \pi_0\mathcal{F}(\mathcal{C}) \arrow[d] \\
\pi_1|\,\mathrm{Span}^{\mathcal{F}}(\mathcal{C}^2)| & \pi_1|\,\mathrm{Span}^{\mathcal{F}}(\mathrm{Ar}\,\mathcal{C})| \arrow[l] \arrow[r] & \pi_1|\,\mathrm{Span}^{\mathcal{F}}(\mathcal{C})|,
\end{tikzcd}$$

where s, t and cof take the source, target, and cofibre of a morphism. The additivity theorem implies that the lower left horizontal map is an isomorphism. Inverting it produces a commutative square

$$\begin{tikzcd}
\pi_0\mathcal{F}(\mathrm{Ar}\,\mathcal{C}) \arrow[r, "\mathrm{t}"] \arrow[d, "(\mathrm{s},\mathrm{cof})"'] & \pi_0\mathcal{F}(\mathcal{C}) \arrow[d] \\
\pi_0\mathcal{F}(\mathcal{C})^2 \arrow[r] & \pi_1|\,\mathrm{Span}^{\mathcal{F}}(\mathcal{C})|
\end{tikzcd}$$

of abelian monoids natural in both $\mathcal{C}$ and $\mathcal{F}$.

2.7.7. **Proposition.** *This square is cocartesian for every stable* $\mathcal{C}$ *and every additive* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$.

In particular, for $\mathcal{F} = \mathrm{Cr}$ we recover the standard fact that $\mathrm{K}_0(\mathcal{C})$ is given by $\pi_0\mathrm{Cr}(\mathcal{C})$ modulo extensions.

Proof. While a proof internal to the Q-construction is certainly possible, the quickest route is through the well-known subdivision equivalence $|\mathcal{F}\,\mathrm{Q}(\mathcal{C})| \simeq |\mathcal{F}\mathrm{S}(\mathcal{C})|$ between Quillen's Q- and Segal's S-constructions (the latter famously developed by Waldhausen); we briefly review this equivalence in Section B.1 below. In $\mathrm{S}(\mathcal{C})$ the 0-,1- and 2-simplices are given by $*$, $\mathcal{C}$ and $\mathrm{Cof}(\mathcal{C})$, respectively, where $\mathrm{Cof}(\mathcal{C})$ denotes the ∞-category of cofibre sequences in $\mathcal{C}$. This is equivalent to $\mathrm{Ar}(\mathcal{C})$ and under this identification the boundary maps of $\mathrm{S}(\mathcal{C})$ are given by source, target and cofibre. Thus we find $\pi_1|\mathcal{F}\mathrm{S}(\mathcal{C})|$ given by $\pi_0\mathcal{F}(\mathcal{C})$ modulo the relation $s(f) + \mathrm{cof}(f) = t(f)$ for every $f \in \pi_0\mathcal{F}(\mathrm{Ar}(\mathcal{C}))$, which is exactly the pushout above. □

Finally, let us mention that $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$ is in fact Verdier localising and not just additive. Again this essentially goes back to work of Waldhausen, and (idempotent completions issues aside) is first explicit in the work of Blumberg, Gepner and Tabuada. A direct proof in the present setting can be found in [HLS22].

3. THE STRUCTURE OF ADDITIVE FUNCTORS

The objective of this section is to derive the fundamental theorems of Grothendieck-Witt theory from the additivity theorem. We will, however, do so in the generality of arbitrary additive functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$. Even when only interested in Grothendieck-Witt spectra this additional layer of generality is useful, for example it enters our proof of the universal property of $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$. The reader is encouraged to keep the two fundamental examples

$$\mathrm{Pn} \quad \text{and} \quad |\mathrm{Cob}(-)|\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$$

in mind throughout. In §4 below, we will specialise the results of this section to define Grothendieck-Witt theory and conclude the main theorems of this paper.

We begin by introducing the notion of a cobordism between Poincaré functors, and use this to establish some fundamental results for group-like additive functors. Chief among these is the agreeance of their values on hyperbolic and metabolic categories. In the case of $|\mathrm{Cob}(-)|$, we already proved this claim in Proposition 2.3.13 by explicit identification of both sides. Using the general statement as a base case, we develop a general theory of isotropic decompositions of Poincaré ∞-categories, which allows for the computations of the values of a group-like additive functor $\mathcal{F}$ applied to many Poincaré ∞-categories $(\mathcal{C}, \Omega)$ of interest, e.g. $\mathrm{Q}_n(\mathcal{C}, \Omega)$ for all n, in terms of hyperbolic pieces and parts that are often simpler than the original $(\mathcal{C}, \Omega)$.

We then use this machinery to establish precise relationships between additive functors taking values in the ∞-categories of E_∞-monoids, E_∞-groups and spectra, in particular constructing left adjoints to the evident forgetful functors. The adjoint passing from E_∞-monoid- to E_∞-group-valued functors, the *group-completion*, is given by $\mathcal{F} \mapsto \Omega|\mathrm{Cob}^{\mathcal{F}}(-)| = \Omega|\mathcal{F}\,\mathrm{Q}(-^{[1]})|$, using the $\mathcal{F}$-based cobordism ∞-category from section §2, and the adjoint from E_∞-group-valued to spectrum-valued functors, the *spectrification*, is given by iterating the Q-construction on $\mathcal{F}$. This generalises the work of Blumberg-Gepner-Tabuada on the universality of algebraic K-theory [BGT13]. Many of our constructions also have geometric precursors in the work of Bökstedt-Madsen on the connection between iterated cobordism categories and algebraic K-theory [BM14]. We will expand on these analogies in §4.

We then turn to a more detailed analysis of spectrum-valued additive functors. To this end we introduce the notion of a bordism-invariant functor (i.e. one that vanishes on metabolic categories), the principal example being $\mathrm{L}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$, the L-theory functor of Ranicki and Lurie. We show that the inclusion of bordism invariant functors into all additive functors also admits a left adjoint bord. It will then follow for rather formal reasons that there always is a natural bicartesian square

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C},\Omega) \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Omega) \arrow[d] \\
\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^{\mathrm{hC}_2} \arrow[r] & \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^{\mathrm{tC}_2}
\end{tikzcd}$$

of spectra, which can in principle be used to compute $\mathcal{F}$ from its *hyperbolisation* $\mathcal{F}^{\mathrm{hyp}} = \mathcal{F} \circ \mathrm{Hyp}$ and its *bordification* $\mathcal{F}^{\mathrm{bord}}$, each of which may be easier to understand than $\mathcal{F}$. We also provide two direct formulas for $\mathcal{F}^{\mathrm{bord}}$, which again have precursors in manifold theory. We use these in §4 to identify the bordification of Grothendieck-Witt theory with L-theory, completing the proof of the main theorem.

3.1. **Cobordisms of Poincaré functors.** In the previous section, we introduced the concept of cobordism in a Poincaré ∞-category. When applied to the Poincaré ∞-category of exact functors between two Poincaré ∞-categories, this yields a natural notion of a cobordism between functors:

3.1.1. **Definition.** Let $(\mathcal{C},\Omega)$ and $(\mathcal{D},\Phi)$ be two Poincaré ∞-categories and let $f,g\colon (\mathcal{C},\Omega) \to (\mathcal{D},\Phi)$ be two Poincaré functors. By a *cobordism* from f to g we shall mean a cobordism in the Poincaré ∞-category $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))$ between the Poincaré objects corresponding to f and g.

We note that the data of such a cobordism can equivalently be encoded by a Poincaré functor $\phi\colon (\mathcal{C},\Omega) \to \mathrm{Q}_1(\mathcal{D},\Phi)$ such that $d_0\phi = f$ and $d_1\phi = g$.

Our first goal is to describe the behaviour of group-like additive functors under such cobordisms. Recall from Definition 1.5.8 that an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ into an ∞-category admitting finite products is called group-like if its canonical lift $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E})$ arising from the semi-additivity of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ actually takes values in the full subcategory $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E}) \subseteq \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{E})$. Regarded as a functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$, the functor $\mathcal{F}$ is then again additive, since limits in $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$ are computed in $\mathcal{E}$, and group-like, since $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$ is additive.

We start by analysing the universal case of a Poincaré cobordism between functors with target $(\mathcal{C},\Omega)$. It is given by the two Poincaré functors $d_0, d_1\colon\ \mathrm{Q}_1(\mathcal{C},\Omega) \to \mathrm{Q}_0(\mathcal{C},\Omega) = (\mathcal{C},\Omega)$, which are equipped with a canonical cobordism between them.

To this end, consider the functor

$$i\colon \mathcal{C} \longrightarrow \mathrm{Q}_1(\mathcal{C}), \quad x \longmapsto \big[0 \leftarrow x \to x\big] \tag{29}$$

and its right adjoint

$$p\colon\ \mathrm{Q}_1(\mathcal{C}) \longrightarrow \mathcal{C}, \quad \big[x \leftarrow w \to y\big] \longmapsto \mathrm{fib}(w \to x).$$

Note that the unit transformation $\mathrm{id} \Rightarrow pi$ is an equivalence, so i is fully-faithful. By the universal property of the hyperbolic construction, Corollary [I].7.2.21, we obtain a pair of Poincaré functors

$$\mathrm{Hyp}(\mathcal{C}) \xrightarrow{i_{\mathrm{hyp}}} \mathrm{Q}_1(\mathcal{C},\Omega) \xrightarrow{p^{\mathrm{hyp}}} \mathrm{Hyp}(\mathcal{C}) \tag{30}$$

which is a retract diagram in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. We also note that $i_{\mathrm{hyp}}\colon\ \mathrm{Hyp}(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C},\Omega)$ factors through $\mathrm{Met}(\mathcal{C},\Omega) \subseteq \mathrm{Q}_1(\mathcal{C},\Omega)$; the corresponding restriction of i_{hyp} agrees with $\mathrm{can}\colon\ \mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C},\Omega)$ (see the recollection section for a review of notation). Similarly, the restriction of p^{hyp} to $\mathrm{Met}(\mathcal{C},\Omega) \subseteq \mathrm{Q}_1(\mathcal{C},\Omega)$ is exactly

$\mathrm{lag}\colon \mathrm{Met}(\mathcal{C},\Qoppa) \to \mathrm{Hyp}(\mathcal{C})$. In particular, we obtain the square

$$\begin{tikzcd}
& (\mathcal{C},\Qoppa) \arrow[d, "\mathrm{cyl}"] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{Met}(\mathcal{C},\Qoppa) \arrow[r] \arrow[dr, "\mathrm{lag}"'] & \mathrm{Q}_1(\mathcal{C},\Qoppa) \arrow[r, "d_1"] \arrow[d, "p^{\mathrm{hyp}}"] & (\mathcal{C},\Qoppa) \\
& \mathrm{Hyp}(\mathcal{C}) &
\end{tikzcd} \tag{31}$$

with cyl the inclusion of constant functors and p^{hyp} split (as a Poincaré functor) by i^{hyp}.

3.1.2. **Lemma.** *For $(\mathcal{C},\Qoppa)$ a Poincaré ∞-category, both the horizontal and vertical sequences of (31) are split Poincaré-Verdier sequences.*

Proof. For the horizontal sequence, this is immediate from Lemma 2.2.8. For the vertical sequence, we shall check that p satisfies the assumptions of Lemma 1.4.1 to conclude that p^{hyp} is a split Poincaré-Verdier projection; the kernel of p^{hyp} is evidently given by the diagrams $\mathrm{TwAr}(\Delta^1) \to \mathrm{Cr}\mathcal{C}$, and since $|\mathrm{TwAr}(\Delta^1)|$ is contractible these are exactly the constant diagrams, which embed $\mathcal{C}$ fully faithfully into $\mathrm{Q}_1(\mathcal{C})$ and evidently $\mathrm{cyl}^*\Qoppa_1 \simeq \Qoppa$.

We already recorded above that p admits a fully faithful left adjoint i taking x to $0 \leftarrow x \rightarrow x$, and

$$\Qoppa_1(0 \leftarrow x \rightarrow x) \simeq \Qoppa(0) \simeq 0.$$

A right adjoint r to p is readily checked to be given by the formula

$$x \longmapsto [\Sigma x \leftarrow 0 \rightarrow 0]$$

and since

$$\mathrm{D}_\Qoppa\big([\Sigma x \leftarrow 0 \rightarrow 0]\big) \simeq [\Omega \mathrm{D}_\Qoppa x \leftarrow \Omega \mathrm{D}_\Qoppa x \rightarrow 0]$$

we also find $\Qoppa(\mathrm{D}_\Qoppa(rx)) \simeq 0$ for all $x \in \mathcal{C}$ as desired. □

Applying Proposition 1.5.11 to (31) we thus obtain:

3.1.3. **Corollary.** *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ be a group-like additive functor. Then the following holds:*

i) The Poincaré functor $\mathrm{cyl}\colon (\mathcal{C},\Qoppa) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ *and the inclusion* $\mathrm{Met}(\mathcal{C},\Qoppa) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ *induce an equivalence*

$$\mathcal{F}(\mathcal{C},\Qoppa) \times \mathcal{F}(\mathrm{Met}(\mathcal{C},\Qoppa)) \longrightarrow \mathcal{F}(\mathrm{Q}_1(\mathcal{C},\Qoppa)),$$

and $\mathcal{F}$ sends the horizontal sequence of (31) to a bifibre sequence in $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$.

ii) The functors $\mathrm{cyl}\colon (\mathcal{C},\Qoppa) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ *and* $i_{\mathrm{hyp}}\colon \mathrm{Hyp}(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ *induce an equivalence*

$$\mathcal{F}(\mathcal{C},\Qoppa) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}(\mathrm{Q}_1(\mathcal{C},\Qoppa)),$$

and $\mathcal{F}$ sends the vertical sequence of (31) to a bifibre sequence in $\mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{E})$.

iii) The functors $d_1\colon \mathrm{Q}_1(\mathcal{C},\Qoppa) \to (\mathcal{C},\Qoppa)$ *and* $p^{\mathrm{hyp}}\colon \mathrm{Q}_1(\mathcal{C},\Qoppa) \to \mathrm{Hyp}(\mathcal{C})$ *induce an equivalence*

$$\mathcal{F}(\mathrm{Q}_1(\mathcal{C},\Qoppa)) \longrightarrow \mathcal{F}(\mathcal{C},\Qoppa) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{C})).$$

As a consequence of the above, we obtain the following corollary, which plays a fundamental role throughout this paper.

3.1.4. **Corollary.** *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ be a group-like additive functor. Then the functors* $\mathrm{lag}\colon \mathrm{Met}(\mathcal{C},\Qoppa) \to \mathrm{Hyp}(\mathcal{C})$ *and* $\mathrm{can}\colon \mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C},\Qoppa)$ *induce inverse equivalences*

$$\mathcal{F}(\mathrm{Met}(\mathcal{C},\Qoppa)) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C})).$$

Proof. The composite

$$(\mathcal{C},\Qoppa) \times \mathrm{Met}(\mathcal{C},\Qoppa) \xrightarrow{(\mathrm{cyl},\mathrm{inc})} \mathrm{Q}_1(\mathcal{C},\Qoppa) \xrightarrow{(d_1,p_{\mathrm{hyp}})} (\mathcal{C},\Qoppa) \times \mathrm{Hyp}(\mathcal{C})$$

is equivalent to the map $\mathrm{id}_{(\mathcal{C},\Qoppa)} \times \mathrm{lag}$. Since both constituents of this composite become equivalences after applying $\mathcal{F}$ by the previous corollary, $\mathcal{F}(\mathrm{lag})$ is a retract of an equivalence and therefore an equivalence itself. Since the functor can is a one-sided inverse to lag at the level of Poincaré ∞-categories, it must induce the inverse equivalence after applying $\mathcal{F}$. □

Applying Corollary 3.1.4 to the group-like additive functor $(\mathcal{C}, \Qoppa) \mapsto |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$ for $\mathcal{F}$ a not necessarily group-like additive functor, we deduce immediately:

3.1.5. **Corollary.** *The functors* lag *and* can *induce inverse equivalences*

$$|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))| \simeq |\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}(\mathcal{C}))|,$$

for every additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$.

This in particular gives an alternative proof of the second half of Proposition 2.3.13 that does not use the algebraic Thom construction. As explained in §2.7, it is furthermore a direct analogue to Waldhausen's additivity theorem in the non-hermitian setting.

To exploit Corollary 3.1.3 further, we need:

3.1.6. **Construction.** Given two Poincaré ∞-categories $(\mathcal{C}, \Qoppa), (\mathcal{D}, \Phi)$ and an exact functor $f\colon \mathcal{C} \to \mathcal{D}$ between the underlying ∞-categories, we obtain a Poincaré functor $\mathrm{N}f\colon (\mathcal{C}, \Qoppa) \to (\mathcal{D}, \Phi)$ by forming the composition

$$(\mathcal{C}, \Qoppa) \xrightarrow{f^{\mathrm{hyp}}} \mathrm{Hyp}(\mathcal{D}) \xrightarrow{\mathrm{id}_{\mathrm{hyp}}} (\mathcal{D}, \Phi)$$

using that the hyperbolic construction is both a left and a right adjoint to the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$. We refer to $\mathrm{N}f$ as the *norm* of f.

Unwinding this construction, we find $(\mathrm{N}f)(x) \simeq f(x) \oplus \mathrm{D}_{\Phi} f^{\mathrm{op}}(\mathrm{D}_{\Qoppa} x)$. Applying Corollary 3.1.3 to a general bordism between Poincaré functors, we then obtain:

3.1.7. **Proposition.** *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ *be a group-like additive functor. Let* $(\mathcal{C}, \Qoppa)$ *and* $(\mathcal{D}, \Phi)$ *be Poincaré* ∞*-categories and let*

$$f \longleftarrow h \longrightarrow g \tag{32}$$

be a cobordism between two Poincaré functors $f, g\colon (\mathcal{C}, \Qoppa) \to (\mathcal{D}, \Phi)$*. Let* $k\colon \mathcal{C} \to \mathcal{D}$ *be the exact functor given by the formula* $k(x) = \mathrm{fib}(h(x) \to f(x))$ *and let* $\mathrm{N}k\colon (\mathcal{C}, \Qoppa) \to (\mathcal{D}, \Phi)$ *be its norm. Then there is a canonical homotopy*

$$\mathcal{F}(g) - \mathcal{F}(f) \sim \mathcal{F}(\mathrm{N}k)\colon \mathcal{F}(\mathcal{C}, \Qoppa) \longrightarrow \mathcal{F}(\mathcal{D}, \Phi)$$

of maps $\mathcal{F}(\mathcal{C}, \Qoppa) \to \mathcal{F}(\mathcal{D}, \Phi)$.

Proof. By corollary 3.1.3, we have a pair of equivalences

$$\mathcal{F}(\mathcal{D}, \Phi) \oplus \mathcal{F}(\mathrm{Hyp}(\mathcal{D})) \xrightarrow{\mathcal{F}(s) \oplus \mathcal{F}(i_{\mathrm{hyp}})} \mathcal{F}(\mathrm{Q}_1(\mathcal{D}, \Phi)) \xrightarrow{(\mathcal{F}(d_0), \mathcal{F}(p^{\mathrm{hyp}}))} \mathcal{F}(\mathcal{D}, \Phi) \oplus \mathcal{F}(\mathrm{Hyp}(\mathcal{D})). \tag{33}$$

These equivalences are inverse to each other: indeed, the composite equivalence

$$\mathcal{F}(\mathcal{D}, \Phi) \oplus \mathcal{F}(\mathrm{Hyp}(\mathcal{D})) \xrightarrow{\simeq} \mathcal{F}(\mathcal{D}, \Phi) \oplus \mathcal{F}(\mathrm{Hyp}(\mathcal{D}))$$

is equivalent to the identity since $p^{\mathrm{hyp}} i_{\mathrm{hyp}}$ and $d_0 s$ are equivalent to the respective identity functors while $d_0 i_{\mathrm{hyp}}$ and $p^{\mathrm{hyp}} s$ are equivalent to the respective zero functors. The equivalences (33) then determine a homotopy between the identity map $\mathrm{id}\colon \mathcal{F}(\mathrm{Q}_1(\mathcal{D}, \Phi)) \to \mathcal{F}(\mathrm{Q}_1(\mathcal{D}, \Phi))$ and the sum $\mathcal{F}(s d_0) + \mathcal{F}(i_{\mathrm{hyp}} p^{\mathrm{hyp}})$, and hence a homotopy

$$\mathcal{F}(\phi) \sim \mathcal{F}(s d_0 \phi) + \mathcal{F}(i_{\mathrm{hyp}} p^{\mathrm{hyp}} \phi) = \mathcal{F}(s f) + \mathcal{F}(i_{\mathrm{hyp}} k^{\mathrm{hyp}})$$

of maps $\mathcal{F}(\mathcal{C}, \Qoppa) \to \mathcal{F}(\mathrm{Q}_1(\mathcal{D}, \Phi))$. Post composing with the map $\mathcal{F}(d_1)\colon \mathcal{F}(\mathrm{Q}_1(\mathcal{D}, \Phi)) \to \mathcal{F}(\mathcal{D}, \Phi)$, we obtain a homotopy

$$\mathcal{F}(g) = \mathcal{F}(d_1 \phi) \sim \mathcal{F}(d_1 s f) + \mathcal{F}(d_1 i_{\mathrm{hyp}} k^{\mathrm{hyp}}) = \mathcal{F}(f) + \mathcal{F}(\mathrm{N}k)$$

of maps $\mathcal{F}(\mathcal{C}, \Qoppa) \to \mathcal{F}(\mathcal{D}, \Phi)$, as desired. □

3.1.8. **Corollary.** *For a group-like additive functor* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$*, the inversion map on* $\mathcal{F}(\mathcal{C}, \Qoppa)$ *is induced by the sum of the endofunctors* $(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{\Qoppa})$ *and* $\mathrm{N}\Omega$ *of* $(\mathcal{C}, \Qoppa)$.

Proof. We resurrect the bent cylinder $\mathrm{bcyl}\colon (\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}_1(\mathcal{C}, \Qoppa)$ with underlying functor

$$X \longmapsto [X \oplus X \xleftarrow{\Delta} X \to 0]$$

from Construction 2.3.9. By construction, it is a nullcobordism of $\mathrm{id}_{(\mathcal{C},\Qoppa)} + (\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{\Qoppa})$. We obtain the conclusion from Proposition 3.1.7 by observing that the fibre of the diagonal $X \to X \oplus X$ is naturally equivalent to ΩX. ☐

Next, we use Corollary 3.1.4 to determine the fundamental group of $|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$. We base the calculation on the well-known analogue for the ∞-categories $\mathrm{Span}^{\mathcal{G}}(\mathcal{C})$ for a small stable ∞-category $\mathcal{C}$ and an additive functor $\mathcal{G}\colon \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{S}$ (i.e. one that sends split Verdier squares to cartesian squares), that we recalled in Proposition 2.7.7.

Analogous to the construction in the non-hermitian case, we consider the diagram

$$\begin{tikzcd}
\pi_0\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \arrow[d] & \pi_0\mathcal{F}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \arrow[l, "\mathrm{lag}"'] \arrow[r, "\mathrm{met}"] \arrow[d] & \pi_0\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[d] \\
\pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Hyp}(\mathcal{C}))| & \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))| \arrow[l] \arrow[r] & \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|,
\end{tikzcd}$$

with the vertical maps induced by various instances of

$$\mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)}(0, 0) \simeq \mathcal{F}(\mathcal{C}, \Qoppa).$$

The lower left horizontal map is an isomorphism by Corollary 3.1.4. Inverting it gives the square in the following:

3.1.9. **Theorem.** *For a Poincaré ∞-category* $(\mathcal{C}, \Qoppa)$ *and an additive functor* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$*, the natural square*

$$\begin{tikzcd}
\pi_0\mathcal{F}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \arrow[r, "\mathrm{met}"] \arrow[d, "\mathrm{lag}"] & \pi_0\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[d] \\
\pi_0\mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \arrow[r] & \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|
\end{tikzcd}$$

of commutative monoids is cocartesian.

Since the map lag is (split) surjective, this in particular describes $\pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$ as the quotient monoid of $\pi_0\mathcal{F}(\mathcal{C}, \Qoppa)$ identifying all metabolic objects with the hyperbolic objects on their Lagrangians. We thus, in particular, obtain an isomorphism

$$\pi_1|\mathrm{Cob}(\mathcal{C}, \Qoppa)| \cong \mathrm{GW}_0(\mathcal{C}, \Qoppa)$$

with the Grothendieck-Witt group constructed in §[I].2.5. We discuss this further in §4 below.

For the proof we need:

3.1.10. **Proposition.** *The boundary map of the algebraic Genauer sequence*

$$|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})| \longrightarrow |\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))| \longrightarrow |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$$

participates in a diagram

$$\begin{tikzcd}
 & \mathcal{F}(\mathcal{C}, \Qoppa) \arrow[dl] \arrow[dr] & \\
\Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \arrow[rr, "\partial"] & & |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})|
\end{tikzcd}$$

with the right hand map arising from the inclusion into the core, and the left hand map from the inclusion as the endomorphism of $0 \in \mathcal{F}(\mathcal{C}, \Qoppa^{[1]})$.

Since the map $\pi_0\mathcal{F}(\mathcal{C}, \Qoppa) \to \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|$ is surjective by Theorem 3.1.9, this in particular determines the effect of the boundary map $\pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \to \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})|$. Before giving the proof, let us denote the higher metabolic ∞-categories from Definition 2.4.8 as

$$\mathrm{Null}(\mathcal{C}, \Qoppa^{[1]}) = \mathrm{fib}\left(\mathrm{dec}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]}) \longrightarrow (\mathcal{C}, \Qoppa^{[1]})\right),$$

and record that from Lemma 2.4.7 and the discussion thereafter, we have:

3.1.11. **Lemma.** *For a Poincaré* ∞*-category* $(\mathcal{C}, \Qoppa)$*, an additive* $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *and* $X \in \mathcal{F}(\mathcal{C}, \Qoppa^{[1]})$*, we have*

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)_{X/} \simeq \mathrm{fib}\left(\mathrm{dec}(\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})) \longrightarrow \mathcal{F}(\mathcal{C}, \Qoppa^{[1]})\right)$$

where the arrow extracts the object positioned at $(0 \leq 0)$ *and thus in particular*

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)_{0/} \simeq \mathcal{F}(\mathrm{Null}(\mathcal{C}, \Qoppa^{[1]})).$$

Proof of Proposition 3.1.10. Recall that one way to describe the boundary map in a fibre sequence $A \to B \to C$ is as the induced map on pullbacks of

$$\left[\Omega C \to \mathrm{P}C \leftarrow \mathrm{P}B\right] \quad \Longrightarrow \quad \left[\,* \to C \leftarrow B\right],$$

where P denotes the spaces of paths starting at the basepoints, and the transformation from left to right is given by evaluation at the endpoint. For the Bott-Genauer sequence, the left side is given by

$$\mathrm{Hom}_{|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)|}(0,0) \longrightarrow |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|_{0/} \xleftarrow{\partial} |\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa)|_{0/}$$

and the right is

$$0 \longrightarrow |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \xleftarrow{\partial} |\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa)|$$

with maps induced by the canonical projections. The composition from the statement is then given by mapping

$$\mathcal{F}(\mathcal{C}, \Qoppa) \longrightarrow |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|_{0/} \xleftarrow{\partial} |\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa)|_{0/}$$

to the former of these diagrams via

$$\mathcal{F}(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)}(0,0) \longrightarrow \mathrm{Hom}_{|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa)|}(0,0).$$

But this composite transformation completes to a transformation of cartesian squares

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r, dashed] \arrow[d, "\mathrm{id}"] & {|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))|_{0/}} \arrow[d, "\mathrm{met}"] & & {|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})|} \arrow[r] \arrow[d] & {|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))|} \arrow[d, "\mathrm{met}"] \\
\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] & {|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|_{0/}} & & 0 \arrow[r] & {|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)|}
\end{tikzcd}$$

as follows: Using Lemma 3.1.11, the dashed map is given by the Poincaré functor

$$(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}_1(\mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}))$$

which sends x to the diagram

$$\begin{tikzcd}
0 \arrow[d] & x \arrow[l] \arrow[r] \arrow[d] & x \arrow[d] \\
0 & x \arrow[l] \arrow[r] & 0,
\end{tikzcd}$$

representing another bent cylinder, whose forms are by definition given by the limit of

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 \arrow[r] & \Qoppa^{[1]}x & \Qoppa^{[1]}x \arrow[l] \\
0 \arrow[r] \arrow[u] & \Qoppa^{[1]}x \arrow[u] & 0 \arrow[l] \arrow[u]
\end{tikzcd}$$

which is $\Qoppa x$, giving the hermitian structure. It is readily checked that this functor is Poincaré.

Finally, rewriting the squares above as the realisations of

$$\begin{tikzcd}
\mathrm{const}\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{F}\,\mathrm{Null}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \arrow[d, "\mathrm{met}"] & & \mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathcal{F}\,\mathrm{Q}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \arrow[d, "\mathrm{met}"] \\
\mathrm{const}\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] & \mathcal{F}\,\mathrm{Null}(\mathcal{C}, \Qoppa) & & 0 \arrow[r] & \mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)
\end{tikzcd}$$

using Lemma 3.1.11, one finds a transformation from the left to the right via inclusion as the 0-simplices in the top left corner and

$$\mathrm{Null} \Longrightarrow \mathrm{dec}(\mathrm{Q}) \overset{d_0}{\Longrightarrow} \mathrm{Q}$$

on the right hand side. □

Proof of Theorem 3.1.9. Denote by $G(\mathcal{C}, \Qoppa)$ the pushout of the diagram

$$\pi_0\mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \longleftarrow \pi_0\mathcal{F}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \longrightarrow \pi_0\mathcal{F}(\mathcal{C}, \Qoppa),$$

and similarly $W(\mathcal{C}, \Qoppa)$ the pushout of

$$0 \longleftarrow \pi_0\mathcal{F}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \longrightarrow \pi_0\mathcal{F}(\mathcal{C}, \Qoppa),$$

giving a canonical map $G(\mathcal{C}, \Qoppa) \to W(\mathcal{C}, \Qoppa)$. By construction, there is a natural map $G(\mathcal{C}, \Qoppa) \to \pi_1|\mathrm{Cob}(\mathcal{C}, \Qoppa)|$. Now, the discussion of the non-Poincaré case in Proposition 2.7.7 implies that this map is an equivalence for hyperbolic categories: The square in Theorem 3.1.9 for $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ and input $\mathrm{Hyp}(\mathcal{C})$ becomes that for $\mathcal{F} \circ \mathrm{Hyp} \colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}$ and input ∞-category $\mathcal{C}$, under the equivalences $\mathrm{Met}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Hyp}(\mathrm{Ar}(\mathcal{C}))$ and $\mathrm{Hyp}(\mathcal{C})^2 \simeq \mathrm{Hyp}(\mathrm{Hyp}(\mathcal{C}))$ from Corollary [I].2.4.9 and Remark [I].7.4.15.

Let us now construct a diagram

$$\begin{tikzcd}
\pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})| \arrow[r] & \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))| \arrow[r] & \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \arrow[r, two heads] & \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa^{[-1]})| \\
G(\mathcal{C}, \Qoppa^{[-1]}) \arrow[r] \arrow[u] & G(\mathrm{Hyp}(\mathcal{C})) \arrow[r] \arrow[u, "\cong"] & G(\mathcal{C}, \Qoppa) \arrow[r, two heads] \arrow[u, two heads] & W(\mathcal{C}, \Qoppa) \arrow[u, "\cong"]
\end{tikzcd},$$

whose upper sequence is induced by the metabolic fibre sequence via additivity and thus exact. Furthermore, the rightmost map of the top sequence is surjective as indicated, since the next term in the sequence is $\pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))|$, which vanishes by Corollary 2.3.11. The vertical maps are the evident ones (see Corollary 2.3.10 for the right most one), except the second one, which is the composition

$$G(\mathrm{Hyp}(\mathcal{C})) \xrightarrow{\mathrm{can}} G(\mathrm{Met}(\mathcal{C}, \Qoppa)) \longrightarrow \pi_1|\mathrm{Cob}^{\mathcal{F}}(\mathrm{Met}(\mathcal{C}, \Qoppa))|.$$

The left two horizontal maps in the lower sequence are

$$G(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow G(\mathrm{Met}(\mathcal{C}, \Qoppa)) \xrightarrow{\mathrm{lag}} G(\mathrm{Hyp}(\mathcal{C}))$$

and

$$\mathrm{hyp} \colon G(\mathrm{Hyp}(\mathcal{C})) \xrightarrow{\mathrm{can}} G(\mathrm{Met}(\mathcal{C}, \Qoppa)) \xrightarrow{\mathrm{met}} G(\mathcal{C}, \Qoppa),$$

respectively. The right one is that constructed above. The second vertical map is an isomorphism by Corollary 3.1.4 and the claim for hyperbolic categories established above.

Now the middle square commutes by construction, the left one by Corollary 3.1.4 and the right by Proposition 3.1.10. Furthermore, the lower sequence is exact at $G(\mathcal{C}, \Qoppa)$ in the sense that two elements $x, y \in G(\mathcal{C}, \Qoppa)$ have the same image in $W(\mathcal{C}, \Qoppa)$ if and only if there are elements w, z in the image of $G(\mathrm{Hyp}(\mathcal{C}))$ such that $x + w = z + y$: By the surjectivity of $\pi_0\mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \to G(\mathrm{Hyp}\,\mathcal{C})$ this follows straight from the cocartesian diagram

$$\begin{tikzcd}
\pi_0\mathcal{F}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \pi_0\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[d] \\
\pi_0\mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \arrow[r] & G(\mathcal{C}, \Qoppa)
\end{tikzcd}$$

by taking horizontal cokernels. It then follows formally that $G(\mathcal{C}, \Qoppa)$ is in fact a group: Since $W(\mathcal{C}, \Qoppa)$ is one, there is for every $a \in G(\mathcal{C}, \Qoppa)$ an element $a' \in G(\mathcal{C}, \Qoppa)$ such that $a + a'$ maps to 0 in $W(\mathcal{C}, \Qoppa)$. But then by exactness, there are $b, b' \in G(\mathrm{Hyp}(\mathcal{C}))$ with $a + a' + b' = b$, from which we can subtract b to get an inverse to a, since $G(\mathrm{Hyp}(\mathcal{C}))$ is group.

Furthermore, the composition

$$G(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow G(\mathrm{Hyp}(\mathcal{C})) \longrightarrow G(\mathcal{C}, \Qoppa)$$

vanishes: By construction, the map $\mathrm{met}\colon \mathcal{G}(\mathrm{Met}(\mathcal{C}, Ϙ)) \to \mathcal{G}(\mathcal{C}, Ϙ)$ factors as

$$\mathcal{G}(\mathrm{Met}(\mathcal{C}, Ϙ)) \xrightarrow{\mathrm{lag}} \mathcal{G}(\mathrm{Hyp}(\mathcal{C})) \xrightarrow{\mathrm{hyp}} \mathcal{G}(\mathcal{C}, Ϙ)$$

which identifies the composition above with

$$\mathcal{G}(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathcal{G}(\mathrm{Met}(\mathcal{C}, Ϙ)) \xrightarrow{\mathrm{met}} \mathcal{G}(\mathcal{C}, Ϙ)$$

which vanishes already at the level of ∞-categories. It is a bit tedious to check that the lower sequence is in fact exact at $\mathcal{G}(\mathrm{Hyp}(\mathcal{C}))$. Luckily, we get away without doing so directly:

We deduce Theorem 3.1.9 by two applications of the 4-lemma. Applying one half of it to the right three columns (extended by 0 to the right) gives surjectivity of the map $\mathcal{G}(\mathcal{C}, Ϙ) \to \pi_1|\mathrm{Cob}^F(\mathcal{C}, Ϙ)|$ for every $(\mathcal{C}, Ϙ)$, in particular also for the left most column. This formally implies exactness at $\mathcal{G}(\mathrm{Hyp}(\mathcal{C}))$ by a short diagram chase, whence the other half of the 4-lemma gives injectivity and thus the claim. ☐

3.2. **Isotropic decompositions of Poincaré ∞-categories.** We now describe a rather general situation which gives rise to cobordisms of Poincaré functors. We will use it to analyse the ∞-categories $\mathrm{Q}_n(\mathcal{C}, Ϙ)$; see Proposition 3.2.16 below.

Let now $(\mathcal{C}, Ϙ)$ be a Poincaré ∞-category. Given a full subcategory $\mathcal{L} \subseteq \mathcal{C}$, we denote by $\mathcal{L}^\perp \subseteq \mathcal{C}$ the full subcategory spanned by the objects $y \in \mathcal{C}$ such that $\mathrm{B}_Ϙ(x, y) \simeq 0$ for every $x \in \mathcal{L}$. Using $\mathrm{B}_Ϙ(x, y) \simeq \mathrm{Hom}_{\mathcal{C}}(x, \mathrm{D}_Ϙ(y))$, we immediately see that $\mathrm{D}_Ϙ(\mathcal{L}^\perp) \subseteq \mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of the objects $z \in \mathcal{C}$ that are right orthogonal to $\mathcal{L}$, i.e. for which $\mathrm{Map}_{\mathcal{C}}(x, z) \simeq 0$ for every $x \in \mathcal{L}$.

3.2.1. **Definition.** By an *isotropic subcategory* of $(\mathcal{C}, Ϙ)$ we shall mean a full stable subcategory $\mathcal{L} \subseteq \mathcal{C}$ with the following properties:

i) $Ϙ$ vanishes on $\mathcal{L}$.
ii) The composite functor

$$\mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{D}_Ϙ} \mathcal{C}/\mathcal{L}^\perp$$

is an equivalence.

The first condition in particular implies $\mathcal{L} \subseteq \mathcal{L}^\perp$, and the second expresses a unimodularity condition on $\mathcal{L}$. In the ordinary theory of quadratic forms, the analogue of this condition is equivalent to the requirement that an isotropic subspace be a direct summand. It admits a convenient reformulation:

3.2.2. **Lemma.** *For a stable subcategory $\mathcal{L} \subseteq \mathcal{C}$ the composite functor $\mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{D}_Ϙ} \mathcal{C}/\mathcal{L}^\perp$ is an equivalence if and only if the inclusion of $\mathcal{L}$ into $\mathcal{C}$ admits a right adjoint. Furthermore, in this case $\mathcal{L} = (\mathcal{L}^\perp)^\perp$.*

In particular, Lagrangians as considered in Definition [I].7.3.10, are examples of isotropic subcategories; we recall their definition in Definition 3.2.7 below.

Proof. The composite being an equivalence is clearly equivalent to $\mathcal{L} \to \mathcal{C} \to \mathcal{C}/\mathrm{D}_Ϙ(\mathcal{L}^\perp)$ being one. Since $\mathrm{D}_Ϙ(\mathcal{L}^\perp)$ consists exactly of the right orthogonal of $\mathcal{L}$, it is closed under retracts in $\mathcal{C}$ and thus gives a Verdier inclusion into $\mathcal{C}$ by Proposition A.1.9. Both the equivalence of the conditions in the statement and the last statement are then instances of Corollary A.2.8. ☐

3.2.3. **Remark.** Applying the remainder of Corollary A.2.8 in the situation at hand, we find that the kernel of the right adjoint $p\colon \mathcal{C} \to \mathcal{L}$ is given by $\mathrm{D}_Ϙ(\mathcal{L}^\perp)$, and thus $\mathcal{L}^\perp$ is the kernel of $p \circ \mathrm{D}_Ϙ$.

3.2.4. **Remark.** The condition that $\mathcal{L} = (\mathcal{L}^\perp)^\perp$ or even $\mathcal{L} = \mathcal{L}^\perp$ does not imply condition ii) of the definition of an isotropic category. For a concrete counterexample, take $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(K[T])$, for K a field of characteristic different from 2. The involution sending T to $-T$ provides $K[T]$ with the structure of a ring with involution, and we can consider the symmetric Poincaré structure this involution provides.

Fix then an $a \in K$, $a \neq 0$, and consider the subcategory $\mathcal{L} = \mathcal{D}^{\mathrm{p}}(K[T])_{T-a}$ spanned by those complexes that become contractible after inverting $T - a$, i.e. whose homology is $(T - a)$-power torsion. We first claim that $Ϙ^{\mathrm{s}}$ vanishes on $\mathcal{L}$: For example from the universal coefficient sequence, one finds that the homology of

$\mathrm{D}_{\Qoppa^{\mathrm{s}}}(X)$ is $(T+a)$-power torsion for $X \in \mathcal{L}$. But then, since $T+a$ and $T-a$ generate the unit ideal, $T-a$ acts invertibly on $\mathrm{D}_{\Qoppa^{\mathrm{s}}}X$, so

$$\Qoppa^{\mathrm{s}}(X) \simeq \mathrm{B}_{\Qoppa^{\mathrm{s}}}(X,X)^{\mathrm{hC}_2} \simeq \mathrm{Hom}_{K[T]}(X, \mathrm{D}_{\Qoppa^{\mathrm{s}}}(X))^{\mathrm{hC}_2} \simeq 0.$$

Similarly, $X \in \mathcal{L}^\perp$ if and only if X is left orthogonal to all perfect $(T+a)$-torsion complexes. Since every $(T+a)$-torsion complex is a colimit of perfect ones by Example 1.4.2, the object X is thus left orthogonal to the entirety of $\left(\mathrm{Mod}_{K[T]}\right)_{T+a}$. But the (non-small) Verdier sequence

$$\left(\mathrm{Mod}_{K[T]}\right)_{T+a} \longrightarrow \mathrm{Mod}_{K[T]} \longrightarrow \mathrm{Mod}_{K[T,(T+a)^{-1}]}$$

is split, with left adjoint to the localisation given by the inclusion $\mathrm{Mod}_{K[T,(T+a)^{-1}]} \to \mathrm{Mod}_{K[T]}$. The image of this left adjoint is the left orthogonal to the Verdier kernel by Lemma A.2.3. In total $\mathcal{L}^\perp$ consists exactly of those perfect complexes over $K[T]$ on which $T+a$ acts invertibly. Since this can be checked on homology it follows easily from the classification of finitely generated modules over the principal ideal domain $K[T]$ that these are exactly the perfect $K[T]$-complexes that become contractible when localised away from the prime ideal $(T+a)$. Repeating the argument above by localising at the complement of $(T-a)$ instead of inverting $T+a$ then shows $\mathcal{L} = (\mathcal{L}^\perp)^\perp$.

But the inclusion of perfect $(T-a)$-power torsion complexes into all perfect $K[T]$-complexes cannot have a right adjoint: If a map $R(M) \to M$ from a $(T-a)$-power torsion module induces an equivalence $\mathrm{Hom}_{K[T]}(X, R(M)) \simeq \mathrm{Hom}_{K[T]}(X, M)$ for all perfect $(T-a)$-power torsion modules X, then this in fact holds for all $(T-a)$-power torsion modules. But then $R(M)$ necessarily agrees with the image of M under the right adjoint to the inclusion $\left(\mathrm{Mod}_{K[T]}\right)_{T-a} \to \mathrm{Mod}_{K[T]}$, which is given by $X \mapsto \mathrm{fib}\left(X \to X[(T-a)^{-1}]\right)$. But even for $X = K[T]$, this is not a perfect $K[T]$-module.

To upgrade this example to one where $\mathcal{L} = \mathcal{L}^\perp$ simply replace $K[T]$ by its localisation at the complement of $(T-a) \cup (T+a)$.

A similar construction generally works for a Dedekind domain with an involution that swaps two maximal ideals.

3.2.5. **Definition.** For an isotropic subcategory $\mathcal{L}$ of a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, we define the *homology ∞-category* $\mathrm{Hlgy}(\mathcal{L})$ to be the cofibre of the inclusion $(\mathcal{L}, \Qoppa) \to (\mathcal{L}^\perp, \Qoppa)$ in $\mathrm{Cat}^{\mathrm{h}}_\infty$.

Thus the underlying ∞-category is $\mathcal{L}^\perp/\mathcal{L}$ and the hermitian structure is the left Kan extension of $\Qoppa_{|(\mathcal{L}^\perp)^{\mathrm{op}}}$ along the projection $(\mathcal{L}^\perp)^{\mathrm{op}} \to (\mathcal{L}^\perp/\mathcal{L})^{\mathrm{op}}$. The next proposition, in particular, shows that $\Qoppa_{|(\mathcal{L}^\perp)^{\mathrm{op}}}$ in fact descends along the projection $(\mathcal{L}^\perp)^{\mathrm{op}} \to (\mathcal{L}^\perp/\mathcal{L})^{\mathrm{op}}$ and gives a Poincaré structure on $\mathrm{Hlgy}(\mathcal{L})$.

3.2.6. **Proposition.** *Let $\mathcal{L}$ be an isotropic subcategory of a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$. Then both $(\mathrm{B}_\Qoppa)_{|(\mathcal{L}^\perp \times \mathcal{L}^\perp)^{\mathrm{op}}}$ and $(\Lambda_\Qoppa)_{|(\mathcal{L}^\perp)^{\mathrm{op}}}$ descend along the projection $(\mathcal{L}^\perp)^{\mathrm{op}} \to (\mathcal{L}^\perp/\mathcal{L})^{\mathrm{op}}$ and give the bilinear and linear part of the hermitian structure on $\mathrm{Hlgy}(\mathcal{L})$, which is Poincaré. The duality on $\mathrm{Hlgy}(\mathcal{L})$ is induced by the functor $(\mathcal{L}^\perp)^{\mathrm{op}} \to \mathcal{L}^\perp$ sending X to $\mathrm{fib}(\mathrm{D}_\Qoppa X \to \mathrm{D}_\Qoppa pX)$, where p denotes the right adjoint to $\mathcal{L} \subseteq \mathcal{C}$ and the arrow is induced by the counit.*

In particular, the composite

$$\mathcal{L}^\perp \cap \mathrm{D}_\Qoppa(\mathcal{L}^\perp) \longrightarrow \mathcal{L}^\perp \longrightarrow \mathcal{L}^\perp/\mathcal{L} = \mathrm{Hlgy}(\mathcal{L}) \tag{34}$$

canonically refines to an equivalence of Poincaré ∞-categories using the restriction of $\Qoppa$ on the source.

In particular, $\mathrm{Hlgy}(\mathcal{L})$ is equivalent to a full Poincaré subcategory of $(\mathcal{C}, \Qoppa)$, which one may think of as the subcategory of harmonic objects for $\mathcal{L}$. We denote by

$$\iota : \mathrm{Hlgy}(\mathcal{L}) \longrightarrow (\mathcal{C}, \Qoppa). \tag{35}$$

the arising fully-faithful Poincaré functor.

Proof. The first two statements follow from the general analysis of Kan-extended hermitian structures: By Lemma [I].1.4.3 the linear and bilinear parts are given by the left Kan-extensions along $(\mathcal{L}^\perp)^{\mathrm{op}} \to (\mathcal{L}^\perp/\mathcal{L})^{\mathrm{op}}$ of the restriction to $\mathcal{L}^\perp$. But they in fact descend along the projection: This is immediate from Condition i) of Definition 3.2.1 for the linear part and from the definition of $\mathcal{L}^\perp$ in the case of the bilinear part. Note that this implies via the decomposition into linear and bilinear parts, that $\Qoppa_{|(\mathcal{L}^\perp)^{\mathrm{op}}}$ also descends along the projection $(\mathcal{L}^\perp)^{\mathrm{op}} \to (\mathcal{L}^\perp/\mathcal{L})^{\mathrm{op}}$, as claimed above. It furthermore implies that the hermitian structure on $\mathrm{Hlgy}(\mathcal{L})$ is also right Kan extended along this map, which we will use below.

For the equivalence of hermitian ∞-categories claimed in the statement, note first that by Lemma A.2.5 and the comments thereafter the cofibre of the counit $pX \to X$ constitutes a right adjoint q to the localisation $\mathcal{L}^\perp \to \mathrm{Hlgy}(\mathcal{L})$. In fact, Lemma A.2.5 implies that q is an equivalence onto the kernel of $p\colon \mathcal{L}^\perp \to \mathcal{L}$, which is $\mathrm{D}_{\Omega}(\mathcal{L}^\perp) \cap \mathcal{L}^\perp$ by Remark 3.2.3. In particular, q is also a right adjoint to the composite

$$c\colon \mathcal{L}^\perp \cap \mathrm{D}(\mathcal{L}^\perp) \to \mathrm{Hlgy}(\mathcal{L})$$

from the statement, which is thus also an equivalence. Now, right Kan extensions are computed by pullback along left adjoints, so the hermitian structure on $\mathrm{Hlgy}(\mathcal{L})$ is given by $\Omega \circ q^{\mathrm{op}}$, which promotes q and thus c to an equivalence of hermitian ∞-categories.

Finally, $\mathcal{L}^\perp \cap \mathrm{D}(\mathcal{L}^\perp)$ is evidently closed under D_{Ω} so forms a Poincaré subcategory of $\mathcal{C}$, whence also $\mathrm{Hlgy}(\mathcal{L})$ is Poincaré. The statement about the duality in $\mathrm{Hlgy}(\mathcal{L})$ then follows from the formula for the inverse q of c. ☐

3.2.7. **Definition.** Let $\mathcal{L} \subseteq \mathcal{C}$ be an isotropic subcategory of a Poincaré ∞-category $(\mathcal{C}, \Omega)$. We say that $\mathcal{L}$ is a *Lagrangian* if $\mathrm{Hlgy}(\mathcal{L}) = 0$. We say that $(\mathcal{C}, \Omega)$ is *metabolic* if it contains a Lagrangian subcategory.

As mentioned, Remark 3.2.2 shows that this definition of Lagrangian agrees with that discussed in Definition [I].7.3.10.

3.2.8. **Remark.** By Lemma A.1.8, an isotropic subcategory $\mathcal{L} \subseteq \mathcal{C}$ is a Lagrangian if and only if the inclusion $\mathcal{L} \subseteq \mathcal{L}^\perp$ is an equivalence. Condition ii) of Definition 3.2.1 therefore yields a Verdier sequence

$$\mathcal{L} \longrightarrow \mathcal{C} \longrightarrow \mathcal{L}^{\mathrm{op}}$$

exhibiting $\mathcal{C}$ as an extension of $\mathcal{L}$ by $\mathcal{L}^{\mathrm{op}}$, where the right functor takes X to $\mathrm{fib}(\mathrm{D}_{\Omega}X \to \mathrm{D}_{\Omega}pX)$ (and p denotes the right adjoint of the inclusion $\mathcal{L} \subseteq \mathcal{C}$). Furthermore Lemma A.2.5 shows that the right functor in this Verdier sequence admits a right adjoint as well.

3.2.9. **Examples.**

i) We showed in Proposition [I].7.3.11 that a Poincaré ∞-category $(\mathcal{C}, \Omega)$ is metabolic if and only if it is of the form $\mathrm{Pair}(\mathcal{D}, \Phi)$ for some hermitian ∞-category $(\mathcal{D}, \Phi)$. In fact, the Lagrangians in $(\mathcal{C}, \Omega)$ are in one-to-one correspondence with representations of $(\mathcal{C}, \Omega)$ as an ∞-category of pairings. Particular examples are the inclusion $\mathcal{C} \to \mathrm{Met}(\mathcal{C}, \Omega)$ as the equivalences, and $\mathcal{C} \times 0 \subseteq \mathrm{Hyp}(\mathcal{C})$.

ii) Extending the Lagrangian of the metabolic category, the full subcategory inclusion $i\colon \mathcal{C} \hookrightarrow \mathrm{Q}_1(\mathcal{C})$ of (29) sending x to $0 \leftarrow x \to x$ gives an isotropic subcategory $\mathcal{L}$, the adjoint p witnessing Condition ii) of Definition 3.2.1 given by

$$[X \leftarrow Y \to Z] \longmapsto [0 \leftarrow \mathrm{fib}(Y \to X) \to \mathrm{fib}(Y \to X)].$$

Thus $\mathrm{D}_{\Omega_1}\mathcal{L}^\perp = \ker(p)$ is spanned by all diagrams with left pointing arrow an equivalence, whereas $\mathcal{L}^\perp$ itself consists of all diagram with right hand arrow an equivalence. Thus $\mathrm{Hlgy}(\mathcal{L}) \simeq (\mathcal{C}, \Omega)$ embedded as the constant diagrams.

iii) More generally, one can consider the inclusion $j_n\colon \mathcal{C} \to \mathrm{Q}_n(\mathcal{C})$ as those diagrams which vanish away from $\{(i \leq n) \mid i \in \{0, \ldots n\}\}$, and are constant on that subposet, i.e.

$$\begin{tikzcd}
 & & 0 \arrow[dl] \arrow[dr] & \cdots & \cdots \arrow[dl] \arrow[dr] & \cdots & X \arrow[dl] \arrow[dr, equal] & & \\
 & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr, equal] & \\
0 & & 0 & & \cdots & & 0 & & X
\end{tikzcd}$$

Formally this can be given by taking the embedding $\mathcal{C} \to \mathrm{Q}_1(\mathcal{C})$ considered in the previous example and composing with the degeneracy $[n] \to [1]$ sending n to 1 and everything else to 0. Using the Segal property of Lemma 2.2.5 it is not difficult to see that the requisite adjoint p is given by taking a diagram $\varphi \in \mathrm{Q}_n(\mathcal{C})$ to image of the fibre of the last left pointing arrow, namely $\varphi(n-1 \leq n) \to \varphi(n-1 \leq n-1)$. It follows that $\mathrm{D}_{\Omega_n}(\mathcal{L}^\perp) = \ker(p)$ consists of all those diagrams φ with the arrow $\varphi(n-1 \leq n) \to \varphi(n-1 \leq n-1)$ an equivalence (and thus all arrows $\varphi(i \leq n) \to \varphi(i \leq n-1)$ equivalences as well).

From the explicit formula for the duality of $\mathrm{Q}_1(\mathcal{C}, Ϙ)$ from Example 2.2.3 i), it then follows that $\mathcal{L}^\perp$ is spanned by the diagrams with the last right pointing arrow $\varphi(n-1 \le n) \to \varphi(n \le n)$ an equivalence, and so in total $\mathrm{Hlgy}(\mathcal{L}) \simeq \mathrm{Q}_{n-1}(\mathcal{C}, Ϙ)$ embedded in $\mathrm{Q}_n(\mathcal{C}, Ϙ)$ via the degeneracy s_{n-1}.

iv) There are several other interesting isotropic subcategories of $\mathrm{Q}_n(\mathcal{C}, Ϙ)$: For example, let $\mathcal{L}_n^+ \subseteq \mathrm{Q}_n(\mathcal{C})$ be the full subcategory spanned by those diagrams $\varphi\colon \mathrm{TwAr}[n]^{\mathrm{op}} \to \mathcal{C}$ for which $\varphi(0 \le 0) = 0$ and $\varphi(0 \le j) \to \varphi(i \le j)$ is an equivalence for $i \le j \in [n]$, i.e.

$$\begin{tikzcd}[column sep=small]
 & & X_2 \arrow[dl] \arrow[dr, equal] & & \cdots \arrow[dl] \arrow[dr, equal] & & X_n \arrow[dl] \arrow[dr, equal] & & \\
 & X_1 \arrow[dl] \arrow[dr, equal] & & X_2 \arrow[dl] \arrow[dr, equal] & & \cdots \arrow[dl] \arrow[dr, equal] & & X_n \arrow[dl] \arrow[dr, equal] & \\
0 & & X_1 & & X_2 & & \cdots & & X_n
\end{tikzcd}$$

Then $\mathcal{L}_n^+ \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ is an isotropic subcategory: To give the right adjoint p_n of the inclusion $\mathcal{L}_n^+ \hookrightarrow \mathrm{Q}_n(\mathcal{C})$, let $\rho_n^+\colon \Delta^n \to \mathrm{TwAr}(\Delta^n)$ denote the functor $k \mapsto (0 \le k)$. Then p_n sends $\varphi\colon \mathrm{TwAr}([n])^{\mathrm{op}} \to \mathcal{C}$ to the left Kan extension along $\rho_n^+\colon [n] \to \mathrm{TwAr}([n])$ of the functor

$$[n] \longrightarrow \mathcal{C} \qquad j \mapsto \mathrm{fib}(\varphi(0 \le j) \to \varphi(0 \le 0)).$$

In particular, the ∞-category $\mathrm{D}(\mathcal{L}_n^+)^\perp$ consists exactly of those diagrams that are right Kan extended from the image of ρ_n^+ and $(\mathcal{L}_n^+)^\perp$ is dually spanned by those diagrams that are left Kan extended from the subposet spanned by the various $(i \le n)$. The homology $\mathrm{Hlgy}(\mathcal{L}_n^+) \subseteq \mathrm{Q}_n(\mathcal{C}, Ϙ)$ is consequently given by the full Poincaré subcategory of constant diagrams.

v) The isotropic subcategory $\mathcal{L}_{n+1}^+ \simeq \mathrm{Fun}(\Delta^n, \mathcal{C})$ from the previous example agrees with that from the proof of Proposition 2.4.9 upon restriction to $\mathrm{Null}_n(\mathcal{C}, Ϙ) \subseteq \mathrm{Q}_{n+1}(\mathcal{C}, Ϙ)$. We showed there, that it is a Lagrangian in $\mathrm{Null}_n(\mathcal{C}, Ϙ)$, and this follows again from the considerations above.

In generalisation of Corollary 3.1.3, we now set out to prove:

3.2.10. **Theorem** (Isotropic decomposition theorem). *Let $(\mathcal{C}, Ϙ)$ be a Poincaré ∞-category and $i\colon \mathcal{L} \to \mathcal{C}$ be the inclusion of an isotropic subcategory. Let $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ be a group-like additive functor. Then the Poincaré functors*

$$i_{\mathrm{hyp}}\colon \mathrm{Hyp}(\mathcal{L}) \longrightarrow (\mathcal{C}, Ϙ) \quad \textit{and} \quad \imath\colon \mathrm{Hlgy}(\mathcal{L}) \longrightarrow (\mathcal{C}, Ϙ)$$

from (35) *induce an equivalence*

$$\mathcal{F}(\mathrm{Hyp}(\mathcal{L})) \times \mathcal{F}(\mathrm{Hlgy}(\mathcal{L})) \longrightarrow \mathcal{F}(\mathcal{C}, Ϙ). \tag{36}$$

We will explicitly construct an inverse to the map appearing in the theorem.

3.2.11. **Construction.** Fix a Poincaré ∞-category $(\mathcal{C}, Ϙ)$ and the inclusion $i\colon \mathcal{L} \to \mathcal{C}$ of an isotropic subcategory with right adjoint p. We note that the counit $ip \to \mathrm{id}_{\mathcal{C}}$ defines a surgery datum on the Poincaré object $\mathrm{id}_{(\mathcal{C}, Ϙ)}$ of $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{C}, Ϙ))$. Performing surgery as in Proposition 2.4.3, we obtain a Poincaré object in $\mathrm{Q}_1(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{C}, Ϙ)))$, in other words, a Poincaré functor $\phi\colon (\mathcal{C}, Ϙ) \to \mathrm{Q}_1(\mathcal{C}, Ϙ)$. By construction, $d_1 \circ \phi = \mathrm{id}$, and we denote by h the composite

$$d_0 \circ \phi\colon (\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ),$$

that is, the result of surgery, giving in total a cobordism

$$\begin{tikzcd}
 & g \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathrm{id} & & h.
\end{tikzcd} \tag{37}$$

By construction

$$\phi(c)\colon \big(c \leftarrow g(c) \to h(c)\big)$$

is obtained first by forming the fibre $g(c) \to c$ of the composite map $c \simeq \mathrm{D}_Ϙ\mathrm{D}_Ϙ(c) \to \mathrm{D}_Ϙ(ip\mathrm{D}_Ϙ(c))$ where the second map is the dual of the counit, and then by forming the cofibre $g(c) \to h(c)$ of the canonically induced map $ip(c) \to g(c)$.

3.2.12. **Lemma.** *The functor $h\colon \mathcal{C} \to \mathcal{C}$ factors through the inclusion $\mathcal{L}^\perp \cap \mathrm{D}_\Qoppa(\mathcal{L}^\perp) \subseteq \mathcal{C}$, and the Poincaré enhancement furnished by Construction 3.2.11 canonically factors as*

$$(\mathcal{C}, \Qoppa) \xrightarrow{\widetilde{h}} \mathrm{Hlgy}(\mathcal{L}) \xrightarrow{\iota} (\mathcal{C}, \Qoppa)$$

and $\widetilde{h} \circ \iota \simeq \mathrm{id}_{\mathrm{Hlgy}(\mathcal{L})}$. In particular, if $\mathcal{L}$ is Lagrangian then $h = 0$.

Proof. For the first part, we observe that both the cofibre of $ipc \to c$ and $\mathrm{D}_\Qoppa ip \mathrm{D}_\Qoppa c$ belong to $\mathrm{D}_\Qoppa(\mathcal{L}^\perp)$: The former because $\mathrm{D}_\Qoppa(\mathcal{L}^\perp) = \ker(p)$ by Remark 3.2.3 and for the latter, we simply note $p\mathrm{D}_\Qoppa X \in \mathcal{L} \subseteq \mathcal{L}^\perp$. Since hc participates in a cofibre sequence

$$hX \longrightarrow \mathrm{cof}(ipc \to c) \longrightarrow \mathrm{D}_\Qoppa(ip\mathrm{D}_\Qoppa(c)),$$

also $hc \in \mathrm{D}_\Qoppa(\mathcal{L}^\perp)$. Since h commutes with the duality, its image is then also contained in $\mathcal{L}^\perp$.

For the second claim, note that $ip(c) \simeq 0$ for $c \in \mathcal{L}^\perp \cap \mathrm{D}_\Qoppa(\mathcal{L}^\perp)$, since $\ker(p) = \mathrm{D}_\Qoppa(\mathcal{L}^\perp)$ by Remark 3.2.3. Thus the cobordism (37) consists of equivalences in this case. The third claim is immediate from Proposition 3.2.6. ☐

3.2.13. **Proposition.** *The functors*

$$p^{\mathrm{hyp}}\colon (\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Hyp}(\mathcal{L}) \quad \textit{and} \quad (\mathcal{C}, \Qoppa) \xrightarrow{\widetilde{h}} \mathrm{Hlgy}(\mathcal{L})$$

combine into a left inverse of the Poincaré functor

$$(i_{\mathrm{hyp}}, \iota)\colon \mathrm{Hyp}(\mathcal{L}) \oplus \mathrm{Hlgy}(\mathcal{L}) \longrightarrow (\mathcal{C}, \Qoppa)$$

from Theorem 3.2.10.

Proof. Consider the composite

$$\mathrm{Hyp}(\mathcal{L}) \oplus \mathrm{Hlgy}(\mathcal{L}) \xrightarrow{(i_{\mathrm{hyp}}, \iota)} (\mathcal{C}, \Qoppa) \xrightarrow{(p^{\mathrm{hyp}}, \widetilde{h})} \mathrm{Hyp}(\mathcal{L}) \oplus \mathrm{Hlgy}(\mathcal{L}).$$

We analyse all four components in turn. That $\widetilde{h}\iota \simeq \mathrm{id}\colon \mathrm{Hlgy}(\mathcal{L}) \to \mathrm{Hlgy}(\mathcal{L})$ is part of Lemma 3.2.12. For the self-map of the $\mathrm{Hyp}(\mathcal{L})$-component, we have that

$$p^{\mathrm{hyp}} i_{\mathrm{hyp}}(x, y) = p^{\mathrm{hyp}}(i(x) \oplus \mathrm{D}_\Qoppa i(y)) = (pi(x) \oplus p\mathrm{D}_\Qoppa i(y), p\mathrm{D}_\Qoppa i(x) \oplus pi(y)).$$

Since $i^*\Qoppa$ vanishes, it follows that $\mathrm{Hom}_{\mathcal{C}}(i(x), \mathrm{D}_\Qoppa i(x)) = 0 = \mathrm{Hom}_{\mathcal{C}}(i(y), \mathrm{D}_\Qoppa i(y))$ and hence $p\mathrm{D}_\Qoppa i(x) = 0 = p\mathrm{D}_\Qoppa i(y)$. We may then conclude that

$$p^{\mathrm{hyp}} i_{\mathrm{hyp}}(x, y) = (pi(x), pi(y)) = \mathrm{Hyp}(pi)\colon \mathrm{Hyp}(\mathcal{L}) \longrightarrow \mathrm{Hyp}(\mathcal{L})$$

and hence the unit equivalence $\mathrm{id}_{\mathcal{L}} \to pi$ induces an equivalence $\mathrm{id}_{\mathrm{Hyp}(\mathcal{L})} \to p^{\mathrm{hyp}} i_{\mathrm{hyp}}$, as desired.

The map

$$p^{\mathrm{hyp}} \circ \iota\colon \mathrm{Hlgy}(\mathcal{L}) \to \mathrm{Hyp}(\mathcal{L})$$

vanishes since $p^{\mathrm{hyp}}\iota = (p\iota)^{\mathrm{hyp}}$ and $\mathcal{L}^\perp \cap \mathrm{D}_\Qoppa \mathcal{L}^\perp \subseteq \ker(p)$ by Remark 3.2.3.

Finally, we similarly have $\widetilde{h} \circ i_{\mathrm{hyp}} \simeq (\widetilde{h} \circ i)_{\mathrm{hyp}}$, and $\widetilde{h} \circ i \simeq 0$, since $p\mathrm{D}_\Qoppa i(x) \simeq 0$ as observed above so that $c \to g(c)$ is an equivalence. ☐

Proof of Theorem 3.2.10. It only remains to show that the composite map

$$\mathcal{F}(\mathcal{C}, \Qoppa) \xrightarrow{(p_*^{\mathrm{hyp}}, \widetilde{h}_*)} \mathcal{F}(\mathrm{Hyp}(\mathcal{L})) \times \mathcal{F}(\mathrm{Hlgy}(\mathcal{C})) \xrightarrow{(i_*^{\mathrm{hyp}}, \iota_*)} \mathcal{F}(\mathcal{C}, \Qoppa)$$

is homotopic to the identity. But this now readily follows by applying Proposition 3.1.7 to the cobordism of Construction 3.2.11 and observing that ip is identified by construction with the fibre of $\beta\colon g \to h$. ☐

In generalisation of Corollary 3.1.4, we thus find:

3.2.14. **Corollary.** *Let $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ be a group-like additive functor and let $(\mathcal{C}, \Qoppa)$ be a Poincaré ∞-category. If $(\mathcal{C}, \Qoppa)$ is metabolic with Lagrangian $\mathcal{L} \subseteq \mathcal{C}$ then $\mathcal{F}(\mathcal{C}, \Qoppa) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{L}))$.*

As a simple application, we have:

3.2.15. **Proposition.** *Given a left split Verdier sequence*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{D} \arrow[l, bend right, "g"'] \arrow[r, "p"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{E}, \arrow[l, bend right, "q"'] \end{tikzcd}$$

the functors

$$\mathcal{F}\,\mathrm{Hyp}(q) + \mathcal{F}\,\mathrm{Hyp}(f) \colon \mathcal{F}\,\mathrm{Hyp}(\mathcal{E}) \times \mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \longleftrightarrow \mathcal{F}\,\mathrm{Hyp}(\mathcal{D}) \ \colon (\mathcal{F}\,\mathrm{Hyp}(p), \mathcal{F}\,\mathrm{Hyp}(g))$$

are inverse equivalences for all group-like additive $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$.

An analogous statement of course holds for right split Verdier sequences. Note, in particular, that applying this proposition to the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{E}$ recovers the fact that any group-like additive functor $\mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{E}$ takes left (or right) split Verdier sequences to split fibre sequences, as it displays $(\mathcal{F}(p), \mathcal{F}(g))$ as a retract of an equivalence. We already noted this with a direct proof in 2.7.6.

Proof. The functor

$$q \times f^{\mathrm{op}} \colon \mathcal{E} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D} \times \mathcal{D}^{\mathrm{op}} = \mathrm{Hyp}(\mathcal{D})$$

is the inclusion of a Lagrangian: It is isotropic since

$$\hom_{\mathcal{D}}(q(e), f(c)) \simeq \hom_{\mathcal{E}}(e, pf(c)) \simeq 0.$$

To see that $(\mathcal{E} \times \mathcal{C}^{\mathrm{op}})^{\perp}$ is no larger, we compute

$$\mathrm{B}_{\Qoppa_{\mathrm{hyp}}}((q(e), f(c)), (y, y')) \simeq \hom_{\mathcal{D}}(q(e), y') \oplus \hom_{\mathcal{D}}(y, f(c)) \simeq \hom_{\mathcal{E}}(e, p(y')) \oplus \hom_{\mathcal{C}}(g(y), c)$$

whence Corollary A.2.8 gives the claim. Thus Theorem 3.2.10 shows that

$$\mathcal{F}\,\mathrm{Hyp}(q) + \mathcal{F}\,\mathrm{Hyp}(f) = (q \times f^{\mathrm{op}})_{\mathrm{hyp}} \colon \mathcal{F}\,\mathrm{Hyp}(\mathcal{E}) \oplus \mathcal{F}\,\mathrm{Hyp}(C) \longleftrightarrow \mathcal{F}\,\mathrm{Hyp}(\mathcal{D})$$

is an equivalence, and the second map in the statement is evidently a right inverse thereof. □

Next, we use Theorem 3.2.10 to analyse the values of $\mathrm{Q}_n(\mathcal{C}, \Qoppa)$ under group-like additive functors. To state the result consider the functor $f_n \colon \mathrm{Q}_n(\mathcal{C}, \Qoppa) \to \mathcal{C}^n$ taking fibres of the left pointing maps along the bottom of a diagram X, i.e.

$$X \longmapsto \big[\mathrm{fib}\big(X(0 \le 1) \to X(0 \le 0)\big), \dots, \mathrm{fib}\big(X(n-1 \le n) \to X(n-1 \le n-1)\big)\big].$$

We then have:

3.2.16. **Proposition.** *The functors*

$$v_n \colon \mathrm{Q}_n(\mathcal{C}, \Qoppa) \to (\mathcal{C}, \Qoppa) \quad \textit{and} \quad f_n^{\mathrm{hyp}} \colon \mathrm{Q}_n(\mathcal{C}, \Qoppa) \to \mathrm{Hyp}(\mathcal{C})^n$$

the former induced by the inclusion $[0] \to [n]$, *combine into an equivalence*

$$\mathcal{F}(\mathrm{Q}_n(\mathcal{C}, \Qoppa)) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^n \oplus \mathcal{F}(\mathcal{C}, \Qoppa)$$

for every group-like additive $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$. *In fact, these equivalences give an identification of the simplicial* E_{∞}*-group* $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ *in* $\mathcal{E}$ *with the bar construction of* $\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ *acting on* $\mathcal{F}(\mathcal{C}, \Qoppa)$ *via the hyperbolisation map* $\mathrm{hyp} \colon \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}(\mathcal{C}, \Qoppa)$.

In particular, it follows that $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa) \simeq \mathrm{asscat}\,\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})$ is a groupoid, whenever $\mathcal{F}$ is group-like (and additive).

Proof. We proceed by induction. For $n = 0$ there is nothing to show. Using the isotropic subcategory $j_{n+1} \colon \mathcal{C} \to \mathrm{Q}_{n+1}(\mathcal{C})$ described in Example 3.2.9 iii), we find an equivalence

$$((j_{n+1})_{\mathrm{hyp}}, s_n) \colon \mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \times \mathcal{F}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa) \longrightarrow \mathcal{F}\,\mathrm{Q}_{n+1}(\mathcal{C}, \Qoppa)$$

as a consequence of Theorem 3.2.10. It is readily checked that this equivalence translates the map $(f^{\mathrm{hyp}}_{n+1}, v_{n+1})$ to the matrix

$$\begin{pmatrix} 0 & f_n^{\mathrm{hyp}} \\ \mathrm{id}_{\mathrm{Hyp}(\mathcal{C})} & 0 \\ 0 & v_n \end{pmatrix} \colon \mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \times \mathcal{F}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa) \longrightarrow \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^n \times \mathcal{F}\,\mathrm{Hyp}(\mathcal{C}) \times \mathcal{F}(\mathcal{C}, \Qoppa).$$

This matrix represents an equivalence by inductive assumption, which implies the first claim.

To obtain an identification with the bar construction, we first note that the bar construction $\mathrm{B}(M, R, N)$ of an action of R on N from the left and on M from the right in a semi-additive ∞-category is the left Kan extension along the inclusion of the coequaliser diagram $(\Delta^{\leq 1}_{\mathrm{inj}})^{\mathrm{op}}$ into Δ^{op} of the diagram

$$M \oplus R \oplus N \rightrightarrows M \oplus N$$

containing the two action maps; this follows directly by evaluation of the pointwise formulae for left Kan extensions. By the calculations above, $d_1 : \mathcal{F}\,\mathrm{Q}_1(\mathcal{C}, \Qoppa)) \to \mathcal{F}(\mathcal{C}, \Qoppa)$ is identified with the projection $\mathcal{F}(\mathrm{Hyp}) \times \mathcal{F}(\mathcal{C}, \Qoppa) \to \mathcal{F}(\mathcal{C}, \Qoppa)$ and it is readily checked that $d_0 : \mathcal{F}\,\mathrm{Q}_1(\mathcal{C}, \Qoppa) \to \mathcal{F}(\mathcal{C}, \Qoppa)$ gets identified with the sum of the identity of $\mathcal{F}(\mathcal{C}, \Qoppa)$ and the hyperbolisation map under the equivalence of Proposition 3.2.16. We therefore obtain a map of simplicial objects

$$\mathrm{B}(0, \mathcal{F}(\mathrm{Hyp}(\mathcal{C})), \mathcal{F}(\mathcal{C}, \Qoppa)) \longrightarrow \mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa))$$

and one readily unwinds the construction to find it given by the maps we just checked to be equivalences. □

Recall the higher metabolic ∞-categories $\mathrm{Null}_n(\mathcal{C}, \Qoppa) = \mathrm{fib}(\mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa) \xrightarrow{d} (\mathcal{C}, \Qoppa))$, where d is induced by the inclusion $[0] \to [1+n]$, from Definition 2.4.8:

3.2.17. **Corollary.** *The functors* $f^{\mathrm{hyp}}_{n+1} : \mathrm{Null}_n(\mathcal{C}, \Qoppa) \to \mathrm{Hyp}(\mathcal{C})^n$ *induce an equivalence*

$$\mathcal{F}\,\mathrm{Q}(\mathrm{Null}_n(\mathcal{C}, \Qoppa)) \simeq \mathrm{B}\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$$

for every group-like additive $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$.

Proof. Simply note that the sequence defining $\mathrm{Null}_n(\mathcal{C}, \Qoppa)$ is in fact a split Poincaré-Verdier sequence by Lemma 2.2.8, so gives rise to a fibre sequence after applying $\mathcal{F}$. The result then follows immediately from Proposition 3.2.16. □

3.2.18. **Remarks.**

i) The identification from Proposition 3.2.16 also shows that for group-like $\mathcal{F}$, the Segal space $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ is complete if and only if $\mathcal{F}(\mathrm{Hyp}\,\mathcal{C})$ vanishes (see §3.5 below for a detailed discussion of such functors): For a bar construction as above, the entirety of $\mathrm{B}(M, R, N)_1 = M \oplus R \oplus N$ consists of equivalences, so it is complete if and only if $R = 0$.

ii) We based the proof of Proposition 3.2.16 on the isotropic decomposition theorem 3.2.10, but Proposition 3.2.16 can also be obtained directly using the Segal property of the simplicial space $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$ and the bar construction, together with the computation of $\mathcal{F}\,\mathrm{Q}_1(\mathcal{C}, \Qoppa)$ from Corollary 3.1.3; we leave the details to the reader.

iii) In Example 3.2.9 v) we constructed a Lagrangian $\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Null}_n(\mathcal{C}, \Qoppa)$ and Theorem 3.2.10 therefore directly yields

$$\mathcal{F}\mathrm{Null}_n(\mathcal{C}, \Qoppa) \simeq \mathcal{F}\,\mathrm{Hyp}\,\mathrm{Fun}(\Delta^n, \mathcal{C}).$$

This formula also implies Corollary 3.2.17 by an iterative application of the splitting lemma; a similar discussion applies to $\mathcal{F}(\mathrm{Q}_n(\mathcal{C}, \Qoppa))$, we again leave the details to the reader. Interestingly, our proof of Corollary 3.2.17 does not yield the assertion that $\mathrm{Null}_n(\mathcal{C}, \Qoppa)$ is metabolic; indeed for $n \geq 3$ we are not aware of an isotropically embedded $\mathcal{C}^n \to \mathrm{Q}_n(\mathcal{C}, \Qoppa)$.

3.3. **The group-completion of an additive functor.** Our goal in this section is to study the behaviour of space-valued additive functors under the hermitian Q-construction, or equivalently of the assignment $\mathcal{F} \mapsto |\mathrm{Cob}^{\mathcal{F}}(-)|$. This is based on the following observation: For any additive functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$, there is a natural cartesian square

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)_{0/} \arrow[d] \\
\{0\} \arrow[r] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)
\end{tikzcd}$$

in Cat_∞, since $\mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)}(0, 0) \simeq \mathcal{F}(\mathcal{C}, \Qoppa)$, which is immediate from our discussion of Segal spaces in §2.3. As an application of the isotropic decomposition principle, we saw in Proposition 3.2.16 that $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)$ is a groupoid if $\mathcal{F}$ is assumed group-like, and thus $\mathcal{F} \simeq \Omega|\mathrm{Cob}^{\mathcal{F}}|$ in this case. This allows one to recognise $\Omega|\mathrm{Cob}^{\mathcal{F}}-|$ as the group-completion of a not-necessarily group-like additive $\mathcal{F}$ by means of [Lur09a, Proposition 5.2.7.4 (3)] and explicit inspection. In the present section, we take a slightly different route and show, moreover, that the realisation of the square above exhibits $|\mathrm{Cob}^{\mathcal{F}}|$ as the suspension of $\mathcal{F}$ in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$. The requisite analysis also allows for a somewhat more direct proof that $\mathcal{F} \simeq \Omega|\mathcal{F}\,\mathrm{Q}-|$ whenever $\mathcal{F}$ is group-like, bypassing the translation from Segal spaces to ∞-categories.

We thus consider the corresponding statement at the level of the Segal spaces $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$. We start by constructing the corresponding model for the cartesian square above. Recall the décalage $\mathrm{dec}(S)$ of a simplicial object S, i.e. $\mathrm{dec}(S)_n \simeq S_{1+n}$, and that we have

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)_{0/} \simeq \mathrm{asscat}(\mathcal{F}\mathrm{Null}(\mathcal{C}, \Qoppa))$$

from Lemma 3.1.11, where the higher metabolic ∞-categories $\mathrm{Null}_n(\mathcal{C}, \Qoppa)$ are given as the fibre of

$$\mathrm{dec}(\mathrm{Q}(\mathcal{C}, \Qoppa)) \longrightarrow \mathrm{Q}_0(\mathcal{C}, \Qoppa) = (\mathcal{C}, \Qoppa). \tag{38}$$

Considering the face map $d_0\colon [n] \to [1+n]$ as a natural transformation $\Delta^n \Rightarrow \Delta^{1+n}$ yields a map of simplicial objects

$$\pi\colon \mathrm{Null}(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}(\mathcal{C}, \Qoppa), \tag{39}$$

which models the right vertical map in the square above.

3.3.1. **Lemma.** *The simplicial objects* $\mathrm{Null}(\mathcal{C}, \Qoppa)$ *and* $\mathrm{dec}(\mathrm{Q}(\mathcal{C}, \Qoppa))$ *extend to split simplicial objects over the zero Poincaré ∞-category and* $(\mathcal{C}, \Qoppa)$*, respectively.*

In particular, $|\mathcal{F}\mathrm{dec}(\mathrm{Q}(\mathcal{C}, \Qoppa))| \simeq \mathcal{F}(\mathcal{C}, \Qoppa)$ by [Lur09a, Lemma 6.1.3.16] (which also defines split simplicial objects).

Proof. By construction the augmented simplicial object (38) is split, which gives both results. ☐

Now, to describe the final map from the square, let $\iota_n\colon \mathcal{C} \to \mathrm{Null}_n(\mathcal{C}, \Qoppa)$ be the simplicial map which at level n is given by the exact functor which sends $x \in \mathcal{C}$ to the diagram $\varphi_x\colon \mathrm{TwAr}[n] \to \mathcal{C}$ given by

$$\varphi_x(i \leq j) = \begin{cases} x & 0 = i < j \\ 0 & \text{otherwise} \end{cases}$$

in which all the maps between the various x's are identities. We note that the image of ι_n is contained in the kernel of

$$\pi_n\colon \mathrm{Null}_n(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Q}_n(\mathcal{C}, \Qoppa) \tag{40}$$

3.3.2. **Lemma.** *The functor* $\iota_n\colon \mathcal{C} \to \mathrm{Null}_n(\mathcal{C}, \Qoppa)$ *determines an equivalence of stable ∞-categories between* $\mathcal{C}$ *and the kernel of* (40)*. In addition, the restriction of the quadratic functor of* $\mathrm{Null}_n(\mathcal{C}, \Qoppa)$ *to* $\mathcal{C}$ *along* ι_n *is naturally equivalent to* $\Qoppa^{[-1]}$.

Proof. By definition, the kernel of (40) consists of those $\varphi\colon \mathrm{TwAr}[n+1]^{\mathrm{op}} \to \mathcal{C}$ in $\mathrm{Q}_{n+1}(\mathcal{C}, \Qoppa)$ such that $\varphi(i \leq j) = 0$ if either $(i \leq j) = (0 \leq 0)$ or $i \geq 1$. The only non-zero entries of such a functor are hence $\varphi(0 \leq j)$ for $j \geq 1$, and for $1 \leq i \leq j$ the maps $\varphi(0 \leq j) \to \varphi(0 \leq i)$ are equivalences by the exactness conditions of Definition 2.2.1, since $\varphi(1 \leq i) = \varphi(1 \leq j) = 0$. Conversely, every functor $\varphi\colon \mathrm{TwAr}[n+1]^{\mathrm{op}} \to \mathcal{C}$ which satisfies these vanishing conditions and for which $\varphi(0 \leq j) \to \varphi(0 \leq i)$ are equivalences satisfies all the exactness conditions of Definition 2.2.1. We may hence conclude that ι_{n+1} yields an equivalence between $\mathcal{C}$ and kernel of (40), since the elements $(0 \leq j)$ span a contractible ∞-category. To finish the proof, we note that for $x \in \mathcal{C}$, we have

$$\lim_{(i\leq j)\in\mathrm{TwAr}[n+1]^{\mathrm{op}}} \Qoppa(\varphi_x(i \leq j)) = \lim_{(i\leq j)\in\mathcal{J}^{\mathrm{op}}_{n+1}} \Qoppa(\varphi_x(i \leq j)) \simeq 0 \times_{\Qoppa(x)} 0 = \Omega\Qoppa(x)$$

where $\mathcal{J}_{n+1} \subseteq \mathrm{TwAr}[n+1]$ is the cofinal full subposet of the twisted arrow category spanned by the arrows of the form $(i \leq j)$ for $j \leq i+1$; see Examples 2.2.3. ☐

In light of Lemma 3.3.2, we now obtain a fibre sequence of simplicial Poincaré ∞-categories

$$\mathrm{const}(\mathcal{C}, \Qoppa^{[-1]}) \xrightarrow{\iota} \mathrm{Null}(\mathcal{C}, \Qoppa) \xrightarrow{\pi} \mathrm{Q}(\mathcal{C}, \Qoppa). \tag{41}$$

3.3.3. **Observation.** *The sequence* (41) *is a split Poincaré-Verdier sequence in each degree: By Lemma 2.2.8, the maps* $d_0 \colon \mathrm{Q}_{1+n}(\mathcal{C}, \mathfrak{P}) \to \mathrm{Q}_n(\mathcal{C}, \mathfrak{P})$ *are split Poincaré-Verdier projections, and the left adjoint of* d_0 *is given via extension by* 0 *and thus factors through the underlying* ∞*-categories of* $\mathrm{Null}_n(\mathcal{C}, \mathfrak{P}) \to \mathrm{Q}_{1+n}(\mathcal{C}, \mathfrak{P})$*, whence Corollary 1.2.3 gives the claim.*

As desired, applying an additive functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ levelwise to the sequence (41) yields a sequence of Segal spaces which corresponds to the fibre sequence of ∞-categories

$$\mathcal{F}(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \mathfrak{P})_{0/} \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \mathfrak{P}).$$

Here, the second functor is the canonical projection as in Lemma 2.4.7 and the first functor is informally given by sending a Poincaré object x to the cobordism $[0 \leftarrow x \rightarrow 0]$. In total, we have thus modelled the square from the start of this section.

We can now formulate the main result of the present section:

3.3.4. **Theorem.** *Let* $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *be an additive functor and consider the square of space valued functors*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & {|\mathcal{F}\mathrm{Null}(-^{[1]})|} \arrow[d] \\
* \arrow[r] & {|\mathcal{F}\,\mathrm{Q}(-^{[1]})|}
\end{tikzcd} \tag{42}$$

obtained from the sequence (41)*. Then we have:*

- *i) The square is cocartesian in* $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$*, and so exhibits* $|\mathcal{F}\,\mathrm{Q}(-^{[1]})| \simeq |\mathrm{Cob}^{\mathcal{F}}(-)|$ *as the suspension of* $\mathcal{F}$ *in* $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$*, since the upper right corner is contractible.*
- *ii) If* $\mathcal{F}$ *is group-like then the square is also cartesian, yielding an equivalence*

$$\tau_{\mathcal{F}} \colon \mathcal{F} \longrightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|.$$

in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$.

Before giving the proof of Theorem 3.3.4, let us give some of its direct consequences. Given an additive functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$, the square (42) determines a natural map

$$\mathcal{F} \longrightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)| \tag{43}$$

in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$. The codomain of (43), being the loop of another additive functor, is always group-like. Our goal is to show that (43) exhibits $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ as universal among group-like additive functors receiving a map from $\mathcal{F}$. In other words, we claim that the association $\mathcal{F} \mapsto \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ realises the group-completion of $\mathcal{F}$ in the semi-additive ∞-category $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$. We need a general lemma:

3.3.5. **Proposition.** *Let* $\mathcal{E}$ *be a semi-additive* ∞*-category which admits suspensions and loops, and let* $\mathcal{E}_{\mathrm{grp}} \subseteq \mathcal{E}$ *be the full subcategory spanned by the group-like objects. Then the following holds:*

- *i) The full subcategory* $\mathcal{E}_{\mathrm{grp}} \subseteq \mathcal{E}$ *is closed under any limits and colimits that exist in* $\mathcal{E}$*, and both the suspension and loop functors* $\Sigma, \Omega \colon \mathcal{E} \to \mathcal{E}$ *have their image contained in* $\mathcal{E}_{\mathrm{grp}}$*. In particular, we may consider the monad* $\Omega\Sigma \colon \mathcal{E} \to \mathcal{E}$ *as a functor from* $\mathcal{E}$ *to* $\mathcal{E}_{\mathrm{grp}}$*.*
- *ii) If the suspension functor* $\Sigma \colon \mathcal{E}_{\mathrm{grp}} \to \mathcal{E}_{\mathrm{grp}}$ *is fully-faithful then the unit map* $u \colon \mathrm{id} \Rightarrow \Omega\Sigma$ *exhibits* $\Omega\Sigma$ *as left adjoint to the inclusion* $\mathcal{E}_{\mathrm{grp}} \to \mathcal{E}$*.*
- *iii) For every object* $A \in \mathcal{E}$*, the suspension of the unit* $\Sigma u \colon \Sigma A \to \Sigma\Omega\Sigma A$ *is an equivalence.*

Proof. The first claim follows from the fact that $x \in \mathcal{E}$ being group-like can be detected on the level of both the represented functor $\mathrm{Map}(-, x)$ and the corepresented functor $\mathrm{Map}(x, -)$ (which automatically take values in monoid objects since $\mathcal{E}$ is semi-additive), and that loop spaces are always group-like.

To prove the second claim, it suffices to check that under the given assumptions the natural transformations $u_{\Omega\Sigma x}, \Omega\Sigma u_x \colon \Omega\Sigma x \to \Omega\Sigma\Omega\Sigma x$ are both equivalences [Lur09a, Proposition 5.2.7.4]. But since $\Omega\Sigma$ is a monad, these two natural transformations admit a common section (the multiplication of the monad) and $\Sigma \colon \mathcal{E}_{\mathrm{grp}} \to \mathcal{E}_{\mathrm{grp}}$ being fully-faithful implies that u is a natural equivalence on all group-like objects of $\mathcal{E}$.

The final claim now follows from the triangle identities. □

Since $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$ is semi-additive by Lemma 1.5.7, we obtain the universal property of the hermitian Q-construction:

3.3.6. **Corollary.** *The natural map* $\mathcal{F} \to \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ *exhibits* $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ *as universal among group-like additive functors receiving a map from* $\mathcal{F}$*; that is, the operation* $\mathcal{F} \mapsto \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is left adjoint to the inclusion*

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{S})) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{S})) \simeq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}),$$

of group-like additive functors inside all additive functors.

We will therefore also denote $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ as $\mathcal{F}^{\mathrm{grp}}$ and refer to it as the group-completion of $\mathcal{F}$. Let us mention that the inclusion in the statement also admits a right adjoint simply given by taking units pointwise; taking units preserves all limits (and thus in particular additive functors) since it is itself a right adjoint. However, this right adjoint clearly annihilates both Pn and Cr, so we have little use for it.

Proof. Note only that Part ii) of Theorem 3.3.4 implies that the unit $\mathrm{id} \Rightarrow \Omega|\mathrm{Cob}^{(-)}|$ is an equivalence on all group-like additive functors. Thus $|\mathrm{Cob}^{(-)}|$ restricts to a fully faithful functor on $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{S}))$ and the previous proposition gives the claim. □

Part iii) of Proposition 3.3.5 together with Theorem 3.3.4 also immediately implies:

3.3.7. **Corollary.** *For every additive functor* $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *and* $(\mathcal{C}, \Qoppa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$*, the natural map*

$$|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)| \longrightarrow \mathrm{Cob}^{\mathcal{F}^{\mathrm{grp}}}(\mathcal{C}, \Qoppa)$$

is an equivalence.

3.3.8. **Remark.** While we described the unit map $\mathcal{F} \to \Omega|\mathcal{F}\,\mathrm{Q}(-^{[1]})|$ of the adjunction arising from Theorem 3.3.4 already at the beginning of this section, the counit $|\Omega\mathcal{F}\,\mathrm{Q}(-^{[1]})| \to \mathcal{F}$ is more elusive. One thing we can say about it is that the composite

$$|\Omega\mathcal{F}\,\mathrm{Q}(-^{[1]})| \to \mathcal{F} \to \Omega|\mathcal{F}\,\mathrm{Q}(-^{[1]})| \tag{44}$$

of the counit and unit can be identified with the *negative* of the canonical limit-colimit interchange map (as we will show below). In case $\mathcal{F}$ is group-like, the unit map is an equivalence by Theorem 3.3.4, so this determines the counit for such $\mathcal{F}$.

Note that in the cases $\mathcal{F} = \mathrm{Pn}, \mathrm{Cr}$ or $\mathrm{Cr}^{\mathrm{hC}_2}$, or more generally any additive $\mathcal{F}$ for which the component of 0 in $\mathcal{F}(\mathcal{C}, \Qoppa)$ is always contractible, the source of the counit simply vanishes. These cases cover all additive functors of interest to us.

To see the claim about the composite map, observe that the limit-colimit interchange map $\sigma \colon |\Omega\mathcal{F}\,\mathrm{Q}(-^{[1]})| \to \Omega|\mathcal{F}\,\mathrm{Q}(-^{[1]})|$ can also be described as the Beck-Chevalley map associated to the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}) \arrow[r, "\mathcal{F} \mapsto |\mathcal{F}\,\mathrm{Q}(-^{[1]})|"] \arrow[d, shift left, "\Sigma"] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}) \arrow[d, shift left, "\Sigma"] \\
\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}) \arrow[r, "\mathcal{F} \mapsto |\mathcal{F}\,\mathrm{Q}(-^{[1]})|"] \arrow[u, bend left, "\Omega"] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}) \arrow[u, bend left, "\Omega"]
\end{tikzcd}$$

where $\Sigma\mathcal{F}$ here denotes the suspension of $\mathcal{F}$ in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$, not the valuewise suspension. By Theorem 3.3.4, this is equivalent to $|\mathcal{F}\,\mathrm{Q}(-^{[1]})|$, but it will be advantageous to keep the notations for the horizontal and vertical arrows separate for a moment. By definition, the Beck-Chevalley map depends on the commutativity data of the square involving the down facing vertical arrows, which is given by the canonical map $\tau \colon \Sigma|\mathcal{F}\,\mathrm{Q}(-^{[1]})| \to |(\Sigma\mathcal{F})\,\mathrm{Q}(-^{[1]})|$ exchanging the order of the colimits. This map is an equivalence, since the geometric realisations occurring on both sides (which by construction are valuewise!) actually compute the colimits in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$: from Proposition 1.4.15, we find that each $\mathcal{F}\,\mathrm{Q}_n(-)$ is additive, and Theorem 2.5.1 implies that the valuewise realisation, which computes the colimit in $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$, already lies in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$. Now, unwinding the definitions using $|\mathcal{F}\,\mathrm{Q}(-^{[1]})| \simeq \Sigma\mathcal{F}$, τ becomes the self equivalence of $\Sigma^2\mathcal{F}$ which switches the two suspension coordinates. In particular, the square involving the down facing vertical arrows can be endowed with two different commutativity structures, corresponding to the swap τ and the identity of $\Sigma^2\mathcal{F}$. These determine two corresponding Beck-Chevalley maps given, respectively, by

$$\Sigma\Omega\mathcal{F} \xrightarrow{u_{\Sigma\Omega\mathcal{F}}} \Omega\Sigma\Sigma\Omega\mathcal{F} \xrightarrow{\Omega\tau_{\Omega\mathcal{F}}} \Omega\Sigma\Sigma\Omega\mathcal{F} \xrightarrow{\Omega\Sigma c_{\mathcal{F}}} \Omega\Sigma\mathcal{F}$$

and

$$\Sigma\Omega\mathcal{F} \xrightarrow{u_{\Sigma\Omega\mathcal{F}}} \Omega\Sigma\Sigma\Omega\mathcal{F} \xrightarrow{\mathrm{id}} \Omega\Sigma\Sigma\Omega\mathcal{F} \xrightarrow{\Omega\Sigma c_{\mathcal{F}}} \Omega\Sigma\mathcal{F},$$

where u and c denote the unit and counit of the adjunction $\Sigma \dashv \Omega$. The second of these is equivalent to (44), since

$$\Omega\Sigma c_{\mathcal{F}} \circ u_{\Sigma\Omega\mathcal{F}} \simeq u_{\mathcal{F}} \circ c_{\mathcal{F}}$$

by naturality. The first, which we showed to be the colimit-limit interchange map above, is its negative since the map $\tau_{\Omega\mathcal{F}} : \Sigma^2\Omega\mathcal{F} \to \Sigma^2\Omega\mathcal{F}$ is homotopic to the negative of the identity map.

We now turn to the proof of Theorem 3.3.4. Part i) requires us to consider the dual Q-construction, denoted $\mathrm{d}\,\mathrm{Q}(\mathcal{C}, \Upsilon)$, which we discuss next.

3.3.9. **Lemma.** *The functors* $\mathrm{Q}_n : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$ *and* $\mathrm{Q}_n : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ *admit left adjoints* $\mathrm{d}\,\mathrm{Q}_n$*, given by tensoring with the poset* $\mathcal{I}_n$*.*

These adjoints participate in the two squares

$$\begin{tikzcd}
\mathrm{Cat}^{\mathrm{p}}_\infty \arrow[r, "\mathrm{d}\,\mathrm{Q}_n"] \arrow[d, "\mathrm{fgt}"] & \mathrm{Cat}^{\mathrm{p}}_\infty \arrow[d, "\mathrm{fgt}"] \\
\mathrm{Cat}^{\mathrm{ex}}_\infty \arrow[r, "\mathrm{d}\,\mathrm{Q}_n"] & \mathrm{Cat}^{\mathrm{ex}}_\infty
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Cat}^{\mathrm{p}}_\infty \arrow[r, "\mathrm{d}\,\mathrm{Q}_n"] & \mathrm{Cat}^{\mathrm{p}}_\infty \\
\mathrm{Cat}^{\mathrm{ex}}_\infty \arrow[r, "\mathrm{d}\,\mathrm{Q}_n"] \arrow[u, "\mathrm{Hyp}"] & \mathrm{Cat}^{\mathrm{ex}}_\infty \arrow[u, "\mathrm{Hyp}"]
\end{tikzcd}$$

obtained by passing to left adjoints everywhere in the analogous diagrams involving the respective Q-constructions.

Proof. Recall that for $[n] \in \Delta$, we have denoted by $\mathcal{I}_n$ the full subposet of $\mathrm{TwAr}(\Delta^n)$ spanned by the arrows of the form $(i \leq j)$ for $j \leq i + 1$. From Examples 2.2.3, we find $\mathrm{Q}_n \simeq (-)^{\mathcal{I}_n}$, which by Proposition [I].6.4.4 has $(-)_{\mathcal{I}_n}$ as a left adjoint when regarded as a functor $\mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$. As an application of Proposition [I].6.6.1, we find that $(\mathcal{C}, \Upsilon)_{\mathcal{I}_n}$ is Poincaré whenever $(\mathcal{C}, \Upsilon)$ is and from Remark [I].6.4.6 and Proposition [I].6.2.2 we then find an equivalence of Poincaré ∞-categories

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Upsilon)_{\mathcal{I}_n}, (\mathcal{D}, \Phi)) \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Upsilon), (\mathcal{D}, \Phi)^{\mathcal{I}_n})$$

which according to Corollary [I].6.2.12 gives the claim by passing to Poincaré objects. □

Now, recall that there is a canonical equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Cat}^{\mathrm{p}}_\infty) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Cat}^{\mathrm{p}}_\infty)^{\mathrm{op}}$ for example as an immediate consequence of Lurie's straightening equivalences, which makes both ∞-categories equivalent to that of bicartesian fibrations over Δ^1 with both fibres identified with $\mathrm{Cat}^{\mathrm{p}}_\infty$; the superscripts L and R indicate left and right adjoint functors, respectively. In particular, as the Q-construction is a simplicial object, the left adjoints above assemble into a cosimplicial object.

3.3.10. **Definition.** Let $(\mathcal{C}, \Upsilon)$ be an hermitian ∞-category. We will denote by $\mathrm{d}\,\mathrm{Q}(\mathcal{C}, \Upsilon)$ the cosimplicial hermitian ∞-category obtained by applying the left adjoint of Q_n in each degree.

3.3.11. **Remark.** The proof of Lemma 3.3.9 does not make the functoriality of $\mathrm{d}\,\mathrm{Q}(\mathcal{C}, \Upsilon)$ very apparent since the ∞-categories $\mathcal{I}_n$ do not form a cosimplicial object.

To remedy this defect, we offer the following description of $\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Upsilon)$: By the discussion in Examples 2.2.3, a diagram $\phi\colon\ \mathrm{TwAr}(\Delta^n) \to \mathcal{C}$ lies in $\mathrm{Q}_n(\mathcal{C}) \subseteq \mathcal{C}^{\mathrm{TwAr}(\Delta^n)}$ if and only if it lies in the image of the right Kan extension along the inclusion $\iota_n : \mathcal{I}_n \to \mathrm{TwAr}(\Delta^n)$. The Poincaré ∞-category $\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Upsilon)$ is dually given by instead considering the quotient in $\mathrm{Cat}^{\mathrm{h}}_\infty$ of $(\mathcal{C}, \Upsilon)_{\mathrm{TwAr}(\Delta^n)}$ by the kernel of the left adjoint $\iota_n^* : \mathcal{C}_{\mathrm{TwAr}[n]} \to \mathcal{C}_{\mathcal{I}_n}$ of the canonical map $(\iota_n)_* : \mathcal{C}_{\mathcal{I}_n} \to \mathcal{C}_{\mathrm{TwAr}[n]}$ on the tensoring construction. Under the identifications $\mathcal{C}_{\mathcal{I}_n} \simeq \mathrm{Fun}(\mathcal{I}_n^{\mathrm{op}}, \mathcal{C})$ of Proposition [I].6.5.8 and its analogue for $\mathrm{TwAr}(\Delta^n)$, the kernel of ι_n^* consists of those $\varphi\colon\ \mathrm{TwAr}[n]^{\mathrm{op}} \to \mathcal{C}$ for which $\varphi(i < j) = 0$ whenever $|j - i| \leq 1$.

One can check that this description directly assembles $\mathrm{d}\,\mathrm{Q}(\mathcal{C}, \Upsilon)$ into a cosimplicial object of $\mathrm{Cat}^{\mathrm{p}}_\infty$, which is left adjoint to $\mathrm{Q}(\mathcal{C}, \Upsilon)$, but we shall not need this description, so we leave the details to the reader.

3.3.12. **Definition.** Let $(\mathcal{C}, \Upsilon)$ be an hermitian ∞-category. We define $\mathrm{dNull}_n(\mathcal{C}, \Upsilon)$ to be the Poincaré-Verdier quotient of $\mathrm{d}\,\mathrm{Q}_{n+1}(\mathcal{C}, \Upsilon)$ by the image of the functor $\mathcal{C} = \mathrm{d}\,\mathrm{Q}_0(\mathcal{C}) \to \mathrm{d}\,\mathrm{Q}_{1+n}(\mathcal{C})$ induced by the inclusion $[0] \to [1 + n]$.

Note that Proposition 1.4.12 shows that there is then indeed a Poincaré-Verdier sequence

$$(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{d}\,\mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{dNull}_n(\mathcal{C}, \Qoppa).$$

3.3.13. **Remark.** The functor dNull_n is by definition the cofibre of the natural transformation $\mathrm{d}\,\mathrm{Q}_{\{0\}} \Rightarrow \mathrm{d}\,\mathrm{Q}_{n+1}$, while the functor $\mathrm{Null}_n(-)$ is the fibre of the natural transformation $\mathrm{Q}_{n+1} \to \mathrm{Q}_0$. We conclude that the association $(\mathcal{C}, \Qoppa) \mapsto \mathrm{dNull}_n(\mathcal{C}, \Qoppa)$ is left adjoint to $(\mathcal{D}, \Phi) \mapsto \mathrm{Null}_n(\mathcal{D}, \Phi)$.

Proof of Theorem 3.3.4. As discussed, the second statement follows from 3.2.16: The associated square of ∞-categories is cartesian (before realisation) even if $\mathcal{F}$ is not group-like: This follows either by direct inspection since one obtains the square from the start of this section, or using the fact that the right vertical map is an isofibration of Segal spaces, see the argument in the proof of the additivity theorem. Proposition 3.2.16 then implies that all occurring ∞-categories already lie in $\mathcal{S}$ if $\mathcal{F}$ is group-like, so realisation has no further effect. We shall also provide another argument below.

For the first statement, we have to show that the square

$$\begin{tikzcd}
\mathrm{Nat}(|\mathcal{F}\,\mathrm{Q}(-^{[1]})|, \mathcal{G}) \arrow[r] \arrow[d] & \mathrm{Nat}(|\mathcal{F}\mathrm{Null}(-^{[1]})|, \mathcal{G}) \arrow[d] \\
* \arrow[r] & \mathrm{Nat}(\mathcal{F}, \mathcal{G})
\end{tikzcd}$$

is cartesian for every additive $\mathcal{G} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$. But we calculate

$$\mathrm{Nat}(|\mathcal{F}\,\mathrm{Q}(-^{[1]})|, \mathcal{G}) \simeq \lim_{[n]\in\Delta} \mathrm{Nat}(\mathcal{F}\,\mathrm{Q}_n(-^{[1]}), \mathcal{G}) \simeq \lim_{[n]\in\Delta} \mathrm{Nat}(\mathcal{F}, \mathcal{G}\mathrm{d}\,\mathrm{Q}_n(-^{[-1]}))$$

and similarly for the upper right hand term. Commuting limits, it thus suffices to show that

$$\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow \mathrm{dNull}_n(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow (\mathcal{C}, \Qoppa) \tag{45}$$

is split Poincaré-Verdier for every Poincaré ∞-category $(\mathcal{C}, \Qoppa)$. Using Remark 1.2.4 this statement can be obtained entirely formally by ($\mathrm{Cat}^{\mathrm{h}}_\infty$-enriched) adjunction from the sequence

$$(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Null}_n(\mathcal{C}, \Qoppa^{[1]}) \longrightarrow \mathrm{Q}_n(\mathcal{C}, \Qoppa^{[1]}) \tag{46}$$

being split Poincaré-Verdier by Observation 3.3.3, but we shall give a more direct argument. It is immediate from adjointness that (45) is a cofibre sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$, so it remains to check that the composite

$$\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa^{[-1]}) \xrightarrow{d_0} \mathrm{d}\,\mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa^{[-1]}) \longrightarrow \mathrm{dNull}_n(\mathcal{C}, \Qoppa^{[-1]})$$

is a Poincaré-Verdier inclusion. But from the equivalence $\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa) \simeq (\mathcal{C}, \Qoppa)_{\mathcal{J}_n}$, we find the first map to be such an inclusion by Proposition 1.4.12. Thus the Poincaré structure on $\mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa^{[-1]})$ is obtained from that on $\mathrm{d}\,\mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa^{[-1]})$ by pullback along d_0, or equivalently by left Kan extension along (the opposite of) the right adjoint $\mathrm{d}\,\mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa^{[-1]}) \to \mathrm{d}\,\mathrm{Q}_n(\mathcal{C}, \Qoppa^{[-1]})$ to d_0. We are therefore done if we show that this right adjoint factors through $\mathrm{dNull}_n(\mathcal{C}, \Qoppa^{[-1]})$. But this follows from the corresponding statement for the left adjoint of $d_0 \colon \mathrm{Q}_{1+n}(\mathcal{C}, \Qoppa^{[1]}) \to \mathrm{Q}_n(\mathcal{C}, \Qoppa^{[1]})$ factoring through $\mathrm{Null}_n(\mathcal{C}, \Qoppa^{[1]})$ in Observation 3.3.3, since the adjunction $\mathrm{d}\,\mathrm{Q}_n \vdash \mathrm{Q}_n$ is compatible with the passage to underlying ∞-categories by the discussion after Lemma 3.3.9 (and the same argument gives the claim for $\mathrm{dNull}_n \vdash \mathrm{Null}_n$). ☐

Alternative to our translation along the equivalence between Segal spaces and ∞-categories and the use of the isotropic decomposition principle in 3.2.16, one can obtain the second claim of Theorem 3.3.4 also by directly applying Rezk's equifibration criterion [Rez14, Proposition 2.4] to the map $\mathcal{F}\mathrm{Null}(\mathcal{C}, \Qoppa) \Rightarrow \mathcal{F}\,\mathrm{Q}(\mathcal{C}, \Qoppa)$:

3.3.14. **Lemma.** *Let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "\tau"] \\
Z \arrow[r] & W
\end{tikzcd}$$

be a cartesian square of functors from some small ∞-category I to $\mathcal{S}$, such that the transformation $\tau\colon Y \Rightarrow W$ is equifibred, i.e. such that

$$\begin{tikzcd}
Y(i) \arrow[r, "\tau_i"] \arrow[d] & W(i) \arrow[d] \\
Y(j) \arrow[r, "\tau_j"] & W(j)
\end{tikzcd}$$

is cartesian for every $i \to j$ in I. Then the square

$$\begin{tikzcd}
\operatorname{colim} X \arrow[r] \arrow[d] & \operatorname{colim} Y \arrow[d, "\tau"] \\
\operatorname{colim} Z \arrow[r] & \operatorname{colim} W
\end{tikzcd}$$

is cartesian as well.

To see that $\mathcal{F}\,\mathrm{Null}(\mathcal{C}, \mathfrak{P}) \to \mathcal{F}\,\mathrm{Q}(\mathcal{C}, \mathfrak{P})$ is equifibred in this sense, one can observe that the Segal condition implies that it suffices to check that the squares

$$\begin{tikzcd}
\mathcal{F}(\mathrm{Null}_2(\mathcal{C}, \mathfrak{P})^{[1]}) \arrow[r] \arrow[d, "d_i"] & \mathcal{F}(\mathrm{Q}_2(\mathcal{C}, \mathfrak{P}^{[1]})) \arrow[d, "d_i"] \\
\mathcal{F}(\mathrm{Null}_1(\mathcal{C}, \mathfrak{P}^{[1]})) \arrow[r] & \mathcal{F}(\mathrm{Q}_1(\mathcal{C}, \mathfrak{P}^{[1]}))
\end{tikzcd} \tag{47}$$

are cartesian for $i = 0, 1, 2$. For $i = 1, 2$, these squares are split Poincaré-Verdier (prior to applying $\mathcal{F}$) by Lemma 2.2.8 and Corollary 1.2.6 and for $i = 0$ the induced map on vertical fibres (over 0, but this suffices since d_0 induces a surjection on π_0) identifies with

$$\mathrm{can}\colon\ \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}(\mathrm{Met}(\mathcal{C}, \mathfrak{P}))$$

which is an equivalence by Corollary 3.1.4. For the reader's convenience let us supply a proof of 3.3.14, as we do not know a reference in the present language.

Proof. By [Lur09a, Lemma 6.1.3.14], we may apply [Lur09a, Theorem 6.1.3.9 (4)] to the ∞-category $\mathcal{S}$ (Lurie calls an equifibred transformation cartesian). This gives us that any extension of τ to the cone of I, such that the extension of W is a colimit cone, is again equifibred if and only if the extension of Y is also a colimit cone. Applying the backwards direction, we find

$$\begin{tikzcd}
Y(i) \arrow[r] \arrow[d] & \operatorname{colim} Y \arrow[d] \\
W(i) \arrow[r] & \operatorname{colim} W
\end{tikzcd}$$

and therefore also

$$\begin{tikzcd}
X(i) \arrow[r] \arrow[d] & \operatorname{colim} Y \arrow[d] \\
Z(i) \arrow[r] & \operatorname{colim} W
\end{tikzcd}$$

is cartesian for every $i \in I$. Cancelling one pullback, it follows that

$$\begin{tikzcd}
X(i) \arrow[r] \arrow[d] & \operatorname{colim} Z \times_{\operatorname{colim} W} \operatorname{colim} Y \arrow[d] \\
Z(i) \arrow[r] & \operatorname{colim} Z
\end{tikzcd}$$

is cartesian as well. But it then follows from [Lur09a, Lemma 6.1.3.2] that the extension of the transformation $X \Rightarrow Z$ to the cone of I via the right hand column of the last diagram is also equifibred. A forward application of [Lur09a, Theorem 6.1.3.9 (4)] now gives the claim. □

3.4. **The spectrification of an additive functor.** In §3.3 we showed that for any additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ the square

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & {|\mathrm{Cob}^{\mathcal{F}}_{0/}|} \arrow[d] \\ 0 \arrow[r] & {|\mathrm{Cob}^{\mathcal{F}}|} \end{tikzcd} \tag{48}$$

exhibits $|\mathrm{Cob}^{\mathcal{F}}|$ as the suspension of $\mathcal{F}$ in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$. Iterating this procedure, we obtain for each additive functor $\mathcal{F}$ a model for the suspension pre-spectrum of $\mathcal{F} \in \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$. To set the stage, we first note that, for each $n \geq 1$, we have an n-fold simplicial object in $\mathrm{Cat}^{\mathrm{p}}_\infty$ given by

$$\mathrm{Q}^{(n)}(\mathcal{C}, \Omega) : (\Delta^{\mathrm{op}})^n \longrightarrow \mathrm{Cat}^{\mathrm{p}}_\infty \qquad ([m_1], ..., [m_n]) \longmapsto \mathrm{Q}_{m_1}\, \mathrm{Q}_{m_2} \ldots \mathrm{Q}_{m_n}(\mathcal{C}, \Omega).$$

By Lemmas 2.2.7 and 2.2.5, $\mathrm{Q}^{(n)}(\mathcal{C}, \Omega)$ is an n-fold Segal object of $\mathrm{Cat}^{\mathrm{p}}_\infty$, the n-fold iterated hermitian Q-construction of $(\mathcal{C}, \Omega)$. As a multiple Segal object, it presents an n-fold ∞-category, though we shall not attempt to make this precise. We simply set:

3.4.1. **Definition.** For $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ additive, we shall call the n-fold Segal space $\mathcal{F}\mathrm{Q}^{(n)}(\mathcal{C}, \Omega^{[n]})$ the $\mathcal{F}$-*based n-extended cobordism ∞-category* $\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, \Omega)$ of $(\mathcal{C}, \Omega)$.

In particular, $\mathrm{Cob}^{\mathcal{F}}_1(\mathcal{C}, \Omega) = \mathcal{F}\mathrm{Q}(\mathcal{C}, \Omega^{[1]})$ really is the Segal space giving rise to the cobordism ∞-category $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$, and $\mathrm{Cob}^{\mathcal{F}}_0(\mathcal{C}, \Omega) = \mathcal{F}(\mathcal{C}, \Omega)$. Furthermore, there are canonical equivalences

$$|\mathrm{Cob}_i^{|\mathrm{Cob}^{\mathcal{F}}_j|}(\mathcal{C}, \Omega)| \simeq |\mathrm{Cob}^{\mathcal{F}}_{j+i}(\mathcal{C}, \Omega)|.$$

The multiple Segal space $\mathrm{Pn}\,\mathrm{Q}^{(n)}(\mathcal{C}, \Omega^{[n]})$ models the n-fold ∞-category informally described as having Poincaré objects of $(\mathcal{C}, \Omega^{[n]})$ as objects, their cobordisms as morphisms, cobordisms between cobordisms as squares and so on up to degree n.

3.4.2. **Remark.** The analogous n-fold topological category Cob^n_d (note the unfortunate index switch) for cobordism categories of d-manifolds first appeared in [BM14], ironically inspired by the iterated version of Quillen's original Q-construction, and served to produce cobordism theoretic deloopings of $|\mathrm{Cob}_d|$. In particular, Bökstedt and Madsen showed that $|\mathrm{Cob}^n_d| \simeq \Omega^{\infty-n}\mathrm{MTSO}(d)$, extending the theorem of Galatius, Madsen, Tillmann and Weiss from the case $n = 1$. They used this description to give an entirely cobordism theoretic model for the spectrum $\mathrm{MTSO}(d)$, which endows it with an interesting map to $\mathcal{A}(\mathrm{BSO}(d))$, studied extensively by Raptis and the 9'th author in [RS14, RS17, RS20], where it was used to give a short proof of the Dwyer-Weiss-Williams index theorem [DWW03]. We will take up the study of the evident refinements of this map in a sequel to the present paper.

The higher categorical incarnations of these extended cobordism categories are of course also the main objects of study in Lurie's (sketch of a) solution to the cobordism hypothesis [Lur09c], and the results of Bökstedt-Madsen have been reproven in the language of higher categories by Schommer-Pries in [SP17].

Now, denote by $\mathcal{PS}p$ the ∞-category of pre-spectra, that is the lax limit of the diagram

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*,$$

consisting of sequences $(X_n)_{n\in\mathbb{N}}$ of pointed spaces together with structure maps $X_n \to \Omega X_{n+1}$. There is a fully faithful inclusion $\mathcal{S}p \subseteq \mathcal{PS}p$, which admits a left adjoint we will refer to as spectrification. It does not affect the homotopy groups. Furthermore, the evaluation functors $ev_n : \mathcal{PS}p \to \mathcal{S}_*$ commute with both limits and colimits, and restrict to the functors $\Omega^{\infty-n} : \mathcal{S}p \to \mathcal{S}_*$ (which still preserve limits, but only filtered colimits).

3.4.3. **Definition.** Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ be a functor. We denote by

$$\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, \Omega) = [\mathrm{Cob}^{\mathcal{F}}_0(\mathcal{C}, \Omega), |\mathrm{Cob}^{\mathcal{F}}_1(\mathcal{C}, \Omega)|, |\mathrm{Cob}^{\mathcal{F}}_2(\mathcal{C}, \Omega)|, \ldots]$$

the corresponding functor from $\mathrm{Cat}^{\mathrm{p}}_\infty$ to pre-spectra with the structure maps determined by the square (48) applied to the functors $|\mathrm{Cob}^{\mathcal{F}}_i|$.

Since the 0-th object in the pre-spectrum $\mathbb{C}\mathrm{ob}(\mathcal{C},\Qoppa)$ is $\mathcal{F}(\mathcal{C},\Qoppa)$ itself, we obtain a natural map

$$\mathcal{F}(\mathcal{C},\Qoppa) \longrightarrow \Omega^\infty \mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa), \tag{49}$$

where the right hand side refers to the 0-th space of the spectrification of $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa)$.

3.4.4. **Remark.**

i) There is another possible definition of the bonding maps of $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa)$: One could take the map $\mathcal{F} \to \Omega|\mathrm{Cob}^{\mathcal{F}}|$ provided by (48) and form

$$|\mathrm{Cob}^{\mathcal{F}}_i| \longrightarrow |\mathrm{Cob}^{|\Omega\mathrm{Cob}^{\mathcal{F}}|}_i| \longrightarrow \Omega|\mathrm{Cob}^{|\mathrm{Cob}^{\mathcal{F}}|}_i| \simeq \Omega|\mathrm{Cob}^{\mathcal{F}}_{1+i}|$$

These differ from the bonding maps we chose by a coordinate flip in the $(1+i)$-fold simplicial object $\mathrm{Cob}^{\mathcal{F}}_{1+i}$. Since iterated application of the Q-construction models the suspension in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S})$ by Theorem 3.3.4, such a coordinate flip induces the negative of the identity on realisations. In particular, this choice of bonding maps gives a pre-spectrum naturally equivalent to $\mathbb{C}\mathrm{ob}(\mathcal{C},\Qoppa)$.

ii) In fact, the coordinate flips endow $\mathbb{C}\mathrm{ob}(\mathcal{C},\Qoppa)$ with the structure of an (∞-categorical version of a) symmetric pre-spectrum, just as the more classical construction of K-theory spectra. We will not have to make use of this observation, which is classically used to produce multiplicative structures on K-spectra, since we argue instead by universal properties to construct multiplicative structures in Paper [IV].

iii) By Theorem 3.3.4, $|\mathrm{Cob}^{\mathcal{F}}_n|$ is a model for the n-fold suspension of $\mathcal{F}$ in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S})$. Considering $\mathbb{C}\mathrm{ob}^{\mathcal{F}}$ as a pre-spectrum object in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S})$, it is thus the suspension pre-spectrum of $\mathcal{F}$.

3.4.5. **Proposition.** *Let $\mathcal{F}$ be an additive functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ and $(\mathcal{C},\Qoppa) \in \mathrm{Cat}^{\mathrm{p}}_\infty$. Then:*

i) The functor $\mathbb{C}\mathrm{ob}^{\mathcal{F}} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{P}\mathcal{S}p$ is again additive and takes values in positive Ω-spectra, i.e. the structure map $|\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C},\Qoppa)| \to \Omega|\mathrm{Cob}^{\mathcal{F}}_{n+1}(\mathcal{C},\Qoppa)|$ is an equivalence for every $n \geq 1$.

ii) If $\mathcal{F}$ is group-like, then $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa)$ is in fact an (Ω-)spectrum, and $\mathbb{C}\mathrm{ob}^{\mathcal{F}}$ is then additive when considered as a functor $\mathbb{C}\mathrm{ob}^{\mathcal{F}} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$.

iii) The natural map $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa) \to \mathbb{C}\mathrm{ob}^{\mathcal{F}^{\mathrm{grp}}}(\mathcal{C},\Qoppa)$ exhibits the right hand side as the spectrification of the left.

In particular, we obtain equivalences

$$\mathcal{F}^{\mathrm{grp}}(\mathcal{C},\Qoppa) \simeq \Omega^\infty \mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa) \quad \text{and} \quad |\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C},\Qoppa)| \simeq \Omega^{\infty-n}\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa)$$

for $n \geq 1$.

From Theorem 3.3.4 and Part ii) of this proposition, we thus obtain the following universal property for the iterated hermitian Q-construction:

3.4.6. **Corollary.** *For a group-like additive functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$, the functor $\mathbb{C}\mathrm{ob}^{\mathcal{F}} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ is the initial additive functor under $\mathbb{S}[\mathcal{F}]$, the pointwise suspension spectrum of $\mathcal{F}$. In other words,*

$$\mathbb{C}\mathrm{ob} : \ \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathrm{E}_\infty}) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p)$$

is left adjoint to the forgetful functor, i.e. composition with Ω^∞. Also,

$$\mathbb{C}\mathrm{ob}\circ(-)^{\mathrm{grp}} : \ \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p)$$

is left adjoint to the forgetful functor.

An explicit description of the counit of the former adjunction is easily derived from Remark 3.3.8.

Proof. For the proof note that transformations $\mathbb{S}[\mathcal{F}] \Rightarrow \mathcal{G}$ of functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ correspond naturally to transformations $\mathcal{F} \Rightarrow \Omega^\infty\mathcal{G}$ of functors to both E_∞-monoids and plain spaces by Lemma 1.5.7. On the other hand, the space of transformations $\mathbb{C}\mathrm{ob}^{\mathcal{F}} \Rightarrow \mathcal{G}$ is given by

$$\lim_{n\in\mathbb{N}} \mathrm{Nat}(\Omega^{\infty-n}\mathbb{C}\mathrm{ob}^{\mathcal{F}}, \Omega^{\infty-n}\mathcal{G}) \simeq \lim_{n\in\mathbb{N}} \mathrm{Nat}(|\mathrm{Cob}^{\mathcal{F}}_n|, \Omega^{\infty-n}\mathcal{G}).$$

But since $|\mathrm{Cob}^{\mathcal{F}}_n|$ is the n-fold suspension of $\mathcal{F}$ by Theorem 3.3.4, this colimit system is constant with value $\mathrm{Nat}(\mathcal{F},\Omega^\infty\mathcal{G})$, which gives the claim. □

Proof of Proposition 3.4.5. By Proposition 2.3.7, we have that $|\mathrm{Cob}^{\mathcal{F}}_n| \simeq |\mathrm{Cob}^{|\mathrm{Cob}^{\mathcal{F}}|}_{n-1}|$ is group-like as soon as $n \geq 1$, and hence in this case the structure map $|\mathrm{Cob}^{\mathcal{F}}_n| \to \Omega|\mathrm{Cob}^{\mathcal{F}}_{n+1}|$ is an equivalence by Theorem 3.3.4 ii). Of course, if $\mathcal{F}$ is group-like then this holds also at the 0-th level. Furthermore, since by Theorem 2.5.1 all functors $|\mathrm{Cob}^{\mathcal{F}}_n|$ are additive so is $\mathbb{C}\mathrm{ob}^{\mathcal{F}}$, as fibre sequences in (pre-)spectra are detected degreewise. This gives the first two statements.

To obtain the third statement, just observe that by Part ii) the spectrification of $\mathbb{C}\mathrm{ob}(\mathcal{C}, Ϙ)$ is given by

$$\left[\Omega|\mathrm{Cob}^{\mathcal{F}}_1(\mathcal{C}, Ϙ)|, |\mathrm{Cob}^{\mathcal{F}}_1(\mathcal{C}, Ϙ)|, |\mathrm{Cob}^{\mathcal{F}}_2(\mathcal{C}, Ϙ)|, \dots\right]$$

which by Corollaries 3.3.6 and 3.3.7 agrees with

$$\left[\mathcal{F}^{\mathrm{grp}}(\mathcal{C}, Ϙ), |\mathrm{Cob}^{\mathcal{F}^{\mathrm{grp}}}_1(\mathcal{C}, Ϙ)|, |\mathrm{Cob}^{\mathcal{F}^{\mathrm{grp}}}_2(\mathcal{C}, Ϙ)|, \dots\right],$$

since $|\mathrm{Cob}^{\mathcal{F}}_{n+1}| = |\mathrm{Cob}^{|\mathrm{Cob}^{\mathcal{F}}|}_n|$. □

Part iii) of Proposition 3.4.5 identifies the non-negative homotopy groups of $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ with those of $\mathcal{F}^{\mathrm{grp}}(\mathcal{C}, Ϙ)$. While these are generally very difficult to understand, we can determine the negative homotopy groups of the spectrum $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ much more easily:

3.4.7. **Proposition.** *For every additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$, Poincaré ∞-category $(\mathcal{C}, Ϙ)$, $n \geq 1$ and $0 \leq k < n$, the iterated bonding maps of the pre-spectrum $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ induce isomorphisms*

$$\pi_k|\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, Ϙ)| \cong \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ^{[n-k-1]})| \quad \textit{and} \quad \pi_{-n}\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \cong \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ^{[n-1]})|.$$

In other words, $\pi_k|\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, Ϙ)|$ for $k < n$ is just the $\mathcal{F}$-based cobordism group of $(\mathcal{C}, Ϙ^{[n-k]})$ and similarly for the negative homotopy groups of $\mathbb{C}\mathrm{ob}^{\mathcal{F}}$.

Proof. Part i) of Proposition 3.4.5 reduces the claim about the left hand side to the case $k = 0$. By realising the n-fold simplicial object $\mathrm{Cob}^{\mathcal{F}}_n$ iteratively, this case follows from Corollaries 2.3.10 and 2.3.11 by induction on n. The statement for the right hand side is now immediate from Proposition 3.4.5 iii) and Corollary 3.3.7. □

3.4.8. **Corollary.** *For any group-like additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ the spectrum $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ is connective whenever $(\mathcal{C}, Ϙ)$ admits a Lagrangian subcategory, in particular $\mathbb{C}\mathrm{ob}^{\mathcal{F}}\,\mathrm{Met}(\mathcal{D}, \Phi)$ and $\mathbb{C}\mathrm{ob}^{\mathcal{F}}\,\mathrm{Hyp}(\mathcal{E})$ are always connective.*

In fact, the functor

$$\mathbb{C}\mathrm{ob}\colon\ \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathrm{E}_\infty}) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p)$$

is fully faithful and its essential image consists precisely of the functors whose values on all metabolic Poincaré ∞-categories $(\mathcal{C}, Ϙ)$ are connective.

The essential image of $\mathbb{C}\mathrm{ob}$ can equivalently be described by the condition that $\mathcal{F}\,\mathrm{Met}(\mathcal{D}, \Phi)$ be connective for all Poincaré ∞-categories $(\mathcal{D}, \Phi)$ or that $\mathcal{F}\,\mathrm{Hyp}(\mathcal{E})$ be connective for all small stable $\mathcal{E}$: For the latter condition this is immediate from Corollary 3.2.14, and for the former it then follows from $\mathcal{F}\,\mathrm{Hyp}(\mathcal{E})$ being a retract of $\mathcal{F}\,\mathrm{Hyp}(\mathrm{Hyp}(\mathcal{E})) \simeq \mathcal{F}(\mathrm{Met}(\mathrm{Hyp}(\mathcal{E})))$.

Proof. The first part is a consequence of Proposition 3.4.7, Corollary 3.2.14 and Corollary 2.3.11. That $\mathbb{C}\mathrm{ob}$ is fully faithful follows from Corollary 3.4.6, since the unit $\mathcal{F} \Rightarrow \Omega^\infty\mathbb{C}\mathrm{ob}^{\mathcal{F}}$ is an equivalence by Proposition 3.4.5 if $\mathcal{F}$ is group-like. To see the statement about the essential image, note that the counit $\mathbb{C}\mathrm{ob}^{\Omega^\infty\mathcal{F}} \to \mathcal{F}$ of the adjunction is an equivalence after applying Ω^∞ by the triangle identities, and therefore an equivalence on non-negative homotopy groups. Applying this counit transformation to the metabolic fibre sequence

$$(\mathcal{C}, Ϙ) \to \mathrm{Met}(\mathcal{C}, Ϙ^{[1]}) \to (\mathcal{C}, Ϙ^{[1]}),$$

we conclude inductively on i that the transformation is an equivalence on π_{-i} for all $i \geq 0$. □

3.4.9. **Remark.** Completely analogous definitions and arguments work in the non-hermitian set-up to give the n-fold Segal spaces $\mathrm{Span}^{\mathcal{F}}_n(\mathcal{C})$ and (pre-)spectra $\mathbb{S}\mathrm{pan}^{\mathcal{F}}(\mathcal{C})$, with the K-theory functor $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}p$ being the (pointwise) spectrification of $\mathbb{S}\mathrm{pan}^{\mathrm{Cr}}$ or equivalently $\mathbb{S}\mathrm{pan}^{\mathrm{Cr}^{\mathrm{grp}}}$. As a consequence of Proposition 2.7.3, one here finds that $\mathbb{S}\mathrm{pan}^{\mathcal{F}}(\mathcal{C})$ is always a connective (pre-)spectrum. The analogue of the above corollary is the statement that

$$\mathbb{S}\mathrm{pan}\colon\ \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathrm{Grp}_{\mathrm{E}_\infty}) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathcal{S}p)$$

is fully faithful with essential image the functors taking values in connective spectra. In particular, the non-connectivity of the iterated Q-construction is an entirely hermitian phenomenon.

Let us also record the relationship between the Q-construction and suspension in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathcal{S}p)$. To this end, consider again the squares

$$\begin{tikzcd} \mathrm{const}\,\mathrm{Met}(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & \mathrm{dec}(\mathrm{Q}(\mathcal{C},\Qoppa)) \arrow[d] & \mathrm{const}(\mathcal{C},\Qoppa^{[-1]}) \arrow[r] \arrow[d] & \mathrm{Null}(\mathcal{C},\Qoppa) \arrow[d] \\ \{0\} \arrow[r] & \mathrm{Q}(\mathcal{C},\Qoppa) & \{0\} \arrow[r] & \mathrm{Q}(\mathcal{C},\Qoppa) \end{tikzcd} \tag{50}$$

consisting of split Poincaré-Verdier sequences in each simplicial degree. Applying an additive $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$, one obtains levelwise cartesian squares of simplicial spectra. As these are also cocartesian by stability, it follows that also

$$\begin{tikzcd} \mathcal{F}\mathrm{Met}(\mathcal{C},\Qoppa) \arrow[r] \arrow[d] & {|\mathcal{F}\mathrm{dec}(\mathrm{Q}(\mathcal{C},\Qoppa))|} \arrow[d] & \mathcal{F}(\mathcal{C},\Qoppa^{[-1]}) \arrow[r] \arrow[d] & {|\mathcal{F}\mathrm{Null}(\mathcal{C},\Qoppa)|} \arrow[d] \\ \{0\} \arrow[r] & {|\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)|} & \{0\} \arrow[r] & {|\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)|} \end{tikzcd}$$

are bicartesian squares of spectra. As the simplicial objects in the top right corner are split by Lemma 3.3.1 over $(\mathcal{C},\Qoppa)$ and 0, respectively, we obtain a canonical equivalence

$$\mathbb{S}^1 \otimes \mathcal{F}(\mathcal{C},\Qoppa^{[-1]}) \longrightarrow |\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)| \tag{51}$$

from the right square and a natural bifibre sequence

$$\mathcal{F}\,\mathrm{Met}(\mathcal{C},\Qoppa) \xrightarrow{\mathrm{met}} \mathcal{F}(\mathcal{C},\Qoppa) \longrightarrow |\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)|$$

from the left one. The first half gives:

3.4.10. **Corollary.** *The endofunctor* $\mathcal{F} \mapsto |\mathcal{F}\,\mathrm{Q}(-^{[1]})|$ *on* $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathcal{S}p)$ *is the internal suspension functor, or equivalently postcomposition with the suspension functor in* $\mathcal{S}p$*.*

As the right square canonically maps to the left one, one finds that under the equivalence (51) the fibre sequence issued from the second square is a rotation of the metabolic fibre sequence

$$\mathcal{F}(\mathcal{C},\Qoppa^{[-1]}) \longrightarrow \mathcal{F}\,\mathrm{Met}(\mathcal{C},\Qoppa) \xrightarrow{\mathrm{met}} \mathcal{F}(\mathcal{C},\Qoppa),$$

a fact we will use below.

3.4.11. **Remark.** The geometric realisation $|\mathcal{F}\,\mathrm{Q}\,|$ occurring in the previous statement may be taken both objectwise as a geometric realisation of simplicial spectra, or as a colimit in the ∞-category $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathcal{S}p)$ itself, since we noted above that the objectwise colimit (which is also the colimit in the ∞-category of all functors $\mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$) is already additive.

However, we warn the reader that in general $|\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)|$ cannot be computed levelwise via the realisation of the simplicial spaces $\Omega^{\infty-n}\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)$: Consider for example the functor $\mathbb{K}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$ extracting the non-connective K-theory of the underlying stable ∞-category $\mathcal{C}$. Then $\Omega^\infty|\mathbb{K}\,\mathrm{Q}(\mathcal{C})| \simeq \Omega^{\infty-1}\mathbb{K}(\mathcal{C})$ need not be connected, whereas

$$|\Omega^\infty\mathbb{K}\,\mathrm{Q}(\mathcal{C})| \simeq |\,\mathcal{K}(\mathrm{Q}(\mathcal{C})^\natural)| \simeq |\,\mathcal{K}(\mathrm{Q}(\mathcal{C}^\natural))| \simeq \Omega^{\infty-1}\,\mathrm{K}(\mathcal{C}^\natural)$$

is always connected.

The notable exception to this discrepancy are the functors $\mathcal{F} = \mathbb{C}\mathrm{ob}^{\mathcal{G}}$ for some group-like additive $\mathcal{G}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}$: For these functors the colimit of $\mathcal{F}\,\mathrm{Q}(\mathcal{C},\Qoppa)\colon \Delta^{\mathrm{op}} \to \mathcal{P}\mathcal{S}p$ (which is formed levelwise) is automatically an Ω-spectrum, and thus also a colimit in spectra. To see this, observe that by switching the order of the realisations we find

$$(|\mathbb{C}\mathrm{ob}^{\mathcal{G}}\,\mathrm{Q}(\mathcal{C},\Qoppa)|)_n = ||\mathcal{G}\,\mathrm{Q}^{(n)}(\mathrm{Q}(\mathcal{C},\Qoppa^{[n]}))|| = \mathbb{C}\mathrm{ob}^{|\mathcal{G}\,\mathrm{Q}\,-|}(\mathcal{C},\Qoppa)_n$$

and the latter terms form an Ω-spectrum by Proposition 3.4.5, so ultimately by the additivity theorem.

Finally, we use these observations to study the effect of shifting the Poincaré structure on the C_2-equivariant spectrum $\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ acted on by the duality of $Ϙ$. This will ultimately lead to our generalisation of Karoubi's periodicity theorem in Corollary 4.5.5 below. To this end, recall that the composite functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty \xrightarrow{\mathrm{Hyp}} \mathrm{Cat}^{\mathrm{p}}_\infty$ refines to a functor $\mathcal{H}\mathrm{yp}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Fun}(\mathrm{BC}_2, \mathrm{Cat}^{\mathrm{p}}_\infty)$ via the action of the duality, see Remark [I].7.4.14.

3.4.12. **Definition.** Given a functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ define the *hyperbolisation* $\mathcal{F}^{\mathrm{hyp}}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Fun}(\mathrm{BC}_2, \mathcal{E})$ of $\mathcal{F}$ as $\mathcal{F}\circ\mathcal{H}\mathrm{yp}$.

3.4.13. **Proposition** (Naive Karoubi periodicity)**.** *There is a canonical equivalence of* C_2*-spectra*

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ^{[-1]}) \simeq \mathbb{S}^{\sigma-1} \otimes \mathcal{F}^{\mathrm{Hyp}}(\mathcal{C}, Ϙ),$$

natural in the Poincaré ∞*-category* $(\mathcal{C}, Ϙ)$ *and the additive functor* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$*. Furthermore, under this equivalence, the boundary map*

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ) \to \mathbb{S}^1 \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ^{[-1]})$$

of the metabolic fibre sequence is induced by the inclusion $S^0 \to S^\sigma$ *as the fixed points.*

Here, $\mathbb{S}^\sigma$ denotes the C_2-spectrum equivalently described as the suspension spectrum of S^σ, the 1-sphere with complex conjugation action, or the functor

$$\mathrm{BC}_2 = \mathrm{BO}(1) \to \mathrm{BO} \xrightarrow{J} \mathrm{Pic}(\mathbb{S}) \subseteq \mathcal{S}p.$$

Proof. We recall that under the equivalence $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ induced by $\mathrm{can}\colon \mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C}, Ϙ)$, the map $\mathrm{met}\colon \mathrm{Met}(\mathcal{C}, Ϙ) \to \mathcal{C}$ identifies with $\mathrm{hyp}\,:\, \mathrm{Hyp}(\mathcal{C}) \to (\mathcal{C}, Ϙ)$. Using the metabolic Poincaré-Verdier sequence, we may therefore identify $\mathbb{S}^1 \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ^{[-1]})$ with the cofibre of the map

$$\mathcal{F}\mathrm{hyp}\colon \mathcal{F}^{\mathrm{hyp}}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ).$$

Now, there is a natural equivalence

$$\mathcal{H}\mathrm{yp}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Hyp}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Hyp}(\mathcal{C}) \otimes C_2$$

which translates the action of $D_Ϙ$ on the left into the flip action on the right, see Remark [I].7.4.15. We may then identify the map $\mathcal{F}\mathrm{hyp}$ with the map

$$\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \otimes C_2 \longrightarrow \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$$

obtained from the map $C_2 \to *$ of C_2-spaces, whose cofibre is S^σ. We therefore obtain a natural equivalence

$$\mathbb{S}^1 \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ^{[-1]}) \simeq \mathbb{S}^\sigma \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, Ϙ)$$

which is the claim. □

3.5. **Bordism invariant functors.**

In the next two subsections, we introduce the notion of a bordism invariant functor out of $\mathrm{Cat}^{\mathrm{p}}_\infty$, the main examples being various flavours of L-theory. We then show that each additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ admits an initial bordism invariant functor $\mathcal{F}^{\mathrm{bord}}$ equipped with a map $\mathcal{F} \to \mathcal{F}^{\mathrm{bord}}$, the bordification of $\mathcal{F}$, and that any group-like $\mathcal{F}$ can then be described in terms of this bordification and the hyperbolisation $\mathcal{F}^{\mathrm{hyp}} = \mathcal{F}\circ\mathrm{Hyp}$ from the previous section. This yields a version of the first part of our Main Theorem for arbitrary additive functors in Corollary 3.6.7; it will be specialised to $\mathcal{F} = \mathrm{Pn}$ in section 4.

To get started recall the notion of a cobordism between Poincaré functors from Definition 3.1.1: It is a Poincaré functor $(\mathcal{C}, Ϙ) \to Q_1(\mathcal{C}', Ϙ')$ projecting correctly to the endpoints of Q_1.

3.5.1. **Definition.** A Poincaré functor $(F, \eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ is called a *bordism equivalence* if there exists a Poincaré functor $(G, \vartheta)\colon (\mathcal{C}', Ϙ') \to (\mathcal{C}, Ϙ)$ such that the composites $(F, \eta)\circ(G, \theta)$ and $(G, \theta)\circ(F, \eta)$ are cobordant to the respective identities.

3.5.2. **Example.** Let $(\mathcal{C}, Ϙ)$ be a Poincaré ∞-category and $\mathcal{L} \subseteq \mathcal{C}$ an isotropic subcategory (see Definition 3.2.1). Then the inclusion $\mathrm{Hlgy}(\mathcal{L}) \subseteq (\mathcal{C}, Ϙ)$ of the homology ∞-category is a bordism equivalence. This follows directly from Construction 3.2.11.

3.5.3. **Definition.** Given an ∞-category with finite products $\mathcal{E}$, we say that an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is *bordism invariant* if it sends bordism equivalences of Poincaré ∞-categories to equivalences in $\mathcal{E}$. We shall denote by $\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ the full subcategory of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ spanned by the bordism invariant functors.

In particular, such a functor vanishes on all metabolic Poincaré ∞-categories, i.e. those that admit a Lagrangian. For (group-like) additive functors, and these are the only ones we investigate in any detail here, the converse holds as well:

3.5.4. **Lemma.** *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ be a group-like additive functor. Then the following are equivalent:*

- *i)* *$\mathcal{F}$ is bordism invariant.*
- *ii)* *$\mathcal{F}$ takes the degeneracy map $s\colon (\mathcal{C}, \mathfrak{Q}) \to \mathrm{Q}_1(\mathcal{C}, \mathfrak{Q})$ to an equivalence for every Poincaré ∞-category $(\mathcal{C}, \mathfrak{Q})$.*
- *iii)* *$\mathcal{F}$ vanishes on all metabolic Poincaré ∞-categories.*
- *iv)* *$\mathcal{F}(\mathrm{Met}(\mathcal{C}, \mathfrak{Q})) \simeq *$ for any Poincaré ∞-category $(\mathcal{C}, \mathfrak{Q})$.*
- *v)* *$\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \simeq *$ for any stable ∞-category $\mathcal{C}$.*

Proof. The functors in ii) are bordism equivalences (essentially by definition) so i) $\Rightarrow$ ii) and it follows immediately from Corollary 3.1.3 that ii) $\Rightarrow$ iv). By Example 3.5.2, all metabolic categories are bordism equivalent to 0, so i) $\Rightarrow$ iii) and since $\mathrm{Met}(\mathcal{C}, \mathfrak{Q})$ really is metabolic, we have iii) $\Rightarrow$ iv). To obtain iv) $\Rightarrow$ v) observe that $\mathcal{F}\,\mathrm{Hyp}(\mathcal{C})$ is a retract of $\mathcal{F}\,\mathrm{Hyp}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}})$, which by Corollary 3.1.4 is equivalent to $\mathcal{F}\,\mathrm{Met}(\mathrm{Hyp}(\mathcal{C})) \simeq *$. Finally, by Proposition 3.1.7, if $\mathcal{F}$ vanishes on hyperbolics, then cobordant Poincaré functors induce homotopic maps after applying $\mathcal{F}$, and so $\mathcal{F}$ is bordism invariant giving v) $\Rightarrow$ i). □

3.5.5. **Example.** The L-theory space provides a bordism invariant functor $\mathcal{L}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$. The fact that it is invariant under bordism equivalences can be seen by direct analysis of its homotopy groups: In degree n they are given by bordism classes of Poincaré objects in $(\mathcal{C}, \mathfrak{Q}^{[-n]})$, see [Lur11, Lecture 7, Theorem 9] for a proof in the present language. Thus, by definition, the two maps $d_0, d_1\colon \mathcal{L}(\mathrm{Q}_1(\mathcal{C}, \mathfrak{Q})) \to \mathcal{L}(\mathcal{C}, \mathfrak{Q})$ induce the same map on L-groups. Consequently, so do any two cobordant functors, and thus bordism equivalences induce inverse isomorphisms on L-groups, compare §[I].2.3. The same statements apply to the L-theory spectrum. We discuss this example and additivity of both functors $\mathcal{L}$ and L in detail in §4.4.

To discuss the second important example, recall from Definition 3.4.12, the hyperbolisation $\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \mathfrak{Q}) = \mathcal{F}(\mathcal{H}\mathrm{yp}(\mathcal{C}))$ taking values in the ∞-category $\mathcal{S}p^{\mathrm{hC}_2} = \mathrm{Fun}(\mathrm{BC}_2, \mathcal{S}p)$ of C_2-spectra via the action of the duality $\mathrm{D}_{\mathfrak{Q}}$ on $\mathrm{Hyp}(\mathcal{C})$.

3.5.6. **Example.** Given an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$, the Tate construction $(-)^{\mathrm{tC}_2}\colon \mathcal{S}p^{\mathrm{hC}_2} \to \mathcal{S}p$ produces a functor $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ which is bordism invariant; to see this, we invoke the natural equivalence

$$\mathcal{H}\mathrm{yp}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathrm{Hyp}(\mathcal{C}) \otimes \mathrm{C}_2 \tag{52}$$

from [I].7.4.15 again, which shows that $\mathcal{F}^{\mathrm{hyp}}(\mathrm{Hyp}(\mathcal{C}))$ is an induced C_2-spectrum. It then follows that for every stable ∞-category $\mathcal{C}$, we have $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}\,\mathrm{Hyp}(\mathcal{C}) \simeq 0$, since the Tate construction generally vanishes on induced C_2-spectra.

Let us also record for later use:

3.5.7. **Lemma.** *If $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is arbitrary and $\mathcal{G}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is bordism-invariant, then the spaces* $\mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}, \mathcal{G}), \mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}, \mathcal{G}), \mathrm{Nat}(\mathcal{G}, \mathcal{F}^{\mathrm{hyp}})$ *and* $\mathrm{Nat}(\mathcal{G}, (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2})$ *are contractible (assuming $\mathcal{E}$ admits sufficient (co)limits to form the homotopy orbits and fixed points appearing).*

Proof. Since $\mathrm{Hyp}\colon \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$ is both left and right adjoint to the forgetful functor by Corollary [I].7.2.21, the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{fgt}} \mathrm{Cat}^{\mathrm{ex}}_{\infty} \xrightarrow{\mathrm{Hyp}} \mathrm{Cat}^{\mathrm{p}}_{\infty}$ is both left and right adjoint to itself. Hence the association $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{hyp}}$ is both left and right adjoint to itself as well. Since $\mathcal{G}^{\mathrm{hyp}} \simeq *$ for any bordism invariant functor, it follows that the mapping space from $\mathcal{F}^{\mathrm{hyp}}$ to and from any bordism invariant functor is trivial.

The computations

$$\mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}, \mathcal{G}) \simeq \mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}, \mathcal{G})^{\mathrm{hC}_2} \simeq * \quad \text{and} \quad \mathrm{Nat}(\mathcal{G}, (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}) \simeq \mathrm{Nat}(\mathcal{G}, \mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2} \simeq *$$

give the second claim. ☐

For the next statement, recall the metabolic Poincaré-Verdier sequence

$$(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, Ϙ) \longrightarrow (\mathcal{C}, Ϙ)$$

from Example 1.2.5.

3.5.8. **Proposition.** *Suppose that $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is a bordism invariant additive functor. Then the natural map*

$$\Omega\mathcal{F}(\mathcal{C}, Ϙ) \longrightarrow \mathcal{F}(\mathcal{C}, Ϙ^{[-1]})$$

arising from the metabolic Poincaré-Verdier sequence is an equivalence. In particular, $\mathcal{F}$ is automatically group-like.

If $\mathcal{E}$ is stable then the converse holds in the sense that an additive functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is bordism invariant if and only if this map is an equivalence for all Poincaré ∞-categories $(\mathcal{C}, Ϙ)$.

In particular, we find $\pi_i\mathcal{F}(\mathcal{C}, Ϙ) = \pi_0\mathcal{F}(\mathcal{C}, Ϙ^{[-i]})$ for every space or spectrum valued bordism invariant functor. Furthermore, by Corollary 3.1.8, the inversion map on $\mathcal{F}(\mathcal{C}, Ϙ)$ is induced by the Poincaré functor $(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{Ϙ})$.

Proof. By Lemma 3.5.4, $\mathcal{F}$ is bordism invariant if and only if $\mathcal{F}\,\mathrm{Met}(\mathcal{C}, Ϙ) \simeq *$ for all Poincaré ∞-categories $(\mathcal{C}, Ϙ)$, from which we obtain a fibre sequence

$$\mathcal{F}(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow * \longrightarrow \mathcal{F}(\mathcal{C}, Ϙ),$$

which gives the first claim. Conversely, if $\mathcal{E}$ is stable, then the map in question being an equivalence implies that $\mathcal{F}\,\mathrm{Met}(\mathcal{C}, Ϙ)$ vanishes for every Poincaré ∞-category. ☐

In particular, bordism invariant functors can be delooped simply by shifting the Poincaré structure, i.e. by considering

$$\left[\mathcal{F}(\mathcal{C}, Ϙ), \mathcal{F}(\mathcal{C}, Ϙ^{[1]}), \mathcal{F}(\mathcal{C}, Ϙ^{[2]}), \dots\right]$$

with the structure maps provided by Proposition 3.5.8. We next show that this delooping agrees with that from the previous section. In fact, we have as the main result of this subsection:

3.5.9. **Theorem.** *The forgetful functor*

$$\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p) \longrightarrow \mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}),$$

i.e. postcomposition with Ω^∞, is an equivalence with inverse

$$\mathcal{F} \longmapsto \mathbb{C}\mathrm{ob}^{\mathcal{F}}.$$

In particular, any additive bordism invariant functor $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ admits an essentially unique lift to another such functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$.

The same is not true for arbitrary group-like additive $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ as the examples

$$(\mathcal{C}, Ϙ) \longmapsto \mathrm{K}(\mathcal{C}^{\mathrm{idem}}) \quad \text{and} \quad \mathbb{K}(\mathcal{C}),$$

which have equivalent infinite loop spaces, show.

For the proof, we need:

3.5.10. **Remark.** If $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ is additive and bordism invariant, then so is $|\mathrm{Cob}^{\mathcal{F}}|$. This follows straight from the definitions, as a cobordism of Poincaré functors $(\mathcal{C}, Ϙ) \to \mathrm{Q}_1(\mathcal{C}', Ϙ')$, induces one

$$\mathrm{Q}_n(\mathcal{C}, Ϙ) \to \mathrm{Q}_n(\mathrm{Q}_1(\mathcal{C}', Ϙ')) \cong \mathrm{Q}_1(\mathrm{Q}_n(\mathcal{C}', Ϙ'))$$

so a bordism equivalence $(\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ gives an equivalence of simplicial objects $\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ) \to \mathcal{F}\,\mathrm{Q}(\mathcal{C}', Ϙ')$, and thus an equivalence on realisations.

Proof of Theorem 3.5.9. That the essential image of Ω^∞ is contained in the bordism invariant functors is clear. If now $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ is bordism invariant, then so is $\mathbb{C}\mathrm{ob}^{\mathcal{F}}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$: By Proposition 3.5.8, $\mathcal{F}$ is group-like, so by Proposition 3.4.5, $\mathbb{C}\mathrm{ob}^{\mathcal{F}}$ takes values in spectra and is additive. To check bordism invariance it suffices, by induction and the equivalences

$$\Omega^{\infty-n}\mathbb{C}\mathrm{ob}^{\mathcal{F}} \simeq |\mathrm{Cob}^{\mathcal{F}}_n| \simeq |\mathrm{Cob}^{|\mathrm{Cob}^{\mathcal{F}}|}_{n-1}|$$

to show that $|\mathrm{Cob}^{\mathcal{F}}|$ is again bordism invariant, which we did above. Thus the adjunction between Ω^∞ and $\mathbb{C}\mathrm{ob}$ restricts as claimed and Ω^∞ is essentially surjective.

Finally, to obtain full faithfulness of Ω^∞, we check that the counit $c\colon \mathbb{C}\mathrm{ob}^{\Omega^\infty\mathcal{F}}(\mathcal{C},\Qoppa) \Rightarrow \mathcal{F}(\mathcal{C},\Qoppa)$ is an equivalence for every bordism invariant $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ and Poincaré ∞-category $(\mathcal{C},\Qoppa)$. By Proposition 3.4.5, $\Omega^\infty c$ is an equivalence for all $(\mathcal{C},\Qoppa)$, but as both domain and target of c are bordism invariant functors, Proposition 3.5.8 then implies that $\Omega^{\infty-n}c$ is an equivalence for all $n \geq 0$, which gives the claim. ☐

3.6. **The bordification of an additive functor.**

In this subsection, we establish the following theorem and deduce a formal version of our Theorem Main Theorem in Corollary 3.6.7.

3.6.1. **Theorem.** *The inclusions*

$$\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}p) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}p) \quad \textit{and} \quad \mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S})$$

of the bordism invariant into all additive functors admit left and right adjoints.

3.6.2. **Definition.** We will refer to these left adjoint functors as *bordification* and denote their values on an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ or $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ by $\mathcal{F}^{\mathrm{bord}}$. The right adjoint we shall call *cobordification* and denote by $(-)^{\mathrm{cbord}}$.

The existence of the left adjoints, Theorem 3.5.9 and Corollary 3.4.6 may be summarised by the following square of forgetful functors and their dotted left adjoints, whose left hand vertical arrows are inverse equivalences, as follows:

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}p) \arrow[r] \arrow[d, shift left] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}p) \arrow[l, dotted, bend right, "(-)^{\mathrm{bord}}"'] \arrow[d, shift left] \\
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}) \arrow[r] \arrow[u, dotted, shift left, "\mathbb{C}\mathrm{ob}"] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty,\mathcal{S}) \arrow[l, dotted, bend right, "(-)^{\mathrm{bord}}"'] \arrow[u, dotted, shift left, "\mathbb{C}\mathrm{ob}\circ(-)^{\mathrm{grp}}"]
\end{tikzcd}$$

Thus the existence of the upper horizontal adjoint implies the existence of the lower one, or more precisely for $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ additive, we have

$$\mathcal{F}^{\mathrm{bord}} \simeq \Omega^\infty((\mathbb{C}\mathrm{ob}^{\mathcal{F}^{\mathrm{grp}}})^{\mathrm{bord}}).$$

Similarly, we find

$$\mathcal{F}^{\mathrm{cbord}} \simeq \Omega^\infty((\mathbb{C}\mathrm{ob}^{\mathcal{F}^\times})^{\mathrm{cbord}}),$$

where $\mathcal{F}^\times$ denotes the pointwise units of $\mathcal{F}$: For $\mathcal{G}$ bordism invariant one computes

$$\begin{aligned}
\mathrm{Nat}(\mathcal{G},\Omega^\infty((\mathbb{C}\mathrm{ob}^{\mathcal{F}^\times})^{\mathrm{cbord}}) &\simeq \mathrm{Nat}(\mathbb{C}\mathrm{ob}^{\mathcal{G}},(\mathbb{C}\mathrm{ob}^{\mathcal{F}^\times})^{\mathrm{cbord}}) \\
&\simeq \mathrm{Nat}(\mathbb{C}\mathrm{ob}^{\mathcal{G}},\mathbb{C}\mathrm{ob}^{\mathcal{F}^\times}) \\
&\simeq \mathrm{Nat}(\mathcal{G},\mathcal{F}^\times) \\
&\simeq \mathrm{Nat}(\mathcal{G},\mathcal{F})
\end{aligned}$$

where the first and third identities follow from Corollary 3.4.6. In particular, the (co)bordifications at the spectrum level determine those at the space level, so we will restrict attention to the case of functors taking values in spectra in this section.

3.6.3. **Remark.** Comparing universal properties, it is easy to see that for additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ we have

$$\Omega^\infty(\mathcal{F}^{\mathrm{cbord}}) \simeq (\Omega^\infty\mathcal{F})^{\mathrm{cbord}}$$

and consequently $\mathcal{F}^{\mathrm{cbord}} \simeq \mathbb{C}\mathrm{ob}^{(\Omega^\infty\mathcal{F})^{\mathrm{cbord}}}$ by Theorem 3.5.9. We explicitly warn the reader that generally neither of the analogous assertions hold true for bordifications instead of cobordifications (unless $\mathcal{F} = \mathbb{C}\mathrm{ob}^{\mathcal{G}}$).

The distinction will play a major role in our discussion of Karoubi localising invariants in Paper [IV]; a concrete counter-example is the Karoubi-Grothendieck-Witt functor $\mathbb{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ discussed there.

We give three distinct formulae for the spectral bordification functor in Proposition 3.6.6, Corollary 3.6.13 and Corollary 3.6.19, and it really is the comparison between these that is most relevant for our work. While this comparison can be established by direct calculations, that route does not lead to shorter arguments and the present framework allows for a more conceptual interpretation; see 3.6.8. Along the way we also establish the cobordification in Proposition 3.6.6.

Before getting started, let us already record the following special cases:

3.6.4. **Lemma.** *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *be additive. Then we have* $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{bord}} \simeq 0$ *and* $(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2})^{\mathrm{bord}} \simeq 0$*, whereas the natural map* $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2} \Rightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}$ *descends to an equivalence*

$$((\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2})^{\mathrm{bord}} \simeq (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}.$$

Dually, $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{cbord}} \simeq 0$*,* $((\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2})^{\mathrm{cbord}} \simeq 0$ *and* $(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2})^{\mathrm{cbord}} \simeq \mathbb{S}^{-1} \otimes (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}$.

To interpret the statement, one can either assume the existence of a (co)bordification functor already (the present lemma will not enter the proof of existence below), or better one can simply interpret the definition of (co)bordifications as a pointwise statement about left and right adjoint objects. In that case, the present lemma, in particular, provides the existence of (co)bordifications for the functors $\mathcal{F}^{\mathrm{hyp}}$, $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}$ and $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$.

Proof. The first two statements are immediate from Lemma 3.5.7. But then consider the cofibre sequence

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2} \longrightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}.$$

For some bordism invariant $\mathcal{G}$, it induces a fibre sequence

$$\mathrm{Nat}((\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}, \mathcal{G}) \longrightarrow \mathrm{Nat}((\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}, \mathcal{G}) \longrightarrow \mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}, \mathcal{G})$$

But the right hand term vanishes by Lemma 3.5.7, whereas $(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}$ is already bordism invariant by Example 3.5.6. The claim follows and those about cobordifications are dual. ☐

Recall from Corollary [I].7.4.18 that the hyperbolic and forgetful maps refine to C_2-equivariant maps

$$\mathcal{H}\mathrm{yp}(\mathcal{C}) \longrightarrow (\mathcal{C}, \Qoppa) \longrightarrow \mathcal{H}\mathrm{yp}(\mathcal{C}),$$

where $(\mathcal{C}, \Qoppa)$ is considered with the trivial C_2-action. It then follows that the induced natural maps

$$\mathcal{F}^{\mathrm{hyp}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\mathrm{hyp}} \tag{53}$$

refine to maps of the form

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow \mathcal{F} \longrightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}. \tag{54}$$

As part of [I].7.4.18, we also showed that the composite of these two maps coincides with the norm map $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \to (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ associated to the C_2-action on $\mathcal{F}^{\mathrm{hyp}}$ (we showed this at the level of Poincaré ∞-categories and it persists since $\mathcal{F}$ preserves finite products).

3.6.5. **Example.** If $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}p$ is additive, then the (co)bordification of the composite $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}p$ vanishes: In that case, the C_2-spectrum $\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Qoppa)$ itself is (co)induced from $\mathcal{F}(\mathcal{C})$, whence the maps $\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Qoppa)_{\mathrm{hC}_2} \to \mathcal{F}(\mathcal{C}) \to \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Qoppa)^{\mathrm{hC}_2}$ are equivalences. This implies for example that the bordifications of Cr, K, $\mathbb{K}$, THH, TC and similar functors all vanish.

Expressed differently, (co)bordification is a genuinely hermitian concept that has no classical counterpart.

Either from 3.6.4 or 3.5.7, we find

$$\mathrm{Nat}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}, \mathcal{G}) \simeq *$$

if $\mathcal{G}$ is bordism invariant. In particular, assuming the existence of a bordification, there must be a sequence

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\mathrm{bord}} \tag{55}$$

whose composition admits an essentially unique null-homotopy. There is a universal way to produce such a sequence:

3.6.6. **Proposition.** *Consider the functor* $\Phi\colon \ \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p)$ *given by the formula*

$$\Phi\mathcal{F} = \mathrm{cof}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow \mathcal{F}).$$

Then, the canonical transformations $\mathcal{F} \Rightarrow \Phi\mathcal{F}$ *exhibit* Φ *as a bordification. Dually,* $\mathrm{fib}(\mathcal{F} \to (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2})$ *is a cobordification of* $\mathcal{F}$*.*

Proof. Let $\mathcal{F}$ be an additive functor. We first verify that $\Phi\mathcal{F}$ is bordism invariant. By lemma 3.5.4, we need to check that $\Phi\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \simeq 0$, or, equivalently, that the canonical transformation

$$\mathcal{F}^{\mathrm{Hyp}}_{\mathrm{hC}_2}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \tag{56}$$

is an equivalence. Indeed, by the equivalence (52) we can identify (56) with a map of the form

$$[\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \otimes \mathrm{C}_2)]_{\mathrm{hC}_2} \longrightarrow \mathcal{F}(\mathrm{Hyp}(\mathcal{C})). \tag{57}$$

It will therefore suffice to check that the pre-composition of (57) with the equivalence $\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \to [\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \otimes \mathrm{C}_2)]_{\mathrm{hC}_2}$ given by the inclusion of a component is an equivalence; by direct inspection it is the identity.

Now suppose that $\mathcal{G}$ is any bordism invariant functor. We need to show that the induced map

$$\mathrm{Nat}(\Phi\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{G})$$

is an equivalence. Indeed, by construction, we have a fibre sequence

$$\mathrm{Nat}(\Phi\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Nat}((\mathcal{F}^{\mathrm{Hyp}})_{\mathrm{hC}_2}, \mathcal{G})$$

and $\mathrm{Nat}((\mathcal{F}^{\mathrm{Hyp}})_{\mathrm{hC}_2}, \mathcal{G}) \simeq *$ by Lemma 3.5.7.

The argument for the final statement is entirely dual. ☐

Applying bordification to the natural map $\mathcal{F} \to (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ and using Lemma 3.6.4, we find an abstract version of our main result, the *fundamental fibre square*:

3.6.7. **Corollary.** *For every additive functor* $\mathcal{F}\colon \ \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ *and Poincaré* ∞*-category* $(\mathcal{C}, \Qoppa)$*, there is a fibre sequence*

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}(\mathcal{C}, \Qoppa) \xrightarrow{\mathrm{hyp}} \mathcal{F}(\mathcal{C}, \Qoppa) \xrightarrow{\mathrm{bord}} \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Qoppa),$$

which canonically extends to a bicartesian square

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Qoppa) \arrow[d] \\
(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}(\mathcal{C}, \Qoppa) \arrow[r] & (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}(\mathcal{C}, \Qoppa).
\end{tikzcd}$$

Proof. It suffices to check that the induced map on horizontal fibres is an equivalence. But both of these are given by $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}$ and the induced map is necessarily the identity by Lemma 3.5.7 ☐

This result can be recast more abstractly as follows:

3.6.8. **Proposition.** *The diagram*

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p) \arrow[r] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p) \arrow[l, bend right=40, "\mathrm{bord}"'] \arrow[l, bend left=40, "\mathrm{cbord}"] \arrow[r] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{ex}}_{\infty}, \mathcal{S}p)^{\mathrm{hC}_2}, \arrow[l, bend right=40, "(-\circ \mathrm{fgt})_{\mathrm{hC}_2}"'] \arrow[l, bend left=40, "(-\circ \mathrm{fgt})^{\mathrm{hC}_2}"]
\end{tikzcd}$$

where the right unlabelled map is pullback along $\mathrm{Hyp}\colon (\mathrm{Cat}^{\mathrm{ex}}_\infty)_{\mathrm{hC}_2} \to \mathrm{Cat}^{\mathrm{p}}_\infty$*, constitutes a stable recollement, whose associated fracture square is precisely the fundamental fibre square from the previous corollary. The images of the left and right adjoints on the right hand side consist of those additive functors* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *such that*

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow \mathcal{F} \quad or \quad \mathcal{F} \longrightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$$

is an equivalence, respectively.

For the notion of stable recollements and their associated fracture squares, see §A.2, particularly Definition A.2.10 and the discussion thereafter.

One might call functors satisfying the properties characterising the essential images above *left* and *right hyperbolic*, whence we find that $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \to \mathcal{F}$ is the terminal approximation to $\mathcal{F}$ by a left hyperbolic, and $\mathcal{F} \to (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ is the initial approximation by a right hyperbolic functor.

Proof. Recall from the discussion above (or see [I].7.4.18) that the functor

$$\mathrm{Hyp}\colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$$

is a both sided adjoint to $\mathrm{fgt}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ and that this adjunction is equivariant for the trivial action of C_2 on $\mathrm{Cat}^{\mathrm{p}}_\infty$ and the opponing action on $\mathrm{Cat}^{\mathrm{ex}}_\infty$ (i.e. it is a relative adjunction over BC_2). It follows formally that

$$\mathrm{Nat}((\mathcal{F}\circ \mathrm{fgt})_{\mathrm{hC}_2}, \mathcal{G}) \simeq \mathrm{Nat}((\mathcal{F}\circ \mathrm{fgt}), \mathcal{G})^{\mathrm{hC}_2} \simeq \mathrm{Nat}(\mathcal{F}, \mathcal{G}\circ \mathrm{Hyp})^{\mathrm{hC}_2}$$

and similarly $\mathrm{Nat}(\mathcal{G}, (\mathcal{F}\circ \mathrm{fgt})^{\mathrm{hC}_2}) \simeq \mathrm{Nat}(\mathcal{G}\circ \mathrm{Hyp}, \mathcal{F})^{\mathrm{hC}_2}$, so that we obtain the right hand adjunctions. Unwinding definitions, one finds that in the former case the counit evaluated at some stable ∞-category $\mathcal{C}$ is the canonical equivalence

$$\mathcal{F}(\mathcal{C}) \longrightarrow (\mathrm{C}_2 \otimes \mathcal{F}(\mathcal{C}))^{\mathrm{hC}_2}$$

since the inclusion $\mathcal{F}(\mathcal{C}) \to \mathcal{F}(\mathcal{C}) \oplus \mathcal{F}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathcal{F}(\mathrm{fgt}(\mathrm{Hyp}(\mathcal{C})))$ gives rise to an equivalence $\mathrm{C}_2 \otimes \mathcal{F}(\mathcal{C}) \simeq \mathcal{F}(\mathrm{fgt}(\mathrm{Hyp}(\mathcal{C})))$ by direct inspection. Thus $(-)^{\mathrm{hyp}}\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{ex}}_\infty, \mathcal{S}p)^{\mathrm{hC}_2}$ has a fully faithful adjoint, and is therefore a split Verdier projection. Its kernel is precisely $\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p)$ by Lemma 3.5.4, whence Proposition A.2.11 and the discussion thereafter give the recollement.

The claim about the Tate square is true by construction, and the conditions given in the final statement unwind to the counit and unit being equivalences, respectively, which characterise the essential images by the triangle identities. □

The construction of (co)bordifications via the hyperbolisation map $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \to \mathcal{F}$ or forgetful map $\mathcal{F} \to (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ discussed so far are, however, not very suitable for computations of $\mathcal{F}^{\mathrm{bord}}$ or $\mathcal{F}^{\mathrm{cbord}}$. Therefore we present two more formulae for the bordification, both of which we put to use in the next section. To verify that these really give bordifications we employ the following criterion:

3.6.9. **Lemma.** *Suppose that* $\mathrm{B}\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p)$ *is a functor equipped with a natural transformation* $\beta\colon \mathrm{id} \Rightarrow \mathrm{B}$. *Suppose the following conditions hold:*

i) B *commutes with colimits;*
ii) if $\mathcal{F}$ *is bordism invariant then* $\beta_{\mathcal{F}}\colon \mathcal{F} \Rightarrow \mathrm{B}\mathcal{F}$ *is an equivalence;*
iii) $\mathrm{B}(\mathcal{F}^{\mathrm{hyp}}) \simeq 0$ *for every additive* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$.

Then β *exhibits* B *as a bordification functor.*

Let us explicitly point out that we do not assume a priori that B takes values in bordism invariant functors. The price is that we have to invest that we already know that there exists a bordification functor into the proof. Direct arguments are also certainly possible, but slightly more cumbersome.

Proof. Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ be an additive functor. Applying B to the fibre sequence $\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \to \mathcal{F} \to \mathcal{F}^{\mathrm{bord}}$ from Corollary 3.6.7 yields a commutative rectangle

$$\begin{tikzcd}
\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{bord}} \arrow[d] \\
\mathrm{B}(\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2}) \arrow[r] & \mathrm{B}\mathcal{F} \arrow[r] & \mathrm{B}(\mathcal{F}^{\mathrm{bord}})
\end{tikzcd} \tag{58}$$

in which both rows are bifibre sequences and the vertical maps are all the respective components of β. By definition, $\mathcal{F}^{\mathrm{bord}}$ is bordism invariant and hence, by property ii), we get that the right most vertical map in (58) is an equivalence. This implies that the left square is bicartesian. On the other hand, by properties i) and iii), the lower left corner of (58) is equivalent to 0, hence the lower right map is an equivalence as well. The right hand square thus exhibits B as equivalent to bord under the identity of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p)$. ☐

Our second formula for bordification is modelled on the classical definition of L-theory spectra via ad-spaces. Its starting point is the *ρ-construction*: For $[n] \in \Delta$ we denote by $\mathcal{T}_n = \mathcal{P}_0([n])^{\mathrm{op}}$ the opposite of the poset of nonempty subsets of $[n]$. We observe that $\mathcal{T}_n$ depends functorially on $[n] \in \Delta$, giving rise to a simplicial ∞-category $\rho(\mathcal{C}, \Omega)$: Given a Poincaré ∞-category $(\mathcal{C}, \Omega)$ denote

$$\rho_n(\mathcal{C}, \Omega) = (\mathrm{Fun}(\mathcal{T}_n, \mathcal{C}), \Omega^{\mathcal{T}_n})$$

the cotensor of $(\mathcal{C}, \Omega)$ by $\mathcal{T}_n$. Since $\mathcal{T}_n$ is the reverse face poset of Δ^n, we find from Proposition [I].6.6.1 that the hermitian ∞-categories $\rho_n\mathcal{C}$ are Poincaré for every $[n] \in \Delta$ and from Proposition [I].6.6.2 that the hermitian functor $\sigma^*\colon \rho_n\mathcal{C} \to \rho_m\mathcal{C}$ is Poincaré for every $\sigma\colon [m] \to [n]$ in Δ. We may hence consider $\rho(\mathcal{C}, \Omega)$ as a simplicial object in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$.

3.6.10. **Definition.** Let $\mathcal{E}$ be an ∞-category with sifted colimits. Given a functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$, we denote by $\mathrm{ad}\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ the functor given by

$$\mathrm{ad}\mathcal{F}(\mathcal{C}, \Omega) = |\mathcal{F}\rho(\mathcal{C}, \Omega)|.$$

Using the functoriality of the cotensor construction, we may promote the association $\mathcal{F} \mapsto \mathrm{ad}\mathcal{F}$ to a functor

$$\mathrm{ad}\colon \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}). \tag{59}$$

The inclusion of vertices then equips ad with a natural transformation $b_{\mathcal{F}}\colon \mathcal{F} \to \mathrm{ad}\mathcal{F}$.

In this section, we consider the ad-construction only in the case when $\mathcal{E} = \mathcal{S}p$, as this entails great simplifications (though the case $\mathcal{E} = \mathcal{S}$ is fundamental for the discussion of L-theory in §4.4). The key is that for a stable $\mathcal{E}$, the collection of additive functors from $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ to $\mathcal{E}$ is closed under colimits inside the ∞-category of all functors. Since in addition, the functor $\rho_n\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$ preserves split Poincaré-Verdier sequences by Proposition 1.4.15, it follows that $\mathrm{ad}\mathcal{F}$ is additive whenever $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ is.

In particular, we may consider ad as a functor

$$\mathrm{ad}\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S}p). \tag{60}$$

3.6.11. **Remark.** The analogous statement with target category $\mathcal{S}$ requires an additivity theorem for the ρ-construction. Lurie showed in [Lur11, Lecture 8, Corollary 9] (see Theorem 4.4.2) that $\mathcal{L} = \mathrm{ad}(\mathrm{Pn})$ is even Verdier-localising, generalising results of Ranicki in more classical language; see e.g. [Ran92, Proposition 13.11].

We now set out to show:

3.6.12. **Proposition.** *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ be an additive functor. Then*

- *i) if $\mathcal{F}$ is bordism invariant, then the map $b_{\mathcal{F}}\colon \mathcal{F} \Rightarrow \mathrm{ad}\mathcal{F}$ is an equivalence.*
- *ii)* $\mathrm{ad}(\mathcal{F}^{\mathrm{hyp}}) \simeq *$.

Combining this with Lemma 3.6.9 and the fact that ad evidently commutes with colimits, we obtain:

3.6.13. **Corollary.** *The natural transformation b exhibits* ad *as a bordification functor.*

For the proof of Proposition 3.6.12, we denote by $\mathcal{P}([n])$ the full power set of $[n]$ and endow $\mathrm{Fun}(\mathcal{P}([n])^{\mathrm{op}}, \mathcal{C})$ with the hermitian structure $\Qoppa^{\mathrm{tf}}$ that sends a cubical diagram $\varphi \colon \mathcal{P}([n])^{\mathrm{op}} \to \mathcal{C}$ to the total fibre of $\Qoppa^{[1]} \circ \varphi^{\mathrm{op}}$; through the isomorphism

$$\mathcal{P}([n])^{\mathrm{op}} \cong \prod\nolimits_{i=0}^{n} [1]$$

the hermitian ∞-category $(\mathrm{Fun}(\mathcal{P}([n])^{\mathrm{op}}, \mathcal{C}), \Qoppa^{\mathrm{tf}})$ is equivalent to $\mathrm{Met}^{(n+1)}(\mathcal{C}, \Qoppa^{[1]})$, but in the form given, it is clear that it assembles into a functor $\mathrm{Cat}^{\mathrm{h}}_{\infty} \to \mathrm{sCat}^{\mathrm{h}}_{\infty}$. Through the identification as an iterated metabolic object, it is, however, easy to check that it restricts to $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{sCat}^{\mathrm{p}}_{\infty}$.

3.6.14. **Lemma.** *The sequence*

$$\rho_n(\mathcal{C}, \Qoppa) \longrightarrow (\mathrm{Fun}(\mathcal{P}([n])^{\mathrm{op}}, \mathcal{C}), \Qoppa^{\mathrm{tf}}) \xrightarrow{\mathrm{ev}_{\emptyset}} (\mathcal{C}, \Qoppa^{[1]}) \tag{61}$$

is a Poincaré-Verdier sequence for all Poincaré ∞-categories $(\mathcal{C}, \Qoppa)$ and $n \in \mathbb{N}$. Furthermore, there are equivalences

$$(\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathrm{Ar}(\mathcal{C})), \Qoppa^{\mathrm{tf}}_{\mathrm{met}}) \simeq \mathrm{dec}(\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathcal{C}), \Qoppa^{\mathrm{tf}}) \simeq \mathrm{Met}(\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathcal{C}), \Qoppa^{\mathrm{tf}})$$

of simplicial Poincaré ∞-categories.

Note that the left term in the second display is just $\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathcal{D}), \Phi^{\mathrm{tf}})$ for $(\mathcal{D}, \Phi) = \mathrm{Met}(\mathcal{C}, \Qoppa)$.

Proof. The map $\mathrm{ev}_{\emptyset}$ is given by evaluation of the cubical diagram φ at $\emptyset$, together with the canonical projection of hermitian functors. Under the equivalence of the middle term with $\mathrm{Met}^{(n+1)}(\mathcal{C}, \Qoppa^{[1]})$, the second map is the $(n+1)$-fold iteration of the map $\mathrm{met} \colon \mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}) \to (\mathcal{C}, \Qoppa^{[1]})$; thus it is a split Poincaré-Verdier projection. Its kernel is equivalent to the first term by restriction along $\mathcal{T}_n = \mathcal{P}_0([n])^{\mathrm{op}} \subset \mathcal{P}([n])^{\mathrm{op}}$ and the equivalence

$$\lim_{\emptyset \neq A \subseteq [n]} \Qoppa \circ \varphi^{\mathrm{op}}(A) \simeq \mathrm{fib}\big(0 \longrightarrow \lim_{\emptyset \neq A \subseteq [n]} \Qoppa^{[1]} \circ \varphi^{\mathrm{op}}(A)\big),$$

since for $\varphi \in \ker(\mathrm{ev}_{\emptyset})$ the second term is equivalent to the total fibre of $\Qoppa^{[1]} \circ \varphi^{\mathrm{op}}$.

For the second claim, note that commuting limits and functor categories gives equivalences

$$\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathrm{Ar}(\mathcal{C})), \Qoppa^{\mathrm{tf}}_{\mathrm{met}}) \simeq \mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}} \times \Delta^1, \mathcal{C}), \Qoppa^{\mathrm{tf}}) \simeq \mathrm{Met}(\mathrm{Fun}(\mathcal{P}([-])^{\mathrm{op}}, \mathcal{C}), \Qoppa^{\mathrm{tf}})$$

and the middle term is the requisite décalage by inspection. □

Proof of Proposition 3.6.12. If $\mathcal{F}$ is bordism invariant, then it vanishes on the middle term of (61) (which is an iterated metabolic construction); we conclude that the left term becomes constant in n, after application of $\mathcal{F}$. This shows i).

To show ii), we note that for a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$

$$\mathcal{F}^{\mathrm{hyp}}\left(\mathrm{Fun}(\mathcal{P}[-], \mathcal{C})\right) = \mathcal{F}\left(\mathrm{Hyp}(\mathrm{Fun}(\mathcal{P}[-], \mathcal{C}))\right) \simeq \mathcal{F}\left(\mathrm{Met}(\mathrm{Fun}(\mathcal{P}[-], \mathcal{C}), \Qoppa^{\mathrm{tf}})\right)$$

by 3.1.4. But this is the décalage of $\mathcal{F}(\mathrm{Fun}(\mathcal{P}[-], \mathcal{C}), \Qoppa^{\mathrm{tf}})$ by Lemma 3.6.14 with augmentation induced by $\mathrm{ev}_{\emptyset}$. Interpreting this map as a map of split simplicial objects, we conclude that its fibre $\mathcal{F}^{\mathrm{hyp}}(\rho_n(\mathcal{C}, \Qoppa))$ is split over 0, and therefore has contractible realisation. □

3.6.15. **Remark.** To see that $\mathcal{F}\rho(\mathcal{C}, \Qoppa)$ is a constant simplicial space if $\mathcal{F}$ is bordism invariant, one can alternatively observe that the degeneracy maps $(\mathcal{C}, \Qoppa) = \rho_0(\mathcal{C}, \Qoppa) \to \rho_n(\mathcal{C}, \Qoppa)$ is the inclusion of the homology ∞-category $\mathrm{Hlgy}(\mathcal{L}^+_n)$ for the following isotropic subcategory $\mathcal{L}^+_n \subseteq \rho_n(\mathcal{C}, \Qoppa)$: Let $\mathcal{T}^0_n \subseteq \mathcal{T}_n$ be the subposet spanned by those $S \subseteq [n]$ which contain 0 and $\mathcal{M}^+_n \subseteq \mathrm{Fun}(\mathcal{T}_n, \mathcal{C})$ be the full subcategory spanned by those diagrams $\varphi \colon \mathcal{T}_n \to \mathcal{C}$ which are left Kan extensions of their restriction to $\mathcal{T}^0_n$, i.e. such that $\varphi(S \cup \{0\}) \to \varphi(S)$ is an equivalence for every $S \subseteq \{1, ..., n\}$. Then $\mathcal{L}^+_n \subseteq \mathcal{M}^+_n$ may be taken to consist of those diagrams which additionally satisfy $\varphi(\{0\}) \simeq 0$.

One readily checks that for $\varphi \in \mathcal{M}^+_n$ there is an equivalence

$$\Qoppa^{\mathcal{T}_n}(\varphi) \simeq \Qoppa(\varphi(0)),$$

so $\mathcal{L}^+_n$ really is isotropic. Furthermore, $(\mathcal{L}^+_n)^{\perp} = \mathcal{M}^-_n$ and $\mathrm{D}_{\Qoppa^{\mathcal{T}_n}}(\mathcal{L}^+_n)^{\perp} = \mathcal{M}^+_n$, where $\mathcal{M}^-_n \subseteq \mathrm{Fun}(\mathcal{T}_n, \mathcal{C})$ is the full subcategory spanned by those diagrams $\varphi \colon \mathcal{T}_n \to \mathcal{C}$ whose restriction to $\mathcal{T}^0_n$ is constant.

Thus $\mathrm{Hlgy}(\mathcal{L}^+_n) \simeq \mathcal{M}^+_n \cap \mathcal{M}^-_n$ consists precisely of the constant diagrams as desired.

In [WW98, Lemma 9.3], Weiss and Williams give a direct verification that $\mathrm{ad}(\mathrm{K}) \simeq 0$, and their proof immediately generalises to give a different argument for the vanishing of bordifications of all additive functors of the form $\mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p$. To use the bordification procedure ad directly in other circumstances, however, one would have to investigate the effect of the ρ-construction on an arbitrary additive functor $\mathcal{F} \colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}$. In particular, one would have to provide an additivity theorem in this generality, to obtain a handle on the geometric realisation occurring in the ad-construction (essentially for the reasons spelled out in Remark 3.4.11). As mentioned in Remark 3.6.11, such a statement was worked out in the case $\mathcal{F} = \mathrm{Pn}$ by Lurie (see [Lur11, Lecture 8, Corollary 9]). We give a detailed account of this case in 4.4.2 below, but refrain from exhibiting further cases in the present paper.

Instead, we present a third bordification procedure more in line with the methods developed here. It is obtained by iterating the boundary map $\mathcal{F}(\mathcal{C}, Ϙ) \to \mathbb{S}^1 \otimes \mathcal{F}(\mathcal{C}, Ϙ^{[-1]})$ of the metabolic fibre sequence.

3.6.16. **Definition.** Let $\mathcal{F} : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$ be an additive functor. We define its *stabilisation* $\mathrm{stab}\mathcal{F}$ by the formula

$$(\mathrm{stab}\mathcal{F})(\mathcal{C}, Ϙ) = \mathrm{colim}(\mathcal{F}(\mathcal{C}, Ϙ) \longrightarrow \mathbb{S}^1 \otimes \mathcal{F}(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathbb{S}^2 \otimes \mathcal{F}(\mathcal{C}, Ϙ^{[-2]}) \longrightarrow \dots),$$

with structure maps the shifts of the boundary map for $\mathcal{F}$, and we denote by

$$\sigma_{\mathcal{F}}^{\infty} : \mathcal{F} \longrightarrow \mathrm{stab}\mathcal{F}$$

the arising natural transformation.

Recall from the discussion preceding Corollary 3.4.10 that the boundary map $\mathcal{F}(\mathcal{C}, Ϙ) \longrightarrow \mathbb{S}^1 \otimes \mathcal{F}(\mathcal{C}, Ϙ^{[-1]})$ of the metabolic fibre sequence

$$\mathcal{F}(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \xrightarrow{\mathrm{met}} \mathcal{F}(\mathcal{C}, Ϙ)$$

is also modelled by the inclusion of vertices

$$\sigma_{\mathcal{F}} \colon \mathcal{F}(\mathcal{C}, Ϙ) \longrightarrow |\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ)|.$$

So we equally well find that

$$\mathrm{stab}\mathcal{F}(\mathcal{C}, Ϙ) \simeq \mathrm{colim}(\mathcal{F}(\mathcal{C}, Ϙ) \xrightarrow{\sigma_{\mathcal{F}}} |\mathcal{F}\,\mathrm{Q}(\mathcal{C}, Ϙ)| \xrightarrow{|\sigma_{\mathcal{F}}\,\mathrm{Q}|} |\mathcal{F}\,\mathrm{Q}^{(2)}(\mathcal{C}, Ϙ)| \longrightarrow \dots),$$

arises from another iteration of the Q-construction.

3.6.17. **Remark.** Again, there is another equally sensible choice for the structure maps in the colimit system in Definition 3.6.16, namely the boundary maps for the functors $\mathbb{S}^i \otimes \mathcal{F}(-^{[-i]})$. These translate to $\sigma_{|\mathcal{F}\,\mathrm{Q}^{(i)}|}$ under the equivalence described above, and thus differ from the ones we choose to employ by a sign $(-1)^i$, compare Remark 3.4.4. Therefore, the choice has no effect on the colimit $\mathrm{stab}\mathcal{F}$.

3.6.18. **Proposition.** *Let $\mathcal{F} : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$ be an additive functor. Then*

i) if $\mathcal{F}$ is bordism invariant, the map $\sigma_{\mathcal{F}}^{\infty} : \mathcal{F} \to \mathrm{stab}\mathcal{F}$ is an equivalence.
ii) $\mathrm{stab}(\mathcal{F}^{\mathrm{hyp}}) \simeq 0$.

Proof. Property i) follows immediately from Corollary 3.5.8. To prove ii), it will suffice to show that for any additive $\mathcal{F} : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}p$ and any stable ∞-category $\mathcal{C}$, the boundary map

$$\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathbb{S}^1 \otimes \mathcal{F}(\mathrm{Hyp}(\mathcal{C})^{[-1]})$$

is null-homotopic. But this follows immediately from the metabolic functor $\mathrm{met} \colon \mathrm{Met}(\mathrm{Hyp}(\mathcal{C})) \to \mathrm{Hyp}(\mathcal{C})$ being split by Corollary [I].2.4.9. □

Since stab evidently commutes with colimits and preserves additivity, we can apply Lemma 3.6.9 and obtain:

3.6.19. **Corollary.** *The transformation σ^{∞} exhibits* stab *as a bordification.*

The filtration provided by the arising equivalence

$$\mathcal{F}^{\mathrm{bord}}(\mathcal{C}, Ϙ) = \mathrm{colim}_{d}\, \mathbb{S}^d \otimes \mathcal{F}(\mathcal{C}, Ϙ^{[-d]})$$

allows us to access the homotopy groups of the bordification of a space-valued $\mathcal{F}$:

3.6.20. **Corollary.** *For every additive* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$*, the structure maps in the colimit of Definition 3.6.16 induce isomorphisms*

$$\pi_i(\mathbb{C}\mathrm{ob}^{\mathcal{F}})^{\mathrm{bord}}(\mathcal{C},\Qoppa) \cong \pi_0|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Qoppa^{[-(i+1)]})|,$$

for all $i \in \mathbb{Z}$*. In particular, the induced maps*

$$\pi_i\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa) \longrightarrow \pi_i\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa)$$

are isomorphisms for $i < 0$ *and for* $i = 0$ *become the canonical projection under the identification Theorem 3.1.9 of the source.*

In other words, the group $\pi_i(\mathbb{C}\mathrm{ob}^{\mathcal{F}})^{\mathrm{bord}}(\mathcal{C},\Qoppa)$ is the $\mathcal{F}$-based cobordism group of $(\mathcal{C},\Qoppa^{[-i]})$. In fact, the proof will show that the colimit description for $\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa)$ stabilises on π_i after step i.

Proof. By Proposition 3.5.8, we need only consider the case $i = -1$ to obtain the first statement and we may furthermore assume $\mathcal{F}$ group-like since both sides of the claimed isomorphism only depend on $\mathcal{F}^{\mathrm{grp}}$ (see Corollary 3.3.7 for the right hand side). But then by Corollary 3.4.8, the spectra $\mathbb{C}\mathrm{ob}^{\mathcal{F}}\,\mathrm{Met}(\mathcal{C},\Qoppa)$ are connective so all maps in the colimit sequence

$$\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa) = \mathrm{colim}(\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa) \longrightarrow \mathbb{S}^1 \otimes \mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa^{[-1]}) \longrightarrow \mathbb{S}^2 \otimes \mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C},\Qoppa^{[-2]}) \longrightarrow \dots),$$

induce isomorphisms on π_{-1}, as their fibres are given by $\mathbb{S}^k \otimes \mathcal{F}(\mathrm{Met}(\mathcal{C},\Qoppa^{[-k]}))$. We conclude using Proposition 3.4.7. The claim about the induced map in π_0 follows from Proposition 3.1.10 by unwinding definitions. □

3.6.21. **Remark.** i) For a general additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$, we do not know how to compute the homotopy groups of $\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa)$ in terms of those of $\mathcal{F}(\mathcal{C},\Qoppa)$. For $i \geq 0$ the canonical map

$$\pi_i\mathcal{F}(\mathcal{C},\Qoppa) \longrightarrow \pi_i\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa)$$

factors canonically as

$$\pi_i\mathcal{F}(\mathcal{C},\Qoppa) \longrightarrow \pi_i((\Omega^{\infty}\mathcal{F})^{\mathrm{bord}}(\mathcal{C},\Qoppa)) \longrightarrow \pi_i\mathcal{F}^{\mathrm{bord}}(\mathcal{C},\Qoppa),$$

and we already noted in Remark 3.6.3 that the right hand map is not an equivalence in general.

ii) One can also check that

$$\lim(\dots \mathbb{S}^{-2} \otimes \mathcal{F}(\mathcal{C},\Qoppa^{[2]}) \longrightarrow \mathbb{S}^{-1} \otimes \mathcal{F}(\mathcal{C},\Qoppa^{[1]}) \longrightarrow \mathcal{F}(\mathcal{C},\Qoppa))$$

is a cobordification of $\mathcal{F}$, but since this limit does not stabilise it is not nearly as useful as the equivalence $\mathcal{F}^{\mathrm{bord}} \simeq \mathrm{stab}(\mathcal{F})$.

Finally, let us mention that one can also use the stab-construction and naive Karoubi periodicity to provide another proof of Corollary 3.6.7 (without even investing that stab is a bordification). We can in fact show directly, that there is a bicartesian square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\sigma^{\infty}_{\mathcal{F}}"] \arrow[d] & \mathrm{stab}\mathcal{F} \arrow[d] \\
(\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2} \arrow[r] & (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{tC}_2}
\end{tikzcd}$$

as follows: Consider the natural transformation $\mathcal{F} \Rightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ for any additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ and apply stab. Using naive Karoubi periodicity Proposition 3.4.13, we find

$$\mathrm{stab}((\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2})(\mathcal{C},\Qoppa) \simeq \operatorname*{colim}_d \mathbb{S}^d \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C},\Qoppa^{[-d]})^{\mathrm{hC}_2} \simeq \operatorname*{colim}_d \left(\mathbb{S}^{d\sigma} \otimes \mathcal{F}^{\mathrm{hyp}}(\mathcal{C},\Qoppa)\right)^{\mathrm{hC}_2}$$

with the structure maps in the final colimit induced by the inclusions $\mathbb{S} \to \mathbb{S}^{\sigma}$ as fixed points. But for any C_2-spectrum, there is a canonical equivalence

$$\operatorname*{colim}_d(\mathbb{S}^{d\sigma} \otimes X)^{\mathrm{hC}_2} \simeq X^{\mathrm{tC}_2},$$

in fact, this is essentially the classical definition of Tate spectra, say in [GM95]; to obtain it from the definition as the cofibre of the norm, recall that $(\mathbb{S}^{d\sigma} \otimes X)^{\mathrm{tC}_2} \simeq X^{\mathrm{tC}_2}$, so that the analogous colimit system of Tate spectra is constant and therefore the colimit for the homotopy orbit spectra vanishes, since then the colimit can be permuted into the orbits and $\mathrm{colim}_d\, \mathbb{S}^{d\sigma} \otimes X \simeq 0$: The colimit is formed along maps $\mathbb{S} \to \mathbb{S}^{\sigma}$, which are (non-equivariantly!) null-homotopic. This produces the Tate square above. To see that it is bicartesian, note that by construction stab preserves cofibre sequences. Now, the cofibre of $\mathcal{F} \Rightarrow (\mathcal{F}^{\mathrm{hyp}})^{\mathrm{hC}_2}$ is easily checked to vanish on hyperbolic categories, so it is bordism invariant by Lemma 3.5.4. Thus by Proposition 3.6.18, $\sigma^{\infty}_{\mathcal{F}}$ induces an equivalence on vertical cofibres of the Tate square. It is therefore cocartesian. This proof is essentially the one given by Schlichting in [Sch17, Section 7] for $\mathcal{F} = \mathrm{GW}$ in the context of differential graded categories on which 2 acts invertibly.

From the fact that the Tate square is bicartesian, one can also obtain the fibre sequence

$$\mathcal{F}^{\mathrm{hyp}}_{\mathrm{hC}_2} \longrightarrow \mathcal{F} \longrightarrow \mathrm{stab}\mathcal{F}$$

and conclude that stab really is a bordification functor, reversing the logic used in the original proof of Corollary 3.6.7.

3.7. The genuine hyperbolisation of an additive functor.

In this final subsection, we recast the fundamental fibre square Corollary 3.6.7 of an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ as the isotropy separation square of a genuine C_2-spectrum, that is a spectral Mackey functor for the group C_2, refining the hyperbolisation $\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Omega) \in \mathcal{S}p^{\mathrm{hC}_2}$. This allows for a convenient way of combining Karoubi periodicity with the shifting behaviour of bordism invariant functors; see Theorem 3.7.7 below. Note, however, that in the end this reformulation does not yield additional information: The ∞-category of genuine C_2-spectra $\mathcal{S}p^{\mathrm{gC}_2}$ participates in a cartesian diagram

$$\begin{tikzcd}
\mathcal{S}p^{\mathrm{gC}_2} \arrow[r, "(-)^{\varphi \mathrm{C}_2} \Rightarrow (-)^{\mathrm{tC}_2}"] \arrow[d, "\mathrm{u}"] & \mathrm{Ar}(\mathcal{S}p) \arrow[d, "t"] \\
\mathcal{S}p^{\mathrm{hC}_2} \arrow[r, "(-)^{\mathrm{tC}_2}"] & \mathcal{S}p,
\end{tikzcd}$$

where $-^{\varphi \mathrm{C}_2}\colon \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$ extracts the geometric fixed points and u the underlying C_2-spectrum; we will give a quick proof of this folklore result in Remark 3.7.5 below. In particular, the data

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Omega) \in \mathcal{S}p^{\mathrm{hC}_2} \quad \text{and} \quad \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Omega) \to \mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Omega)^{\mathrm{tC}_2}$$

can be used to *define* the desired genuine refinement $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)$ of $\mathcal{F}^{\mathrm{hyp}}(\mathcal{C}, \Omega)$. The Mackey functor point of view does, however, have the advantage that the requisite data can be constructed, once and for all, at the level of Poincaré ∞-categories: In Corollary [I].7.4.18, we constructed (pre-)Mackey objects $\mathrm{gHyp}(\mathcal{C}, \Omega)$ in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$, see Theorem 3.7.1 below for the statement. The genuine C_2-spectrum $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)$ just described arises then by simply applying $\mathcal{F}$ to $\mathrm{gHyp}(\mathcal{C}, \Omega)$.

Let us briefly recall the notion of a spectral Mackey functor. For a discrete group G, we denote by $\mathrm{Span}(G)$ the span ∞-category of finite G-sets, introduced for the purposes of equivariant homotopy theory in [Bar17, Df. 3.6] (under the name effective Burnside category).

Then, a Mackey object in an additive ∞-category $\mathcal{A}$ is by definition a product preserving functor $\mathrm{Span}(G) \to \mathcal{A}$. If $\mathcal{A}$ is taken to be $\mathcal{S}p$, the results of [Nar16, Appendix A] or [GM20, Appendix C] show, that the arising ∞-category underlies the model category of orthogonal G-spectra classically used for the definition of genuine G-spectra, see e.g. [Sch20]. We will treat spectral Mackey functors as the definition of the latter objects and therefore put

$$\mathcal{S}p^{\mathrm{gC}_2} = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{C}_2), \mathcal{S}p).$$

Evaluation at the finite C_2-set C_2 then defines the functor $\mathrm{u}\colon \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p^{\mathrm{hC}_2}$, by retaining the action of the span

$$\mathrm{C}_2 \xleftarrow{\mathrm{id}} \mathrm{C}_2 \xrightarrow{\mathrm{flip}} \mathrm{C}_2.$$

Evaluation at the one-point C_2-set defines the genuine fixed points $-^{\mathrm{gC}_2}\colon \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$. A genuine C_2-spectrum thus gives rise to a pair of spectra (E^{gC_2}, E), together with a C_2-action on E and restriction and transfer maps

$$\mathrm{res}\colon E^{\mathrm{gC}_2} \to E^{\mathrm{hC}_2} \qquad \mathrm{tr}\colon E_{\mathrm{hC}_2} \to E^{\mathrm{gC}_2}$$

coming from the spans

$$* \leftarrow \mathrm{C}_2 \xrightarrow{\mathrm{id}} \mathrm{C}_2 \qquad \text{and} \qquad \mathrm{C}_2 \xleftarrow{\mathrm{id}} \mathrm{C}_2 \to * \tag{62}$$

together with a host of coherence data, which in particular identifies the composite $\mathtt{tr}\circ\mathtt{res}\colon E_{\mathrm{hC}_2} \to E^{\mathrm{hC}_2}$ with the norm map of E, and similarly for other target categories.

In Corollary [I].7.4.18, we showed:

3.7.1. **Theorem.** *The construction of hyperbolic categories canonically refines to a functor*

$$\mathrm{gHyp} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \longrightarrow \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{C}_2), \mathrm{Cat}^{\mathrm{p}}_{\infty})$$

together with natural equivalences of Poincaré ∞-categories

$$\mathrm{gHyp}(\mathcal{C}, \Qoppa)^{\mathrm{gC}_2} \simeq (\mathcal{C}, \Qoppa),$$

and C_2*-Poincaré ∞-categories*

$$\mathrm{u}(\mathrm{gHyp}(\mathcal{C}, \Qoppa)) \simeq \mathrm{Hyp}\,\mathcal{C},$$

such that transfer and restriction

$$\mathrm{gHyp}(\mathcal{C}, \Qoppa)_{\mathrm{hC}_2} \to \mathrm{gHyp}(\mathcal{C}, \Qoppa)^{\mathrm{gC}_2} \quad \textit{and} \quad \mathrm{gHyp}(\mathcal{C}, \Qoppa)^{\mathrm{gC}_2} \to \mathrm{gHyp}(\mathcal{C}, \Qoppa)^{\mathrm{hC}_2},$$

are naturally identified with

$$\mathrm{hyp}\colon\ \mathrm{Hyp}(\mathcal{C})_{\mathrm{hC}_2} \to (\mathcal{C}, \Qoppa) \quad \textit{and} \quad \mathrm{fgt}\colon (\mathcal{C}, \Qoppa) \to \mathrm{Hyp}(\mathcal{C})^{\mathrm{hC}_2}.$$

3.7.2. **Definition.** Let $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ be an additive functor. Then we call the composite

$$\mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{gHyp}} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{C}_2), \mathrm{Cat}^{\mathrm{p}}_{\infty}) \xrightarrow{\mathcal{F}} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{C}_2), \mathcal{S}p) = \mathcal{S}p^{\mathrm{gC}_2}.$$

the *genuine hyperbolisation* $\mathcal{F}^{\mathrm{ghyp}}$ of $\mathcal{F}$.

Now any genuine C_2-spectrum X has an associated isotropy separation square

$$\begin{tikzcd}
X^{\mathrm{gC}_2} \arrow[r] \arrow[d] & X^{\varphi\mathrm{C}_2} \arrow[d] \\
X^{\mathrm{hC}_2} \arrow[r] & X^{\mathrm{tC}_2},
\end{tikzcd}$$

where the simplest description of the geometric fixed points $-^{\varphi\mathrm{C}_2}$ for our purposes is as the cofibre of the transfer $X_{\mathrm{hC}_2} \to X^{\mathrm{gC}_2}$.

3.7.3. **Remark.** There are many other more conceptual descriptions of the geometric fixed points. For example [Bar17, B.7] describes $-^{\varphi\mathrm{C}_2} : \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$ as the left Kan extension along the fixed point functor

$$(-)^{\mathrm{C}_2} : \mathrm{Span}(\mathrm{C}_2) \to \mathrm{Span}(\mathrm{Fin}),$$

under the equivalence $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{S}p) \simeq \mathcal{S}p$ and classically they are often defined as the cofibre of $(X \otimes \mathbb{S}[\mathrm{EC}_2])^{\mathrm{gC}_2} \to X^{\mathrm{gC}_2}$, where $\mathrm{EC}_2 \in \mathcal{S}^{\mathrm{C}_2}$ is the unique C_2-space with empty fixed points, whose underlying space is contractible, see e.g. [Sch20, Proposition 7.6]; here $\mathcal{S}^{\mathrm{C}_2}$ is the ∞-category of functors from the opposite of the orbit category $\mathrm{O}(\mathrm{C}_2)$ of C_2 to $\mathcal{S}$ and the genuine suspension functor $\mathbb{S}[-]\colon \mathcal{S}^{\mathrm{C}_2} \to \mathcal{S}p^{\mathrm{gC}_2}$ is given as the composite

$$\mathcal{S}^{\mathrm{C}_2} \xrightarrow{\mathbb{S}[-]} \mathcal{S}p^{\mathrm{C}_2} \xrightarrow{\mathrm{Lan}} \mathcal{S}p^{\mathrm{gC}_2},$$

where the second functor is left Kan extension along the evident inclusion $\mathrm{O}(\mathrm{C}_2)^{\mathrm{op}} \to \mathrm{Span}(\mathrm{C}_2)$ (it is also the left derived functor of the suspension spectrum functor in the classical model category picture). The genuine fixed points of the result are described by tom Dieck's splitting [Sch20, Theorem 6.12]

$$\mathbb{S}[X]^{\mathrm{gC}_2} \simeq \mathbb{S}[X^{\mathrm{gC}_2}] \oplus \mathbb{S}[X_{\mathrm{hC}_2}],$$

which can be recovered from the pointwise formula for the Kan extension.

Geometric fixed points are in fact characterised in terms of this construction as the unique colimit preserving, symmetric monoidal functor $\mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$ participating in a diagram

$$\begin{tikzcd}
\mathcal{S}^{\mathrm{C}_2} \arrow[r, "\mathrm{gC}_2"] \arrow[d, "\mathbb{S}[-]"] & \mathcal{S} \arrow[d, "\mathbb{S}[-]"] \\
\mathcal{S}p^{\mathrm{gC}_2} \arrow[r, "\varphi\mathrm{C}_2"] & \mathcal{S}p,
\end{tikzcd}$$

see [Sch20, Remark 7.15].

Now from the identification of the transfer in Theorem 3.7.1 and Proposition 3.6.6, there results an identification $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)^{\varphi\mathrm{C}_2} \simeq \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Qoppa)$ and by the universal property of bordifications, this determines the entire isotropy separation square. We conclude:

3.7.4. **Corollary.** *The isotropy separation square of the genuine* C_2*-spectrum* $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)$ *is naturally identified with the fundamental fibre square, in symbols*

$$\begin{tikzcd}
\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)^{\mathrm{gC}_2} \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)^{\varphi\mathrm{C}_2} \arrow[d] & & \mathcal{F}(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Qoppa) \arrow[d] \\
\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)^{\mathrm{hC}_2} \arrow[r] & \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)^{\mathrm{tC}_2} & \simeq & \mathcal{F}\,\mathrm{Hyp}(\mathcal{C})^{\mathrm{hC}_2} \arrow[r] & \mathcal{F}\,\mathrm{Hyp}(\mathcal{C})^{\mathrm{tC}_2},
\end{tikzcd}$$

for any additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *and Poincaré* ∞*-category* $(\mathcal{C}, \Qoppa)$.

In particular, combining this with the following remark, we find that the functor $\mathcal{F}^{\mathrm{ghyp}}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p^{\mathrm{gC}_2}$ is additive again (although this is also readily checked straight from the definition).

3.7.5. **Remark.** That the extraction of isotropy separation squares leads to a cartesian square

$$\begin{tikzcd}
\mathcal{S}p^{\mathrm{gC}_2} \arrow[r, "(-)^{\varphi\mathrm{C}_2} \Rightarrow (-)^{\mathrm{tC}_2}"] \arrow[d, "\mathrm{u}"] & \mathrm{Ar}(\mathcal{S}p) \arrow[d, "t"] \\
\mathcal{S}p^{\mathrm{hC}_2} \arrow[r, "(-)^{\mathrm{tC}_2}"] & \mathcal{S}p,
\end{tikzcd}$$

is a direct application of Proposition A.2.12: The forgetful functor $\mathrm{u}\colon \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p^{\mathrm{hC}_2}$ admits both a left and a right adjoint through Kan extension, whose images are often said to consist of the Borel (co)complete C_2-spectra. One readily checks the compositions starting and ending in $\mathcal{S}p^{\mathrm{hC}_2}$ to be the identity. Thus u is a split Verdier projection (of non-small ∞-categories). The results of §A.2 together with some elementary manipulations of the functors involved complete this to a stable recollement

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, "R"] & \mathcal{S}p^{\mathrm{gC}_2} \arrow[l, bend right, "(-)^{\varphi\mathrm{C}_2}"'] \arrow[l, bend left] \arrow[r] & \mathcal{S}p^{\mathrm{hC}_2}, \arrow[l, bend right, "(-)^{\mathrm{q}}"'] \arrow[l, bend left, "(-)^{\mathrm{s}}"]
\end{tikzcd}$$

where R is given by restriction along the fixed point functor $\mathrm{Span}(\mathrm{C}_2) \to \mathrm{Span}(\mathrm{Fin})$, under the identification $\mathcal{S}p \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{S}p)$, and the lower left functor takes X to the fibre of $X^{\mathrm{gC}_2} \to X^{\mathrm{hC}_2}$. The classifying functor of this recollement is given by $-^{\mathrm{tC}_2}\colon \mathcal{S}p^{\mathrm{hC}_2} \to \mathcal{S}p$, so Proposition A.2.12 shows that the square above is cartesian. Furthermore, the resulting bicartesian square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & R(X^{\varphi\mathrm{C}_2}) \arrow[d] \\
(\mathrm{u}X)^{\mathrm{s}} \arrow[r] & R(((\mathrm{u}X)^{s})^{\varphi\mathrm{C}_2})
\end{tikzcd}$$

recovers the isotropy separation square of X upon applying genuine fixed points.

Finally, we use the genuine spectrum $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Qoppa)$ to combine naive periodicity with the behaviour of bordism invariant functors under shifting.

3.7.6. **Lemma.** *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ *be an additive functor and* $\mathcal{C}$ *a stable* ∞*-category. Then the map of genuine* C_2*-spectra*

$$\mathrm{C}_2 \otimes \mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \to \mathcal{F}^{\mathrm{ghyp}}(\mathrm{Hyp}\,\mathcal{C})$$

adjoint to the diagonal

$$\mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \to \mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \oplus \mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \simeq \mathcal{F}(\mathrm{Hyp}\,\mathcal{C} \times \mathrm{Hyp}\,\mathcal{C}) \simeq \mathcal{F}(\mathcal{H}\mathrm{yp}(\mathrm{Hyp}\,\mathcal{C}))$$

is an equivalence. In particular, $\mathcal{F}^{\mathrm{ghyp}}(\mathrm{Met}(\mathcal{C}, \Omega)) \simeq \mathrm{C}_2 \otimes \mathcal{F}(\mathrm{Hyp}\,\mathcal{C})$.

Proof. The map is an equivalence both on underlying spectra and on geometric fixed points: On underlying spectra, this follows immediately from the corresponding statement

$$\mathcal{H}\mathrm{yp}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{C}_2 \otimes \mathrm{Hyp}(\mathcal{C})$$

on underlying C_2-Poincaré ∞-categories from [I].7.4.15. Furthermore, both spectra have vanishing geometric fixed points: The left hand side by the symmetric monoidality of geometric fixed points together with ${\mathrm{C}_2}^{\mathrm{gC}_2} = \emptyset$, the right hand side by bordism invariance. □

As a direct generalisation of Proposition 3.4.13, we then have:

3.7.7. **Theorem** (Genuine Karoubi periodicity)**.** *Let* $(\mathcal{C}, \Omega)$ *be a Poincaré* ∞*-category and* $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ *an additive functor. Then there is a natural equivalence of genuine* C_2*-spectra*

$$\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega^{[-1]})) \simeq \mathbb{S}^{\sigma-1} \otimes \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega),$$

which translates the boundary map

$$\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)) \longrightarrow \mathbb{S}^{1} \otimes \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega^{[-1]}))$$

of the metabolic fibre sequence into the map induced by the inclusion $\mathbb{S} \to \mathbb{S}^{\sigma}$ *as the fixed points.*

In particular, passing to geometric fixed points we recover the equivalence $\mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Omega^{[i]}) \simeq \mathbb{S}^{i} \otimes \mathcal{F}^{\mathrm{bord}}(\mathcal{C}, \Omega)$ from Proposition 3.5.8.

Proof. Given the previous lemma, the proof of Proposition 3.4.13 applies essentially verbatim, when interpreted in the ∞-category of genuine C_2-spectra: Lemma 3.7.6 identifies the once-rotated metabolic fibre sequence

$$\mathcal{F}^{\mathrm{ghyp}}(\mathrm{Met}(\mathcal{C}, \Omega)) \xrightarrow{\mathrm{met}} \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega) \xrightarrow{\partial} \mathbb{S}^{1} \otimes \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega^{[-1]})$$

with

$$\mathrm{C}_2 \otimes \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \longrightarrow \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega) \longrightarrow \mathbb{S}^{\sigma} \otimes \mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)$$

obtained by tensoring $\mathcal{F}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)$ with $\mathbb{S}[\mathrm{C}_2] \to \mathbb{S} \to \mathbb{S}^{\sigma}$. □

Alternatively, the statement of Theorem 3.7.7 can also be deduced from Proposition 3.4.13 together with Proposition 3.5.8, via the interpretation of genuine C_2-spectra as isotropy separation squares: For every genuine C_2-spectrum the canonical map $X \simeq \mathbb{S} \otimes X \to \mathbb{S}^{\sigma} \otimes X$ induces an equivalence on geometric fixed points, for example by monoidality and $(\mathbb{S}^{\sigma})^{\varphi \mathrm{C}_2} \simeq \mathbb{S}$.

Therefore the effect of tensoring with $\mathbb{S}^{\sigma-1}$ on both geometric fixed points and the Tate construction is a shift, which by Proposition 3.5.8 is also the effect of shifting the quadratic functors on these terms. Combined with the statement on underlying spectra Proposition 3.4.13, we obtain the claim.

4. GROTHENDIECK-WITT THEORY

In this section, we define the central object of this paper, the Grothendieck-Witt spectrum $\mathrm{GW}(\mathcal{C}, \Omega)$ associated to a Poincaré ∞-category $(\mathcal{C}, \Omega)$, and discuss its main properties. Most of the results are corollaries of the results of §3.

We start out by defining the Grothendieck-Witt space $\mathcal{GW}(\mathcal{C}, \Omega)$ and record its properties, as specialisations of the general results of the previous section to the case $\mathcal{F} = \mathrm{Pn} \in \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$. We then proceed to analyse the Grothendieck-Witt spectrum in the same manner, in particular identifying its hyperbolisation as K-theory and its bordification as L-theory.

This leads to the identification of the homotopy type of the algebraic cobordism categories in Corollary 4.2.3, the fundamental fibre square in Corollary 4.4.14, localisation sequences for Grothendieck-Witt spectra of discrete rings in Corollary 4.4.17 and our generalisation of Karoubi periodicity in Corollaries 4.3.4 and 4.5.5, constituting the main results of the present paper.

In the final subsection, we spell out the relation of our constructions to the LA-theory of Weiss and Williams from [WW14]. In particular, our results provide a cycle model for the infinite loop spaces of their spectra.

4.1. The Grothendieck-Witt space.

In this section, we define the Grothendieck-Witt space of a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, whose homotopy groups are by definition the higher Grothendieck-Witt groups of $(\mathcal{C}, \Qoppa)$.

Recall that a functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ into an ∞-category with finite limits is called additive if it carries split Poincaré-Verdier squares to cartesian squares, see §1.5. Additive functors automatically take values in E_∞-monoids (with respect to the cartesian product in $\mathcal{E}$) but may well not be group-like; the functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ taking Poincaré objects being the first example. Denoting by $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{E})$ the full subcategory of additive functors, Corollary 3.3.6 asserts that the inclusion

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathrm{E}_\infty}(\mathcal{S})) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$$

admits a left adjoint $(-)^{\mathrm{grp}}$, the group completion functor.

4.1.1. **Definition.** We define the *Grothendieck-Witt space* functor $\mathcal{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathrm{Grp}_{\mathrm{E}_\infty}$ to be the group-completion

$$\mathcal{GW}(\mathcal{C}, \Qoppa) = \mathrm{Pn}^{\mathrm{grp}}(\mathcal{C}, \Qoppa),$$

of the functor $\mathrm{Pn} \in \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$. Furthermore, for a Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, we set

$$\mathrm{GW}_i(\mathcal{C}, \Qoppa) = \pi_i\, \mathcal{GW}(\mathcal{C}, \Qoppa),$$

the *Grothendieck-Witt-groups* of $(\mathcal{C}, \Qoppa)$.

We already introduced a group $\mathrm{GW}_0(\mathcal{C}, \Qoppa)$ explicitly in §[I].2.5 as the quotient of $\pi_0\mathrm{Pn}(\mathcal{C}, \Qoppa)$ given by identifying every metabolic object with the hyperbolisation of its Lagrangian. We will see in Corollary 4.1.8 below that this matches with the definition above. Let us also mention that we frequently regard $\mathcal{GW}$ as a functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ without further comment.

As a direct reformulation of the definition of the Grothendieck-Witt functor, we record:

4.1.2. **Observation.** *The functor* $\mathcal{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *is additive and group-like, and it is the initial such functor under* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$.

We will show in Corollary 4.4.15 that $\mathcal{GW}$ is in fact Verdier localising (and not just additive); that is, it takes all Poincaré-Verdier squares to cartesian squares (and not just the split ones).

4.1.3. **Remark.** Let us explicitly warn the reader that $\mathcal{GW}$ is the group-completion of Pn inside $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S})$, but *not* given as a levelwise group-completion, that is, $\mathcal{GW}(\mathcal{C}, \Qoppa)$ is generally not the group completion of the E_∞-monoid $\mathrm{Pn}(\mathcal{C}, \Qoppa)$. Indeed, the levelwise group completion of Pn does not yield an additive functor. The functor $\mathcal{GW}$ can then be considered as the universal way of fixing this. For the relation between Grothendieck-Witt spectra defined by group-completions of forms and those considered here, see [HS21].

From the results of the previous section, we obtain several formulae for $\mathcal{GW}(\mathcal{C}, \Qoppa)$: Recall from Definition 2.3.2 the cobordism ∞-category $\mathrm{Cob}(\mathcal{C}, \Qoppa)$ associated to the Segal space $\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})$ given by the hermitian Q-construction. From Corollary 3.3.6, we find:

4.1.4. **Corollary.** *There are canonical equivalences*

$$\mathcal{GW}(\mathcal{C}, \Qoppa) \simeq \Omega|\mathrm{Cob}(\mathcal{C}, \Qoppa)| \simeq \Omega|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})|$$

natural in the Poincaré ∞-category $(\mathcal{C}, \Qoppa)$.

These formulae are in accordance with the usual definition of the K-space $\Omega|\operatorname{Span}(\mathcal{C})| \simeq \Omega|\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})|$ of $\mathcal{C}$.

4.1.5. **Corollary.** *The functor* $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *preserves filtered colimits.*

Proof. It is a composite of functors

$$\mathrm{Cat}^{\mathrm{p}}_\infty \xrightarrow{(-)^{[1]}} \mathrm{Cat}^{\mathrm{p}}_\infty \xrightarrow{\mathrm{Q}} \mathrm{sCat}^{\mathrm{p}}_\infty \xrightarrow{\mathrm{Pn}} \mathrm{s}\mathcal{S} \xrightarrow{|-|} \mathcal{S} \xrightarrow{\Omega} \mathcal{S}$$

each of which preserves filtered colimits: The first is an equivalence, for Pn this is [I].6.1.8, for Q_n this follows from the same statement for $\mathrm{Fun}(\mathrm{TwAr}[n], -)$, which in turn follows from the finiteness of $\mathrm{TwAr}[n]$, and for the remaining two functors this is well-known. □

Classically, the Grothendieck-Witt space is often defined as the fibre of the forgetful functor from the hermitian to the usual Q-construction. We obtain such a description from the metabolic fibre sequence: Applying the hermitian Q-construction to the split Poincaré-Verdier sequence

$$(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{Met}(\mathcal{C}, \Qoppa^{[1]}) \longrightarrow (\mathcal{C}, \Qoppa^{[1]})$$

results in the fibre sequence

$$|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa)| \longrightarrow |\mathrm{Pn}\,\mathrm{Q}\,\mathrm{Met}(\mathcal{C}, \Qoppa^{[1]})| \xrightarrow{\mathrm{met}} |\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})|,$$

modelling the algebraic Genauer sequence

$$|\mathrm{Cob}(\mathcal{C}, \Qoppa^{[-1]})| \longrightarrow |\mathrm{Cob}^{\partial}(\mathcal{C}, \Qoppa^{[-1]})| \xrightarrow{\partial} |\mathrm{Cob}(\mathcal{C}, \Qoppa)|;$$

see Theorem 2.5.1 and Corollary 2.5.2. Now from Proposition 2.3.13 and Example 2.3.3, we obtain:

4.1.6. **Corollary.** *There are canonical equivalences*

$$|\mathrm{Pn}\,\mathrm{Q}\,\mathrm{Met}(\mathcal{C}, \Qoppa^{[1]})| \simeq |\mathrm{Pn}\,\mathrm{Q}\,\mathrm{Hyp}(\mathcal{C})| \simeq |\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})|$$

under which the metabolic fibre sequence corresponds to

$$|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa)| \xrightarrow{\mathrm{fgt}} |\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})| \xrightarrow{\mathrm{hyp}} |\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa^{[1]})|.$$

In particular, there are natural equivalences

$$\mathcal{GW}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \simeq \mathcal{K}(\mathcal{C}) \quad \textit{and} \quad \mathcal{GW}(\mathrm{Hyp}(\mathcal{D})) \simeq \mathcal{K}(\mathcal{D})$$

for all Poincaré ∞-categories $(\mathcal{C}, \Qoppa)$ *and stable ∞-categories* $\mathcal{D}$.

We also immediately obtain:

4.1.7. **Corollary.** *There are canonical equivalences*

$$\mathcal{GW}(\mathcal{C}, \Qoppa) \simeq \mathrm{fib}(|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, \Qoppa)| \xrightarrow{\mathrm{fgt}} |\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})|)$$

natural in the Poincaré ∞-category $(\mathcal{C}, \Qoppa)$, *and natural fibre sequences*

$$\mathcal{GW}(\mathcal{C}, \Qoppa^{[-1]}) \xrightarrow{\mathrm{fgt}} \mathcal{K}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathcal{GW}(\mathcal{C}, \Qoppa).$$

This formula for the Grothendieck-Witt space is the transcription of the classical definition for example from [Sch10a] into our framework.

We can also use these formulae for an explicit description of $\mathrm{GW}_0(\mathcal{C}, \Qoppa)$, giving another direct link. From Theorem 3.1.9, we find:

4.1.8. **Corollary.** *The natural map*

$$\pi_0\mathrm{Pn}(\mathcal{C}, \Qoppa) \longrightarrow \mathrm{GW}_0(\mathcal{C}, \Qoppa)$$

exhibits the target as the quotient of the source by the congruence relation generated by

$$[x, q] \sim [\mathrm{hyp}(w)], \tag{63}$$

where (x, q) *runs through the Poincaré objects of* $(\mathcal{C}, \Qoppa)$ *with Lagrangian* $w \longrightarrow x$.

In particular, $\mathrm{GW}_0(\mathcal{C}, \Omega)$ is the quotient of $\pi_0\mathrm{Pn}(\mathcal{C}, \Omega)^{\mathrm{grp}}$ by the subgroup spanned by the differences $[x, q] - [\mathrm{hyp}(w)]$, but part of the statement is that one does not need to complete $\pi_0\mathrm{Pn}(\mathcal{C}, \Omega)$ to a group in order to obtain $\mathrm{GW}_0(\mathcal{C}, \Omega)$ as a quotient. From Corollary 3.1.8 or indeed from Lemma [I].2.5.3, we find that for $[x, q] \in \mathrm{GW}_0(\mathcal{C}, \Omega)$ we have

$$-[x, q] = [x, -q] + \mathrm{hyp}(\Omega X).$$

Finally, we showed in Corollary [I].5.2.8 that the functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{S})$ admits a canonical lax symmetric monoidal structure with respect to the tensor product of Poincaré ∞-categories on the left and the tensor product of E_∞-spaces on the right; informally it is simply given by tensoring Poincaré objects. Since $\pi_0\colon \mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{S}) \to \mathrm{CMon}$ is also lax symmetric monoidal for the tensor products on both sides, the functor $\pi_0\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{CMon}$ acquires a canonical lax symmetric monoidal structure. We showed in Proposition [I].7.5.3:

4.1.9. **Proposition.** *The functor* $\mathrm{GW}_0\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{A}b$ *admits a unique lax symmetric monoidal structure, making the transformation* $\pi_0\mathrm{Pn} \to \mathrm{GW}_0$ *symmetric monoidal.*

In Paper [IV], we will enhance this to a lax symmetric monoidal structure on the functor $\mathcal{GW}$ itself, but for the purposes of the present paper, the above suffices.

4.2. **The Grothendieck-Witt spectrum.** Our next goal is to deloop the Grothendieck-Witt-space into a spectrum valued additive functor. To this end recall from Corollary 3.3.6 that the forgetful functor

$$\Omega^\infty\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathcal{S}p) \longrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathrm{E}_\infty})$$

admits a left adjoint $\mathbb{C}\mathrm{ob}$.

4.2.1. **Definition.** We define the *Grothendieck-Witt spectrum* functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ by

$$\mathrm{GW}(\mathcal{C}, \Omega) = \mathbb{C}\mathrm{ob}^{\mathcal{GW}}(\mathcal{C}, \Omega),$$

and denote by $\mathrm{GW}_*(\mathcal{C}, \Omega)$ its homotopy groups.

We will see in Corollary 4.2.3 below, that for $i \geq 0$ this conforms with Definition 4.1.1.

Again, we list the properties that are immediate from the results of the previous section. As a reformulation of the definition we find:

4.2.2. **Corollary.** *The functor* $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *is additive, and it is the initial such functor equipped a transformation*

$$\mathrm{Pn} \Rightarrow \Omega^\infty\,\mathrm{GW}$$

of functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$.

In fact, we show in Corollary 4.4.15 below, that GW is Verdier-localising and not just additive.

Proof. By Corollary 3.4.6, the functor GW is the initial additive functor to spectra equipped with a transformation $\mathcal{GW} \Rightarrow \Omega^\infty\,\mathrm{GW}$ of E_∞-groups. By the universal property of $\mathcal{GW}$ established in Observation 4.1.2, the functor GW is therefore also the initial additive functor to spectra with a transformation $\mathrm{Pn} \Rightarrow \Omega^\infty\,\mathrm{GW}$ of E_∞-monoids. □

As additive functors to spaces carry unique refinements to $\mathrm{Mon}_{\mathrm{E}_\infty}(\mathcal{S})$, these statements remain true upon dropping the E_∞-structures, and by adjunction GW is also the initial additive functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ under $\mathbb{S}[\mathrm{Pn}]$.

Next, we identify the spaces $\Omega^{\infty-i}\,\mathrm{GW}(\mathcal{C}, \Omega)$. To this end, recall the i-fold simplicial space

$$\mathrm{Cob}_i(\mathcal{C}, \Omega) = \mathrm{Pn}\,\mathrm{Q}^{(i)}(\mathcal{C}, \Omega^{[i]}),$$

given by the iterated hermitian Q-construction from Definition 2.2.1. These model the extended cobordism categories of $(\mathcal{C}, \Omega)$ and by Proposition 3.4.5 form a positive Ω-spectrum $\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\mathcal{C}, \Omega)$. The natural map

$$\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\mathcal{C}, \Omega) \longrightarrow \mathrm{GW}(\mathcal{C}, \Omega)$$

exhibits the right hand side as the spectrification of the left. From Proposition 3.4.5, we also find:

4.2.3. **Corollary.** *For any Poincaré ∞-category* $(\mathcal{C}, \Qoppa)$*, there are canonical equivalences*

$$\mathcal{GW}(\mathcal{C}, \Qoppa) \simeq \Omega^\infty \operatorname{GW}(\mathcal{C}, \Qoppa) \quad \textit{and} \quad |\mathrm{Cob}_i(\mathcal{C}, \Qoppa)| \simeq \Omega^{\infty - i} \operatorname{GW}(\mathcal{C}, \Qoppa)$$

for any $i \geq 1$*, that are natural in* $(\mathcal{C}, \Qoppa)$*. In particular, we obtain isomorphisms*

$$\pi_i \operatorname{GW}(\mathcal{C}, \Qoppa) \cong \operatorname{GW}_i(\mathcal{C}, \Qoppa)$$

for all $i \geq 0$*.*

4.2.4. **Corollary.** *The functor* $\operatorname{GW} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}p$ *preserves filtered colimits.*

Proof. The functor $\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}$ from $\mathrm{Cat}^{\mathrm{p}}_\infty$ to the ∞-category of prespectra does, by our description through the iterated Q-construction, and the spectrification functor preserves filtered colimits. ☐

4.2.5. **Remark.** We propose to view the equivalences

$$|\mathrm{Cob}_i(\mathcal{C}, \Qoppa)| \simeq \Omega^{\infty - i} \operatorname{GW}(\mathcal{C}, \Qoppa)$$

for $i \geq 1$ as a close analogue of the equivalence

$$|\mathrm{Cob}^i_d| \simeq \Omega^{\infty - i}\mathrm{MTSO}(d),$$

established by Galatius, Madsen, Tillmann and Weiss for $i = 1$, and Bökstedt and Madsen in general [GTMW09, BM14]. In particular, the sequence of spectra $\operatorname{GW}(\mathcal{C}, \Qoppa^{[-d]})$ can be considered as an algebraic analogue of the Madsen-Tillmann-spectra $\mathrm{MTSO}(d)$.

Of course, our arguments so far correspond only to the statement that the higher cobordism categories Cob^n_d deloop one another, i.e. that $|\mathrm{Cob}^n_d| \simeq \Omega|\mathrm{Cob}^{n+1}_d|$ for $n \geq 1$. The identification of the resulting spectrum via a Pontryagin-Thom construction has no direct analogue in our work. We describe the homotopy type of $\operatorname{GW}(\mathcal{C}, \Qoppa)$ by different means in Corollary 4.4.14 below.

We shall make this connection more than an analogy in future work by promoting the association of its cochains or stable normal bundle to a manifold into functors

$$\sigma : \mathrm{Cob}_d \longrightarrow \mathrm{Cob}\big((\mathcal{S}p/\mathrm{BSO}(d))^\omega, \Qoppa^{\mathrm{v}}_{-\gamma_d}\big) \longrightarrow \mathrm{Cob}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), (\Qoppa^{\mathrm{s}})^{[-d]}\big)$$

from the geometric to our algebraic cobordism categories. The Grothendieck-Witt spectrum of the middle term has already appeared in manifold topology, see §4.6, and we expect the comma ∞-category of the composite functor over 0 to be closely related to the ∞-category $\mathrm{Cob}^{\mathcal{L}}_{2n+1}$ from [HP19] for $d = 2n + 1$.

Just as the negative homotopy groups of the Madsen-Tillmann spectra are given by the cobordism groups, so are the negative homotopy groups of the Grothendieck-Witt spectrum. From Definition [I].2.3.11, we recall:

4.2.6. **Definition.** The L*-group* $\mathrm{L}_0(\mathcal{C}, \Qoppa)$ of a Poincaré ∞-category is defined as the quotient monoid of $\pi_0\mathrm{Pn}(\mathcal{C}, \Qoppa)$ by the submonoid of forms (x, q) admitting a Lagrangian $w \to x$.

For the definition of a Lagrangian, see Definition [I].2.3.1. Also, $\mathrm{L}_0(\mathcal{C}, \Qoppa)$ really is a group: We showed in Corollary 2.3.10 that there is a canonical isomorphism

$$\pi_0|\mathrm{Cob}(\mathcal{C}, \Qoppa^{[-1]})| \cong \mathrm{L}_0(\mathcal{C}, \Qoppa),$$

and consequently, we get

$$[x, -q] + [x, q] = 0$$

in $\mathrm{L}_0(\mathcal{C}, \Qoppa)$ from Proposition 2.3.7. In other words, $\mathrm{L}_0(\mathcal{C}, \Qoppa)$ is the cobordism group of Poincaré forms in $(\mathcal{C}, \Qoppa)$ and inverses are given be reversing the orientation. From Proposition 3.4.7, we obtain:

4.2.7. **Corollary.** *For* $i > 0$ *there are canonical isomorphisms*

$$\pi_{-i} \operatorname{GW}(\mathcal{C}, \Qoppa) \cong \mathrm{L}_0(\mathcal{C}, \Qoppa^{[i]})$$

natural in the Poincaré ∞-category $(\mathcal{C}, \Qoppa)$*.*

By Proposition [I].7.5.3, we have:

4.2.8. **Proposition.** *The functor* $\mathrm{L}_0 : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{A}b$ *admits a unique lax symmetric monoidal structure, making the transformation* $\pi_0\mathrm{Pn} \to \mathrm{L}_0$ *symmetric monoidal. In fact, this transformation then factors lax symmetric monoidally over* GW_0*.*

We will use this fact in Proposition 4.6.4 below.

4.3. **The Bott-Genauer sequence and Karoubi's fundamental theorem.** In the present section, we analyse the behaviour of the metabolic Poincaré-Verdier sequence

$$(\mathcal{C}, Ϙ^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, Ϙ) \xrightarrow{\mathrm{met}} (\mathcal{C}, Ϙ)$$

under the Grothendieck-Witt functor. From Example 2.3.3 and Corollary 3.1.4, we obtain:

4.3.1. **Corollary.** *The functors* lag : Met$(\mathcal{C}, Ϙ) \leftrightarrow$ Hyp$(\mathcal{C})$: can *induce inverse equivalences*

$$\mathrm{GW}\left(\mathrm{Met}(\mathcal{C}, Ϙ)\right) \simeq \mathrm{GW}\left(\mathrm{Hyp}(\mathcal{C})\right)$$

for every Poincaré ∞-category $(\mathcal{C}, Ϙ)$ *and switching the order of the hyperbolic and* Q*-constructions gives an equivalence*

$$\mathrm{GW}\left(\mathrm{Hyp}(\mathcal{D})\right) \simeq \mathrm{K}(\mathcal{D})$$

for every stable ∞-category $\mathcal{D}$*. In particular, for the hyperbolisation of the Grothendieck-Witt functor, we find*

$$\mathrm{GW}^{\mathrm{hyp}} \simeq \mathrm{K}\,.$$

Now, applying GW to the metabolic sequence gives a fibre sequence

$$\mathrm{GW}(\mathcal{C}, Ϙ^{[-1]}) \xrightarrow{\mathrm{fgt}} \mathrm{K}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, Ϙ),$$

of spectra, which we call the *Bott-Genauer sequence*. It is a general version of the Bott-sequence appearing for example in [Sch17, Section 6].

4.3.2. **Remark.** We chose the present terminology to highlight the analogy with the fibre sequence

$$\mathrm{MTSO}(d+1) \longrightarrow \mathbb{S}[\mathrm{BSO}(d+1)] \longrightarrow \mathrm{MTSO}(d)$$

originally appearing in [GTMW09, Section 3], which complemented Genauer's theorem in [Gen12, Section 6] that

$$|\mathrm{Cob}_d^{\partial}| \simeq \Omega^{\infty-1}\mathbb{S}[\mathrm{BSO}(d)].$$

In particular, in the Bott-Genauer sequence the algebraic K-theory spectrum really arises via the metabolic category, encoding objects with boundary, rather than the hyperbolic category. From this perspective, the connectivity of the algebraic K-theory spectrum corresponds to the fact that the bordism groups of manifolds with boundary vanish.

Finally, we observe that the Bott-Genauer sequence gives a vast extension of Karoubi's fundamental theorem: Following Karoubi and Schlichting [Kar80a, Sch17], we define functors

$$\mathrm{U}(\mathcal{C}, Ϙ) = \mathrm{fib}(\mathrm{K}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, Ϙ)) \quad \text{and} \quad \mathrm{V}(\mathcal{C}, Ϙ) = \mathrm{fib}(\mathrm{GW}(\mathcal{C}, Ϙ) \xrightarrow{\mathrm{fgt}} \mathrm{K}(\mathcal{C})).$$

Karoubi's fundamental theorem [Kar80a, p. 260] compares these functors in the setting of discrete rings with involution. In the setting of Poincaré ∞-categories, this statement is a direct consequence of the Bott-Genauer sequence (we will specialise this abstract version to discrete rings in Corollary 4.3.4 below).

4.3.3. **Corollary** (Karoubi's fundamental theorem)**.** *There is a canonical equivalence*

$$\mathrm{U}(\mathcal{C}, Ϙ^{[2]}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{C}, Ϙ) \tag{64}$$

natural in the Poincaré ∞-category $(\mathcal{C}, Ϙ)$.

Proof. Simply note that the Bott-Genauer sequence allows us to identify both sides with $\mathrm{GW}(\mathcal{C}, Ϙ^{[1]})$. □

Now, we showed in [I].7.4.19 that the Poincaré ∞-category $(\mathcal{C}, Ϙ^{[2]})$ can be described another way: By [I].7.4.17 Ϙ canonically lifts to a functor $\widetilde{Ϙ} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p^{\mathrm{gC}_2}$ with $\widetilde{Ϙ}^{\mathrm{gC}_2} \simeq Ϙ$, and $\Omega^k : \mathcal{C} \to \mathcal{C}$ upgrades to an equivalence

$$(\mathcal{C}, Ϙ^{[2k]}) \simeq (\mathcal{C}, (\mathbb{S}^{k-k\sigma} \otimes \widetilde{Ϙ})^{\mathrm{gC}_2}),$$

where $\mathbb{S}^{\sigma}$ is the suspension spectrum of the sign representation of C_2 on $\mathbb{R}$. While more complicated at first glance, the right hand side is actually much easier to analyse directly, and corresponds to the idea of "inserting a sign" into a Poincaré structure.

Let us spell this out for the Grothendieck-Witt theory of rings. Recall that these are integrated into our set-up via their derived categories of modules. More generally, consider an E_∞-ring k, and an E_1-algebra R over k and an invertible k-module with genuine involution $(M, \alpha\colon N \to M^{\mathrm{tC}_2})$ over R. Then by Proposition [I].3.5.3, the general equivalence above becomes

$$(\mathrm{Mod}^\omega_R, (\Qoppa^\alpha_M)^{[2n]}) \simeq (\mathrm{Mod}^\omega_R, \Qoppa^{\mathbb{S}^n\otimes\alpha}_{\mathbb{S}^{n-n\sigma}\otimes M}).$$

Now, if k is equipped with a 2σ-orientation, i.e. a factorisation of its unit map as

$$\mathbb{S} \longrightarrow (\mathbb{S}^{2-2\sigma} \otimes k)^{\mathrm{hC}_2} \xrightarrow{\mathrm{fgt}} k,$$

then we learn from [I].3.5.13 that the module with involution $\mathbb{S}^{n-n\sigma} \otimes M$ only depends on the parity of n modulo 2 up to a canonical equivalence. We denote the common value for odd n by $-M$, so that we get

$$(\mathrm{Mod}^\omega_R, (\Qoppa^\alpha_M)^{[2]}) \simeq (\mathrm{Mod}^\omega_R, \Qoppa^{\mathbb{S}^1\otimes\alpha}_{-M})$$

as a special case of the above. This situation occurs most importantly, whenever k is discrete, e.g. $k = \mathrm{H}\mathbb{Z}$, and if R and M are discrete as well $-M$ really is given by changing its involution by a sign. The situation also occurs more generally, however, when k is complex oriented (e.g. even periodic) or $2 \in \pi_0(k)^\times$, see [I].3.5.14 for details.

A 2σ-orientation on k in particular produces an equivalence $(-M)^{\mathrm{tC}_2} \simeq \mathbb{S}^1 \otimes M^{\mathrm{tC}_2}$, so we obtain equivalences

$$(\mathrm{Mod}^\omega_R, (\Qoppa^{\mathrm{s}}_M)^{[2]}) \simeq (\mathrm{Mod}^\omega_R, \Qoppa^{\mathrm{s}}_{-M}), \qquad (\mathrm{Mod}^\omega_R, (\Qoppa^{\mathrm{q}}_M)^{[2]}) \simeq (\mathrm{Mod}^\omega_R, \Qoppa^{\mathrm{q}}_{-M})$$

and, whenever R is furthermore connective,

$$(\mathrm{Mod}^\omega_R, (\Qoppa^{\geq m}_M)^{[2]}) \simeq (\mathrm{Mod}^\omega_R, \Qoppa^{\geq m+1}_{-M}).$$

Also note that if k admits a σ-orientation, e.g. it is discrete of characteristic 2 or more generally real oriented, then we even find $M \simeq -M$.

Recall furthermore, that for $\mathrm{c} \in \mathrm{K}_0(R) = \mathrm{K}_0(\mathrm{Mod}^\omega_R)$ a subgroup, we denote by $\mathrm{Mod}^{\mathrm{c}}_R \subseteq \mathrm{Mod}^\omega_R$ the full subcategory spanned by those R-modules X with $[X] \in \mathrm{c}$, the most interesting special cases being

$$\mathrm{Mod}^{\mathrm{K}_0(S)}_{\mathrm{H}S} = \mathcal{D}^{\mathrm{p}}(S) \quad \text{and} \quad \mathrm{Mod}^{\langle \mathrm{H}S\rangle}_{\mathrm{H}S} = \mathcal{D}^{\mathrm{f}}(S)$$

for S an ordinary ring. We shall need to assume that c is closed under the involution induced by M. This is clearly always true in the former case, and in the latter amounts to $[M] = \pm[S] \in \mathrm{K}_0(S)$.

4.3.4. **Corollary.** *For R an E_1-algebra over an E_∞-ring k, which carries a 2σ-orientation, M an invertible k-module with involution over R, and $\mathrm{c} \subseteq \mathrm{K}_0(R)$ a subgroup closed under the involution induced by M, there are canonical equivalences*

$$\mathrm{U}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_M) \quad \textit{and} \quad \mathrm{U}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_M)$$

and if R is furthermore connective, then also

$$\mathrm{U}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\geq m+1}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\geq m}_M)$$

for arbitrary $m \in \mathbb{Z}$.

Now, these equivalence can be specialised further to a discrete ring R, $c = \mathrm{K}_0(R)$ and a discrete invertible module with involution over R, in which case one can take $k = \mathbb{Z}$ together with its canonical 2σ-orientation. Choosing either $m = 1$ or $m = 2$, we then obtain the following extension of Karoubi's fundamental theorem:

4.3.5. **Corollary.** *For a discrete ring R and a discrete invertible module with involution M over R, there are canonical equivalences*

$$\mathrm{U}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{q}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{q}}_M), \quad \mathrm{U}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{s}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{s}}_M).$$

and

$$\mathrm{U}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gq}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{ge}}_M), \quad \mathrm{U}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{ge}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{gs}}_M).$$

Given the comparisons in Appendix B, all of these equivalences collapse into the classical formulation of Karoubi's fundamental theorem upon restricting to discrete rings in which 2 is invertible; if 2 is not assumed invertible they are, however, distinct. We will explore their uses for discrete rings in the third paper of this series.

Let us also apply the abstract fundamental theorem 4.3.3 to the case of a form parameter, as originally defined by Bak [Bak81] and generalised by Schlichting [Sch21]. Recall from [I].4.2 that given a discrete ring R and a discrete invertible module with involution (M, σ) over R, a form parameter λ on M is the data of an abelian group Q, equipped with an action by the multiplicative monoid of R through group homomorphisms, and two R-equivariant homomorphisms

$$M_{\mathrm{C}_2} \xrightarrow{\tau} Q \xrightarrow{\rho} M^{\mathrm{C}_2}$$

whose composition is the norm map (i.e. the map induced by $\mathrm{id}_M + \sigma$) and such that

$$(a+b)x = ax + \tau((a \otimes b)\rho(x)) + bx$$

for all $a, b \in R$ and $x \in Q$. As explained in §[I].4.2, the 2-polynomial functor $\Qoppa^{\lambda}_{\mathrm{proj}} : \mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{A}b$ sending P to its group of λ-hermitian forms, followed by the canonical Eilenberg-Mac Lane inclusion $\mathcal{A}b \to \mathcal{S}p$ extends (essentially uniquely) by non-abelian derivation to a hermitian structure $\Qoppa^{\mathrm{g}\lambda}_{M} : \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$.

Any form parameter (Q, τ, ρ) on M admits a dual form parameter $\check{\lambda}$

$$(-M)_{\mathrm{C}_2} \longrightarrow M/Q \longrightarrow (-M)^{\mathrm{C}_2}$$

on $(-M)$ in which the first map is surjective. For example we have $\check{\pm \mathrm{s}} = \mp\mathrm{ev}$ and $\check{\pm \mathrm{ev}} = \mp\mathrm{q}$, in what is hopefully evident notation. We showed in Proposition [I].4.2.29 that the loop functor refines to an equivalence of Poincaré ∞-categories

$$\big(\mathcal{D}^{\mathrm{p}}(R), (\Qoppa^{\mathrm{g}\lambda}_{M})^{[2]}\big) \longrightarrow \big(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{g}\check{\lambda}}_{-M}\big), \tag{65}$$

whenever ρ is injective in the original form parameter. Applying Corollary 4.3.3 to this equivalence immediately implies:

4.3.6. **Corollary.** *For a discrete ring R, a discrete invertible module with involution M, $c \subseteq \mathrm{K}_0(R)$ a subgroup closed under the involution induced by M and a form parameter $\lambda = (Q, \tau, \rho)$ on M with ρ injective and dual $\check{\lambda}$, there is a canonical equivalence*

$$\mathrm{U}(\mathcal{D}^{c}(R), \Qoppa^{\mathrm{g}\check{\lambda}}_{-M}) \simeq \mathbb{S}^1 \otimes \mathrm{V}(\mathcal{D}^{c}(R), \Qoppa^{\mathrm{g}\lambda}_{M}).$$

Together with [HS21, Theorem A], which identifies the Grothendieck-Witt space of $(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{g}\lambda}_{M})$ considered in the present paper with the Grothendieck-Witt space considered in [Kar09] (which is defined as the group completion of the E_∞-monoid of the corresponding Poincaré forms on projective modules), this proves Conjectures 1 and 2 in §3.4 and §4.3 of loc. cit, respectively. Note for translation purposes that Karoubi's hermitian and quadratic modules for the form parameter λ correspond to Poincaré forms for the form parameters λ and $\check{\lambda}$, respectively, as explained in [Sch21, Example 3.9].

4.4. L-theory and the fundamental fibre square.

In the present section, we prove our main result on the homotopy type of the Grothendieck-Witt spectrum. In §3.6, we studied the bordification of an additive functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ and in Corollary 4.4.14, we produced a fibre square reconstructing $\mathcal{F}$ from its hyperbolisation $\mathcal{F}^{\mathrm{hyp}}$ and its bordification $\mathcal{F}^{\mathrm{bord}}$. In the previous subsection, we obtained an equivalence $\mathrm{GW}^{\mathrm{hyp}} \simeq \mathrm{K}$, and in the present section, we show $\mathrm{GW}^{\mathrm{bord}} \simeq \mathrm{L}$. To set the stage, recall the ρ-construction from Definition 3.6.10.

4.4.1. **Definition.** The L-*theory space* is the functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ given by

$$\mathcal{L}(\mathcal{C}, \Qoppa) = |\mathrm{Pn}\rho(\mathcal{C}, \Qoppa)|$$

obtained by applying the ρ-construction to Pn.

Since $\rho_0(\mathcal{C}, \Omega) = (\mathcal{C}, \Omega)$, there is a canonical map

$$\mathrm{Pn}(\mathcal{C}, \Omega) \to \mathcal{L}(\mathcal{C}, \Omega).$$

and by construction, the 1-skeleta of the ρ and Q construction agree, so, from Corollary 2.3.10, we find that the natural map

$$\pi_0 \mathrm{Pn}(\mathcal{C}, \Omega) \longrightarrow \pi_0 \mathcal{L}(\mathcal{C}, \Omega)$$

descends to an isomorphism

$$\mathrm{L}_0(\mathcal{C}, \Omega) \longrightarrow \pi_0 \mathcal{L}(\mathcal{C}, \Omega)$$

for all Poincaré ∞-categories $(\mathcal{C}, \Omega)$.

But much more is true: Generalising a classical result of Ranicki, Lurie showed in [Lur11, Lecture 7, Theorem 9] that there are canonical isomorphisms

$$\pi_i \mathcal{L}(\mathcal{C}, \Omega) = \mathrm{L}_0(\mathcal{C}, \Omega^{[-i]})$$

for all $i \geq 0$. While analogous to our results on bordifications, this is more difficult and fundamentally rests on the fact that $\mathrm{Pn}\rho(\mathcal{C}, \Omega)$ is a Kan simplicial space. In fact:

4.4.2. **Theorem.** *Given a Poincaré-Verdier sequence* $(\mathcal{C}, \Omega) \to (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$*, the functor* $\mathrm{Pn}\rho(\mathcal{D}, \Phi) \to \mathrm{Pn}\rho(\mathcal{E}, \Psi)$ *is a Kan fibration of simplicial spaces with fibre* $\mathrm{Pn}\rho(\mathcal{C}, \Omega)$.

In particular, the functor $\mathcal{L} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *is Verdier-localising and bordism invariant.*

The above identification of homotopy groups is then a consequence of Proposition 3.5.8. The result itself is the main content of [Lur11, Lectures 8 & Lecture 9]; we give the proof here for completeness' sake. It rests on the following lemma:

4.4.3. **Lemma.** *Given a Poincaré-Verdier projection* $(\mathcal{D}, \Phi) \xrightarrow{(p,\eta)} (\mathcal{E}, \Psi)$*, an object* $x \in \mathcal{D}$*, a map* $f \colon y \to p(x)$ *in* $\mathcal{E}$ *and a diagram*

$$\begin{tikzcd} K \arrow[rr] \arrow[d] & & * \arrow[d, "q"] \\ \Omega^\infty\Phi(x) \arrow[r, "\eta"] & \Omega^\infty\Psi(p(x)) \arrow[r, "f^*"] & \Omega^\infty\Psi(y) \end{tikzcd}$$

with $K \in \mathcal{S}^\omega$*, there exists an arrow* $g \colon z \to x$ *in* $\mathcal{D}$ *lifting* f *together with a lift*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & * \arrow[d, "r"] \\ \Omega^\infty\Phi(x) \arrow[r, "g^*"] & \Omega^\infty\Phi(z) \end{tikzcd}$$

of the original rectangle.

Proof. Since p is essentially surjective by Corollary A.1.7, there exists a v in $\mathcal{D}$ with $p(v) \simeq y$, and applying [NS18, Theorem I.3.3 ii)], we can then modify v to find $h \colon w \to y$ lifting f. From Remark 1.1.8, we furthermore find

$$\Psi(y) \simeq \operatorname*{colim}_{c \in \mathcal{C}_{w/}} \Phi(\mathrm{fib}(w \to c))$$

so, putting $u = \mathrm{fib}(w \to c)$ for an appropriate c, we find a lift $s \in \Omega^\infty\Phi(u)$ lifting q, and the composite $u \to w \to y$ still lifts f. To find a lift of the homotopy of maps $K \to \Omega^\infty\Psi(y)$, note that the colimit above is filtered so, since K is assumed compact, we also have

$$\mathrm{Hom}_{\mathcal{S}}(K, \Omega^{\infty+1}\Psi(y)) \simeq \operatorname*{colim}_{c' \in \mathcal{C}_{u/}} \mathrm{Hom}_{\mathcal{S}}(K, \Omega^{\infty+1}\Phi(\mathrm{fib}(u \to c')),$$

which for appropriate c yields all the desired data on for $z = \mathrm{fib}(u \to c')$ and g the composite $z \to u \to y$. □

Proof of Theorem 4.4.2. We need to show that each solid diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \rho(\mathcal{D}, \Phi) \arrow[d] \\ \Delta^n \arrow[r] & \rho(\mathcal{E}, \Psi) \end{tikzcd}$$

admits a dotted filler up to homotopy (where we regard Δ^n and Λ^n_i as simplicial spaces via the inclusion $\mathrm{Set} \subset \mathcal{S}$).

To unwind this, recall that $\rho_n(\mathcal{D}, \Phi) = (\mathcal{D}, \Phi)^{\mathcal{T}_n}$, where $\mathcal{T}_n = \mathcal{P}_0([n])^{\mathrm{op}}$ is the opposite of the barycentric subdivision $\mathrm{sd}(\Delta^n)$ of Δ^n. Denote then by $H^i_n \subseteq \mathcal{T}_n$ the opposite of the subdivision of the i-horn, i.e. the collection of subsets missing an element besides i. Then the lifting problem above translates to showing that the canonical map

$$\mathrm{Pn}\left((\mathcal{D}, \Phi)^{\mathcal{T}_n}\right) \longrightarrow \mathrm{Pn}\left((\mathcal{E}, \Psi)^{\mathcal{T}_n}\right) \times_{\mathrm{Pn}\left((\mathcal{E},\Psi)^{H^i_n}\right)} \mathrm{Pn}\left((\mathcal{D}, \Phi)^{H^i_n}\right)$$

is surjective on π_0. To this end, we first show the corresponding statement on spaces of hermitian objects, and then explain how to adapt a lift in $\mathrm{Fm}\left((\mathcal{D}, \Phi)^{\mathcal{T}_n}\right)$ to a Poincaré one, provided its images in $\mathrm{Fm}\left((\mathcal{E}, \Psi)^{\mathcal{T}_n}\right)$ and $\mathrm{Fm}\left((\mathcal{D}, \Phi)^{H^i_n}\right)$ are Poincaré. The first claim even holds for boundary inclusions instead of horn inclusions, so denote by B_n the opposite of the subdivision of $\partial\Delta^n$ and consider hermitian objects $(F : \mathcal{T}_n \to \mathcal{E}, q)$ and $(G : B_n \to \mathcal{D}, r)$ and an equivalence between their images in $\mathrm{Fm}\left((\mathcal{E}, \Psi)^{B_n}\right)$. Put then $x \in \mathcal{D}$ as the limit of G. By construction, there is then a canonical map $f : y \to p(x)$, where $y = F(b)$ with b the barycentric vertex $[n]$ in $\mathcal{T}_n$. Furthermore, regarding $r \in \Phi^{B_n}(G)$ as a map $r : * \to \lim_{B^{\mathrm{op}}_n} \Omega^\infty\Phi\circ G^{\mathrm{op}}$ it is adjoint to a transformation $\mathrm{const}_* \Rightarrow \Omega^\infty\Phi\circ G^{\mathrm{op}}$, which gives rise to a map

$$|B^{\mathrm{op}}_n| \simeq \operatorname*{colim}_{B^{\mathrm{op}}_n} * \longrightarrow \operatorname*{colim}_{B^{\mathrm{op}}_n} \Omega^\infty\Phi\circ G^{\mathrm{op}} \longrightarrow \Omega^\infty\Phi(\lim_{B_n} G) = \Omega^\infty\Phi(x)$$

whose composition down to $\Omega^\infty\Psi(y)$ is canonically identified with the constant map with value $q \in \Omega^\infty\Psi^{\mathcal{T}_n}(F) \simeq \Omega^\infty\Psi(F(b))$, since $b = [n]$ is initial in $\mathcal{T}^{\mathrm{op}}_n$, so $|\mathcal{T}^{\mathrm{op}}_n| \simeq *$.

We can therefore apply the previous lemma to obtain a lift $g : z \to x$ of f, together with a lift $s \in \Omega^\infty\Phi(z)$ of q and an identification of the composite

$$|B^{\mathrm{op}}_n| \longrightarrow \Omega^\infty\Phi(x) \xrightarrow{g^*} \Omega^\infty\Phi(z)$$

with the constant map on s, that lifts the identification above. Since $\mathcal{T}_n$ is the cone on B_n, the map g precisely defines an extension of G to a map $\widetilde{G} : \mathcal{T}^{\mathrm{op}}_n \to \mathcal{D}$, on which s defines a hermitian form, and the remainder of the data produced bears witness to $(\widetilde{G}, s)$ being a lift as desired.

For the second step, we need to modify a hermitian lift $(\widetilde{G}, r) \in \mathrm{Fm}\left((\mathcal{D}, \Phi)^{\mathcal{T}_n}\right)$ of $(F, q) \in \mathrm{Pn}\left((\mathcal{E}, \Psi)^{\mathcal{T}_n}\right) \in$ and $(G, s) \in \mathrm{Pn}\left((\mathcal{D}, \Phi)^{H^n_i}\right)$ into a Poincaré lift. This is achieved by performing surgery as follows: The algebraic Thom construction from Corollary [I].2.4.6 gives an equivalence

$$\mathrm{Fm}\left((\mathcal{D}, \Phi)^{\mathcal{T}_n}\right) \simeq \mathrm{Pn}(\mathrm{Met}\left((\mathcal{D}, \Phi^{[1]})^{\mathcal{T}_n}\right)$$

refining the map taking $(\widetilde{G}, s)$ to

$$\mathrm{D}_{\Phi^{\mathcal{T}_n}}\left(\widetilde{G}\right) \longrightarrow \mathrm{cof}\left(\widetilde{G} \xrightarrow{s_\#} \mathrm{D}_{\Phi^{\mathcal{T}_n}}\left(\widetilde{G}\right)\right). \tag{66}$$

In particular, the Poincaré objects in $(\mathcal{D}, \Phi)^{\mathcal{T}_n}$ correspond precisely to those arrows with vanishing target (the target is the boundary of $(\widetilde{G}, s)$ in the sense of Definition 4.4.7 below). Since (F, q) and (G, r) and the boundary maps in the ρ-construction are Poincaré (see the discussion before Definition 3.6.10), it follows that the target in our case already lies in the kernels of both

$$\mathcal{D}^{\mathcal{T}_n} \longrightarrow \mathcal{D}^{H^n_i} \quad \text{and} \quad \mathcal{D}^{\mathcal{T}_n} \longrightarrow \mathcal{E}^{\mathcal{T}_n}.$$

We claim that the intersection of these kernels is equivalent to $\mathrm{Met}(\mathcal{C}, Ϙ^{[1-n]})$ as a Poincaré ∞-category. This is clear on underlying ∞-categories, and follows for the hermitian structures from the iterative formulae for limits of cubical diagrams, i.e.

$$\lim_{\mathcal{T}^{\mathrm{op}}_n} X \simeq X(\{0, ..., n-1\}) \times_{\lim_{\mathcal{T}^{\mathrm{op}}_{n-1}} X} \lim_{\mathcal{T}^{\mathrm{op}}_{n-1}} X\circ(- \cup n),$$

which is easily verified using [Lur09a, Corollary 4.2.3.10] by decomposing $\mathcal{T}_n$ as the pushout of $\mathcal{T}_{n-1}$ and $\mathcal{T}_n \setminus \{0, ..., n-1\}$ over their intersection. We thus find that the cofibre of $s_\#$ admits a Lagrangian L, since objects in metabolic Poincaré ∞-categories are canonically metabolic by Remark [I].7.3.23. We can thus perform surgery on (66) with the surgery datum $0 \to L$, see Proposition 2.4.3. The resulting arrow has vanishing target, and by design the surgery changes neither the image in $\mathcal{E}^{\mathcal{T}_n}$ nor the restriction to $\mathcal{D}^{H^n_i}$.

Translating back along the algebraic Thom construction thus provides the desired Poincaré lift of (F, q) and (G, q).

To deduce the remaining claims, note that the statement about the fibre is immediate from both cotensors and Pn preserving limits. That $\mathcal{L}$ is Verdier-localising now follows, since colimits of simplicial fibre sequences with second map a Kan fibration are again fibre sequences, see e.g. [Lur16, Theorem A.5.4.1].

To finally obtain bordism invariance, one can either proceed by observing that on account of the Kan property the i-th homotopy groups of $\mathrm{L}(\mathcal{C}, Ϙ) = |\mathrm{Pn}\rho(\mathcal{C}, Ϙ)|$ can be described as the quotient of

$$\pi_0 \operatorname{fib}\left(\mathrm{Hom}_{s\mathcal{S}}(\Delta^i, \mathrm{Pn}\rho(\mathcal{C}, Ϙ)) \longrightarrow \mathrm{Hom}_{s\mathcal{S}}(\partial\Delta^i, \mathrm{Pn}\rho(\mathcal{C}, Ϙ))\right)$$

by the equivalence relation generated by a pair of such elements admitting an extension to

$$\pi_0 \operatorname{fib}\left(\mathrm{Hom}_{s\mathcal{S}}(\Delta^1 \times \Delta^i, \mathrm{Pn}\rho(\mathcal{C}, Ϙ)) \to \mathrm{Hom}_{s\mathcal{S}}(\Delta^1 \times \partial\Delta^i, \mathrm{Pn}\rho(\mathcal{C}, Ϙ))\right).$$

This quotient is readily checked to be exactly $\mathrm{L}_0(\mathcal{C}, Ϙ^{[-i]})$. This is the route taken in both [Ran92] and [Lur11].

Alternatively, one can employ Lemma 3.6.14 to see that $\mathrm{L}(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq |\mathrm{Pn}\rho(\mathrm{Met}(\mathcal{C}, Ϙ))|$ is the realisation of a split simplicial object over 0, therefore vanishes, and then conclude by 3.5.4. Let us remark that via the algebraic Thom construction [I].2.4.6, the extra degeneracy of the split simplicial space $\mathrm{Pn}\rho(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq \mathrm{Fm}\rho(\mathcal{C}, Ϙ^{[-1]})$ attains a particularly easy form: It is simply given by extension-by-zero. We leave the necessary unwinding of definitions to the reader. □

It now follows from Theorem 3.5.9 that $\mathcal{L}$ admits an essentially unique lift to a functor with values in spectra.

4.4.4. **Definition.** We define the L-*theory spectrum* $\mathrm{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ by

$$\mathrm{L}(\mathcal{C}, Ϙ) = \mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, Ϙ)$$

with $(\mathcal{C}, Ϙ)$ a Poincaré ∞-category, and denote by $\mathrm{L}_i(\mathcal{C}, Ϙ)$ its homotopy groups.

This definition of the L-groups agrees with Definition 4.2.6, since from Proposition 3.4.5 and Proposition 3.5.8, we obtain:

4.4.5. **Corollary.** *There are canonical equivalences*

$$\Omega^{\infty-i}\,\mathrm{L}(\mathcal{C}, Ϙ) \simeq \mathcal{L}(\mathcal{C}, Ϙ^{[i]})$$

for all $i \in \mathbb{Z}$. In particular, there are isomorphisms

$$\pi_i\,\mathrm{L}(\mathcal{C}, Ϙ) \cong \mathrm{L}_0(\mathcal{C}, Ϙ^{[-i]})$$

also for negative i.

In fact, the definition

$$\mathrm{L}(\mathcal{C}, Ϙ) \simeq \left[\mathcal{L}(\mathcal{C}, Ϙ), \mathcal{L}(\mathcal{C}, Ϙ^{[1]}), \mathcal{L}(\mathcal{C}, Ϙ^{[2]}), \dots\right]$$

with structure maps arising from Proposition 3.5.8 is a direct generalisation of the classical definition of L-theory spectra due to Ranicki; see for example [Ran92, Section 13], and it is rather more elegant than our definition which iterates the Q-construction on top of the ρ-construction. As an important consequence, we obtain:

4.4.6. **Corollary.** *The functor* $\mathrm{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ *is bordism invariant, Verdier-localising. It also preserves filtered colimits.*

One can even directly describe the boundary operator of the long exact sequence on the L-groups of a Poincaré-Verdier sequence.

4.4.7. **Definition.** Given a Poincaré ∞-category $(\mathcal{C}, Ϙ)$ and a hermitian object $(X, q) \in \mathrm{Fm}(\mathcal{C}, Ϙ)$, the *boundary* of (X, q) is the Poincaré object $\partial(X, q) \in \mathrm{Pn}(\mathcal{C}, Ϙ^{[1]})$ obtained as the result of surgery on $(X \to 0, q) \in \mathrm{Surg}_0(\mathcal{C}, Ϙ^{[1]})$.

Note that by the discussion preceding Proposition 2.4.3, the object underlying $\partial(X, q)$ is given by the cofibre of $q_\sharp : X \to \mathrm{D}_Ϙ X$.

4.4.8. **Proposition.** *Given a Poincaré-Verdier sequence*

$$(\mathcal{C},\Qoppa)\xrightarrow{i}(\mathcal{D},\Phi)\xrightarrow{p}(\mathcal{E},\Psi),$$

the boundary operator $\mathrm{L}_i(\mathcal{E},\Psi)\to\mathrm{L}_{i-1}(\mathcal{C},\Qoppa)$ *of the resulting long exact sequence takes a Poincaré object* $(X,q)\in\mathrm{Pn}(\mathcal{E},\Psi^{[-i]})$ *to* $\partial(Y,q')\in\mathrm{Pn}(\mathcal{C},\Qoppa^{[1-i]})$*, where* $(Y,q')\in\mathrm{Fm}(\mathcal{D},\Phi^{[-i]})$ *is any lift of* (X,q)*.*

In particular, the proposition asserts that such a hermitian lift of X can always be found, and its image in $\mathrm{L}_{i-1}(\mathcal{C},\Qoppa)$ is the obstruction against finding a Poincaré lift of X.

Proof. From Proposition 3.1.10, we find that the inverse to the boundary isomorphism $\pi_1\,\mathrm{L}(\mathcal{E},\Psi)\to\pi_0\,\mathrm{L}(\mathcal{E},\Psi^{[-1]})$ takes a Poincaré object X in the target to the loop w represented by $0\leftarrow X\to 0\in\mathrm{Pn}\rho_1(\mathcal{E},\Psi)$. We now compute the map $\mathrm{L}_1(\mathcal{E},\Psi)\to\mathrm{L}_0(\mathcal{C},\Qoppa)$, the case of general $i\in\mathbb{Z}$ follows by shifting the quadratic functor. That any Poincaré object $(X,q)\in\mathrm{Pn}(\mathcal{E},\Psi^{[-1]})$ can be lifted to some $(Y,q')\in\mathrm{Fm}(\mathcal{D},\Phi^{[-1]})$ is an application of Lemma 4.4.3 (with $K=\emptyset$).

Now, regarding the map $(Y\to 0,q')$ as a surgery datum in $\mathrm{Surg}_0(\mathcal{D},\Phi)$, we can apply Proposition 2.4.3 to obtain a cobordism from 0 to the result of surgery, which is $\partial(Y,q')$. We regard this cobordism as an element of $\mathrm{Pn}(\rho_1(\mathcal{D},\Phi))$ and thus as a path in $\mathrm{L}(\mathcal{D},\Phi)$. By construction, this path lifts the loop in $\mathrm{L}(\mathcal{E},\Psi)$ defined by X via the consideration in the first paragraph. Therefore, its endpoint $\mathrm{cof}(Y\to\mathrm{D}_{\Qoppa^{[-1]}}Y)$ represents the image of (X,q) under the boundary map as claimed. □

4.4.9. **Remark.** In [Lur11, Lecture 20], Lurie gives yet another definition of the L-theory spectrum, by directly constructing an excisive functor $\mathcal{S}_*^{\mathrm{fin}}\to\mathcal{S}$, whose value on the one point space is $\mathcal{L}(\mathcal{C},\Qoppa)$. However, while certainly true it is never justified in [Lur11], that the functor constructed evaluates to $\mathcal{L}(\mathcal{C},\Qoppa^{[i]})$ on the i-sphere.

Now, by construction there is a natural transformation $\mathrm{Pn}\Rightarrow\mathcal{L}\simeq\Omega^\infty\,\mathrm{L}$, which uniquely extends to a transformation

$$\mathrm{bord}\colon\ \mathrm{GW}\Rightarrow\mathrm{L}\tag{67}$$

of functors $\mathrm{Cat}_\infty^{\mathrm{p}}\to\mathcal{S}p$ by Corollary 4.2.2. We record, see Corollary 3.6.20:

4.4.10. **Corollary.** *Under the identifications of Theorem 3.1.9 the map*

$$\mathrm{bord}\colon\ \pi_0\,\mathrm{GW}(\mathcal{C},\Qoppa)\to\pi_0\,\mathrm{L}(\mathcal{C},\Qoppa)$$

becomes the canonical projection $\mathrm{GW}_0(\mathcal{C},\Qoppa)\to\mathrm{L}_0(\mathcal{C},\Qoppa)$*. Similarly, for* $i>0$*, the induced map* $\pi_{-i}\,\mathrm{GW}(\mathcal{C},\Qoppa)\to\pi_{-i}\,\mathrm{L}(\mathcal{C},\Qoppa)$ *is identified with the identity of* $\mathrm{L}_0(\mathcal{C},\Qoppa^{[i]})$ *by Proposition 3.4.7.*

4.4.11. **Remark.** While the map $\mathrm{bord}\colon\ \mathrm{GW}(\mathcal{C},\Qoppa)\to\mathrm{L}(\mathcal{C},\Qoppa)$ is most easily constructed via the universal property of GW, it is also easy to obtain a direct map between these spectra when defining them via the Q- and ρ-constructions: Consider the map of cosimplicial objects $\eta\colon\ (\mathrm{sd}\,\Delta^n)^{\mathrm{op}}\to\mathrm{TwAr}(\Delta^n)$, that sends a non-empty subset $T\subseteq[n]$ to the pair $(\min T\le\max T)$. It is an isomorphism in degrees 0 and 1 and in degree 2 it sends a diagram

$$\begin{tikzcd}
& & v\times_y w \arrow[dl] \arrow[dr] & & \\
& v \arrow[dl] \arrow[dr] & & w \arrow[dl] \arrow[dr] & \\
x & & y & & z
\end{tikzcd}$$

in $\mathrm{Q}_2(\mathcal{C},\Qoppa^{[1]})$ to

$$\begin{tikzcd}
& & y & & \\
& v \arrow[ur] \arrow[ddl] & & w \arrow[ul] \arrow[ddr] & \\
& & v\times_y w \arrow[ul] \arrow[ur] \arrow[d] & & \\
x & & v\times_y w \arrow[ll] \arrow[rr] & & z
\end{tikzcd}$$

in $\rho_2(\mathcal{C}, Ϙ^{[1]})$. The analogous operation on manifold cobordisms takes two composable cobordisms to the 2-ad given by the cartesian product of their composition with an interval; the ad-structure is given (after smoothing corners) by decomposing the boundary into the original two cobordisms, represented along the diagonal edges, and their composite given by the lower horizontal edge. In general then, the transformation $\eta\colon\ \mathrm{Q} \Rightarrow \rho$ regards n composable 1-ads as a special case of an n-ad.

Now, η induces a map

$$\mathcal{GW}(\mathcal{C}, Ϙ) = \Omega|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, Ϙ^{[1]})| \xrightarrow{\Omega|\eta|} \Omega|\mathrm{Pn}\rho(\mathcal{C}, Ϙ^{[1]})| \xrightarrow{\partial} |\mathrm{Pn}\rho(\mathcal{C}, Ϙ)| = \mathcal{L}(\mathcal{C}, Ϙ)$$

and thus a map $\eta\colon\ \mathrm{GW} = \mathbb{C}\mathrm{ob}^{\mathcal{GW}} \Rightarrow \mathbb{C}\mathrm{ob}^{\mathcal{L}} = \mathrm{L}$. Using Proposition 3.1.10, it is not difficult to check, that this map satisfies the universal property defining bord. Since we shall not have to make use of that statement, we leave the details to the reader.

We now turn to the main result of this section:

4.4.12. **Theorem.** *The transformation* bord *exhibits* L *as the bordification of* GW. *In particular,* $\mathrm{L}\colon\ \mathrm{Cat}^{\mathrm{p}}_{\infty} \to$ *$\mathcal{S}p$ is the initial bordism invariant, additive functor equipped with a transformation* $\mathrm{Pn} \Rightarrow \Omega^{\infty}\mathrm{L}$ *of functors* $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$.

From Theorem 3.5.9, we also find that $\mathcal{L}\colon\ \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ is the initial bordism invariant, additive functor under either Pn or $\mathcal{GW}$.

Proof. We give two proofs of the first statement. The second is then immediate from Corollary 4.2.2.

The first argument employs the formula of Definition 3.6.10 in terms of the ad-construction for bordifications: The natural equivalence of Poincaré ∞-categories $\rho_n\,\mathrm{Q}_m(\mathcal{C}, Ϙ) \simeq \mathrm{Q}_m\,\rho_n(\mathcal{C}, Ϙ)$ identifies $\mathrm{L}(\mathcal{C}, Ϙ)$ with the geometric realisation of the simplicial spectrum $\mathrm{GW}(\rho(\mathcal{C}, Ϙ))$ in the ∞-category of prespectra. Since the result is already an (Ω-)spectrum we have that $\mathrm{L}(\mathcal{C}, Ϙ)$ is also the geometric realisation of $\mathrm{GW}(\rho(\mathcal{C}, Ϙ))$ in $\mathcal{S}p$. We obtain a natural identification $\mathrm{L} \simeq |\,\mathrm{GW}\,\rho| = \mathrm{ad}(\mathrm{GW})$, which gives the claim by Corollary 3.6.13.

We can also employ the stab-construction: By Proposition 3.5.8, the map bord factors over a map

$$\operatorname*{colim}_{d} \mathbb{S}^{d} \otimes \mathrm{GW}(\mathcal{C}, Ϙ^{[-d]}) \longrightarrow \mathrm{L}(\mathcal{C}, Ϙ).$$

But it follows from Corollary 4.4.10 and Corollary 3.6.20 that this map is an isomorphism on homotopy groups. By Corollary 3.6.19, the claim follows a second time. □

4.4.13. **Remark.** Under the analogy between $\mathrm{GW}(\mathcal{C}, Ϙ^{[-d]})$ and $\mathrm{MTSO}(d)$ (see Remarks 4.2.5 and 4.3.2) the equivalence

$$\operatorname*{colim}_{d} \mathbb{S}^{d} \otimes \mathrm{GW}(\mathcal{C}, Ϙ^{[-d]}) \simeq \mathrm{L}(\mathcal{C}, Ϙ)$$

corresponds to the canonical equivalence

$$\operatorname*{colim}_{d} \mathbb{S}^{d} \otimes \mathrm{MTSO}(d) \simeq \mathrm{MSO},$$

whose proof is an elementary manipulation of Thom spectra (see [GTMW09, Section 3]). In particular, the role of the spectrum MSO is played by $\mathrm{L}(\mathcal{C}, Ϙ)$ in our theory; even its definition in terms of the ρ-construction is modelled on Quinn's construction of the ad-spectrum of manifolds Ω^{SO}, whose homotopy groups by construction are the cobordism groups. The second proof of the above theorem is then a translation of the well-known equivalence

$$\operatorname*{colim}_{d} \mathbb{S}^{d} \otimes \mathcal{C}\mathrm{ob}_{d} \simeq \Omega^{\mathrm{SO}}$$

from geometric topology; using the main result of [Ste21] this identification can in fact be achieved without reference to Thom spectra whatsoever and therefore used to deduce the equivalence $\mathrm{MSO} \simeq \Omega^{\mathrm{SO}}$, i.e. the Pontryagin-Thom theorem, from the equivalences $\mathcal{C}\mathrm{ob}_{d} \simeq \mathrm{MTSO}(d)$ of Bökstedt, Galatius, Madsen, Tillmann and Weiss.

Now since the functor $(\mathcal{C}, Ϙ) \mapsto \mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2}$ is bordism invariant by Example 3.5.6, the composite $\mathrm{GW} \overset{\mathrm{fgt}}{\Rightarrow} \mathrm{K}^{\mathrm{hC}_2} \Rightarrow \mathrm{K}^{\mathrm{tC}_2}$ factors uniquely over a map $\Xi\colon\ \mathrm{L} \Rightarrow \mathrm{K}^{\mathrm{tC}_2}$ and we obtain the main result of this paper:

4.4.14. **Corollary** (The fundamental fibre square)**.** *The natural square*

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{C}, \Omega) \arrow[r, "\mathrm{bord}"] \arrow[d, "\mathrm{fgt}"] & \mathrm{L}(\mathcal{C}, \Omega) \arrow[d, "\Xi"] \\
\mathrm{K}(\mathcal{C}, \Omega)^{\mathrm{hC}_2} \arrow[r] & \mathrm{K}(\mathcal{C}, \Omega)^{\mathrm{tC}_2}
\end{tikzcd} \tag{68}$$

is bicartesian for every Poincaré ∞*-category* $(\mathcal{C}, \Omega)$ *and in particular, there is a natural fibre sequence*

$$\mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, \Omega) \xrightarrow{\mathrm{bord}} \mathrm{L}(\mathcal{C}, \Omega). \tag{69}$$

Proof. Apply Corollary 3.6.7 in combination with Corollary 4.3.1 and Theorem 4.4.12. □

We will exploit this result to give computations of Grothendieck-Witt groups of discrete rings in Paper [III], and solve the homotopy limit problem for number rings. For now, we record:

4.4.15. **Corollary.** *The functor* $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ *is Verdier-localising.*

Proof. Given Corollary 4.4.6, we need only recall that K-theory is a Verdier-localising functor $\mathrm{K} : \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathcal{S}p$ (as by Proposition 1.1.4 the underlying sequence of a Poincaré-Verdier sequence is indeed a Verdier sequence). The statement about K-theory in large parts goes back to Waldhausen's fibration theorem and a direct proof in the present context was recently recorded in [HLS22, Section 6]. □

Now, by construction the composite

$$\mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, \Omega) \xrightarrow{\mathrm{fgt}} \mathrm{K}(\mathcal{C}, \Omega)^{\mathrm{hC}_2}$$

is the norm map of the C_2-spectrum $\mathrm{K}(\mathcal{C}, \Omega) \in \mathcal{S}p^{\mathrm{hC}_2}$. In particular, it is split after inverting 2 by the canonical maps

$$\mathrm{K}(\mathcal{C}, \Omega)^{\mathrm{hC}_2} \longrightarrow \mathrm{K}(\mathcal{C}, \Omega) \longrightarrow \mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2}$$

divided by 2. But then also the fibre sequence

$$\mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}(\mathcal{C}, \Omega) \xrightarrow{\mathrm{bord}} \mathrm{L}(\mathcal{C}, \Omega)$$

splits after inverting 2 and we obtain:

4.4.16. **Corollary.** *There is a canonical equivalence*

$$\mathrm{GW}(\mathcal{C}, \Omega)[\tfrac{1}{2}] \simeq \mathrm{K}(\mathcal{C}, \Omega)[\tfrac{1}{2}]_{\mathrm{hC}_2} \oplus \mathrm{L}(\mathcal{C}, \Omega)[\tfrac{1}{2}]$$

natural in the Poincaré ∞*-category* $(\mathcal{C}, \Omega)$ *and in particular*

$$\mathrm{GW}_i(\mathcal{C}, \Omega)[\tfrac{1}{2}] \cong \mathrm{K}_i(\mathcal{C})[\tfrac{1}{2}]_{\mathrm{C}_2} \oplus \mathrm{L}_i(\mathcal{C}, \Omega)[\tfrac{1}{2}].$$

Proof. Only the final statement remains to be proven, and it follows immediately from the former and the collapse of the homotopy orbit spectral sequence of a C_2-spectrum in which 2 is invertible to its edge. □

As a consequence of Corollary 1.4.9, we obtain localisation properties of Grothendieck-Witt spectra, which will form the basis of our analysis of the Grothendieck-Witt groups of Dedekind rings in the third paper of this series; see Corollary [III].2.2.5.

From Proposition 1.4.8, we immediately obtain:

4.4.17. **Corollary.** *Let* A *be a discrete ring,* M *a discrete invertible module with involution over* A, $\mathrm{c} \subseteq \mathrm{K}_0(A)$ *a subgroup closed under the involution induced by* M *and* $S \subseteq A$ *a multiplicative subset compatible with* M*, such that* (A, S) *satisfies the left Ore condition. Let, furthermore,* $\mathcal{D}^{\mathrm{c}}(A)_S$ *denote the full subcategory of* $\mathcal{D}^{\mathrm{c}}(A)$ *spanned by the* S*-torsion complexes. Then the inclusion and localisation functors fit into fibre sequences*

$$\mathrm{GW}(\mathcal{D}^{\mathrm{c}}(A)_S, \Omega^{\geq m}_M) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{c}}(A), \Omega^{\geq m}_M) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{im(c)}}(A[S^{-1}]), \Omega^{\geq m}_{M[S^{-1}]})$$

for all $m \in \mathbb{Z} \cup \{\pm\infty\}$.

For the compatibility condition between the multiplicative subset and the invertible module confer Definition 1.4.3 and Example 1.4.4.

In particular, one obtains a fibre sequence

$$\mathrm{GW}(\mathcal{D}^{\mathrm{f}}(A)_S, \Qoppa^{\geq m}_M) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{f}}(A), \Qoppa^{\geq m}_M) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{f}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]})$$

though this generally fails for $\mathcal{D}^{\mathrm{p}}$ in place of $\mathcal{D}^{\mathrm{f}}$, but see Remark 4.4.19 below. Upon taking connective covers, the case of commutative A is for example also implied by [Sch17]. We similarly obtain fibre sequences

$$\mathrm{L}(\mathcal{D}^{\mathrm{c}}(A)_S, \Qoppa^{\geq m}_M) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{c}}(A), \Qoppa^{\geq m}_M) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{im(c)}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]}),$$

which, upon investing our identification of the genuine L-spectra in the third instalment of this series, see Theorem [III].1.2.22, recover localisation sequences of Ranicki's; see [Ran81, Section 3.2].

4.4.18. **Remark.** By Corollary 1.4.6, the quadratic variant of Corollary 4.4.17 actually works for an arbitrary E_1-ring spectrum A and an invertible module M with involution over A, but for the symmetric and genuine variants, one has to require further conditions. We leave details to the interested reader, as we shall have no need for that generality.

4.4.19. **Remark.** By the cofinality theorem, the map $\mathrm{K}_i(\mathcal{D}^{\mathrm{c}}(R)) \to \mathrm{K}_i(\mathcal{D}^{\mathrm{p}}(R)) = \mathrm{K}_i(R)$ induces an isomorphism $i > 0$ and is the inclusion $\mathrm{c} \to \mathrm{K}_0(R)$ for $i = 0$. We will show a hermitian analogue in the fourth instalment of this series, namely that for any pair of involution-closed subgroups $\mathrm{c} \subseteq \mathrm{d} \subseteq \mathrm{K}_0(R)$ the squares

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{D}^{\mathrm{d}}(R), \Qoppa) \arrow[d] & & \mathrm{L}(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \arrow[r] \arrow[d] & \mathrm{L}(\mathcal{D}^{\mathrm{d}}(R), \Qoppa) \arrow[d] \\
\mathrm{K}(\mathcal{D}^{\mathrm{c}}(R), \mathrm{D}_\Qoppa)^{\mathrm{hC}_2} \arrow[r] & \mathrm{K}(\mathcal{D}^{\mathrm{d}}(R), \mathrm{D}_\Qoppa)^{\mathrm{hC}_2} & & \mathrm{K}(\mathcal{D}^{\mathrm{c}}(R), \mathrm{D}_\Qoppa)^{\mathrm{tC}_2} \arrow[r] & \mathrm{K}(\mathcal{D}^{\mathrm{d}}(R), \mathrm{D}_\Qoppa)^{\mathrm{tC}_2}
\end{tikzcd}$$

are cartesian, see Theorem Undefined ref. It follows that there are fibre sequences

$$\mathrm{GW}(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{d}}(R), \Qoppa) \longrightarrow \mathrm{H}(\mathrm{d}/\mathrm{c})^{\mathrm{hC}_2}$$

$$\mathrm{L}(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{d}}(R), \Qoppa) \longrightarrow \mathrm{H}(\mathrm{d}/\mathrm{c})^{\mathrm{tC}_2}.$$

In particular, the map $\mathrm{GW}_i(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \longrightarrow \mathrm{GW}_i(\mathcal{D}^{\mathrm{d}}(R), \Qoppa)$ is an isomorphism for positive i and injective for $i = 0$. On the L-theoretic side, we recover Ranicki's *Rothenberg-sequences*

$$\ldots \longrightarrow \mathrm{L}_i(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \longrightarrow \mathrm{L}_i(\mathcal{D}^{\mathrm{d}}(R), \Qoppa) \longrightarrow \widehat{\mathrm{H}}^{-i}(\mathrm{C}_2; d/c) \longrightarrow \mathrm{L}_{i-1}(\mathcal{D}^{\mathrm{c}}(R), \Qoppa) \longrightarrow \ldots$$

[Ran80, Proposition 9.1].

In a similar vein, one can compare localisations along a ring homomorphism:

4.4.20. **Proposition.** *Let $p : A \to B$ be a homomorphism of discrete rings, M and N discrete invertible modules with involution over A and B, respectively, $\eta : M \to N$ a group homomorphism that is $p \otimes p$-linear, $S \subseteq A$ a subset and $m \in \mathbb{Z} \cup \{\pm\infty\}$. Then if*

i) *the map $B \otimes_A M \to N$ induced by η is an isomorphism,*
ii) *the subset S is compatible with M,*
iii) *for every $s \in S$ the induced map $p\colon A /\!\!/ s \to B /\!\!/ p(s)$ on cofibres of right multiplication by s and $p(s)$, respectively, is an equivalence in $\mathcal{D}(A)$,*
iv) *the pairs (S, A) and $(p(S), A)$ both satisfy the left Ore condition, and*
v) *the boundary map $\widehat{\mathrm{H}}^{-m}(\mathrm{C}_2, N[p(S)^{-1}]) \to \widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2, M)$ in Tate cohomology of the short exact sequence*

$$M \xrightarrow{(-\eta,\mathrm{can})} N \oplus M[S^{-1}] \xrightarrow{(\mathrm{can},\eta)} N[p(S)^{-1}]$$

vanishes,

the square

$$\begin{tikzcd}
(\mathcal{D}^{\mathrm{c}}(A), \Qoppa^{\geq m}_M) \arrow[r] \arrow[d] & (\mathcal{D}^{\mathrm{im(c)}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]}) \arrow[d] \\
(\mathcal{D}^{p(\mathrm{c})}(B), \Qoppa^{\geq m}_N) \arrow[r] & (\mathcal{D}^{\mathrm{im}(p(\mathrm{c}))}(B[p(S)^{-1}]), \Qoppa^{\geq m}_{N[p(S)^{-1}]})
\end{tikzcd}$$

is a Poincaré-Verdier square for every subgroup $c \subseteq \mathrm{K}_0(A)$ stable under the involution induced by M, and so in particular becomes cartesian after taking either GW-, K- *or* L-*spectra.*

Here, condition v) is to be interpreted as vacuous if $m = \pm\infty$. Note also that condition iv) is equivalent to requiring that p induces an isomorphism on kernels and cokernels of right multiplication by any $s \in S$.

The K-theoretic part is a classical result of Karoubi, Quillen and Vorst, see [Vor79, Proposition 1.5], and investing the identification of the L-spectra from the third instalment in this series, see Theorem [III].1.2.22, the L-theoretic part recovers an analogous result of Ranicki [Ran81, Section 3.6].

Proof. Let us start out by observing that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[S^{-1}] \arrow[d] \\ B \arrow[r] & B[p(S)^{-1}] \end{tikzcd}$$

is cartesian in $\mathcal{D}(A)$: Denoting the top horizontal fibre by F, this is equivalent to the assertion that $F \to B \otimes^{\mathbb{L}}_A F$ is an equivalence in $\mathcal{D}(A)$, but combining Example 1.4.2 with assumptions iii) and iv), this holds for any object of $\mathcal{D}(A)_S$. Tensoring the square with M (over A) then produces the short exact sequence appearing in v). Furthermore, from the Ore conditions, we also find that the natural map $B \otimes^{\mathbb{L}}_A A[S^{-1}] \to B[p(S)^{-1}]$ is an equivalence. It is then readily checked that $p(S)$ is compatible with N.

Now, the rows of the diagram of Poincaré ∞-categories

$$\begin{tikzcd} (\mathcal{D}^{\mathrm{c}}(A)_S, \Qoppa^{\geq m}_M) \arrow[r] \arrow[d] & (\mathcal{D}^{\mathrm{c}}(A), \Qoppa^{\geq m}_M) \arrow[r] \arrow[d] & (\mathcal{D}^{\mathrm{im(c)}}(A[S^{-1}]), \Qoppa^{\geq m}_{M[S^{-1}]}) \arrow[d] \\ (\mathcal{D}^{p(\mathrm{c})}(B)_{p(S)}, \Qoppa^{\geq m}_N) \arrow[r] & (\mathcal{D}^{p(\mathrm{c})}(B), \Qoppa^{\geq m}_N) \arrow[r] & (\mathcal{D}^{\mathrm{im}(p(\mathrm{c}))}(B[p(S)]^{-1}), \Qoppa^{\geq m}_{N[p(S)^{-1}]}) \end{tikzcd}$$

are Poincaré-Verdier sequences by Proposition 1.4.8, and the vertical maps are Poincaré functors on account of assumption i), see Lemma [I].3.4.3, and the right hand square is Ind-adjointable: The square formed by the horizontal right adjoints on inductive completions identifies with the (a priori only lax) diagram

$$\begin{tikzcd} \mathcal{D}(A) \arrow[d, "B\otimes^{\mathbb{L}}_A -"] & \mathcal{D}(A[S^{-1}]) \arrow[l, "\mathrm{fgt}"'] \arrow[d, "B[p(S)^{-1}]\otimes^{\mathbb{L}}_{A[S^{-1}]} -"] \\ \mathcal{D}(B) & \mathcal{D}(B[p(S)^{-1}]), \arrow[l, "\mathrm{fgt}"'] \end{tikzcd}$$

with Beck-Chevalley transformation given by the natural map $B \otimes^{\mathbb{L}}_A X \to B[p(S)^{-1}] \otimes^{\mathbb{L}}_{A[S^{-1}]} X$ for $X \in \mathcal{D}(A[S^{-1}])$. Since both sides commute with colimits, it suffices to establish that this map is an equivalence for $X = A[S^{-1}]$, which we observed above.

We now claim that the left hand vertical map is an equivalence of Poincaré ∞-categories, whence Lemma 1.5.3 gives the claim. The fact that the underlying functor of stable ∞-categories is an equivalence follows from assumption i): By Example 1.4.2 the ∞-categories $\mathcal{D}^{\mathrm{p}}(A)_S$ and $\mathcal{D}^{\mathrm{p}}(B)_{p(s)}$ are generated by the objects $A /\!\!/ s$ and $B /\!\!/ p(s)$ under shifts, retracts and finite colimits, so the functor is essentially surjective and full faithfulness can be tested on these generators, where we compute

$$\mathrm{Hom}_A(A /\!\!/ s, A /\!\!/ t) \simeq \mathrm{Hom}_A(A /\!\!/ s, B /\!\!/ p(t)) \simeq \mathrm{Hom}_{\mathrm{B}}(B /\!\!/ p(s), B /\!\!/ p(t)).$$

See also [LT19, Proposition 1.17] for an alternative argument that the underlying square of ∞-categories is cartesian. It remains to check that the natural map $\Qoppa^{\geq m}_M(X) \to \Qoppa^{\geq m}_N(p_! X)$ induced by η is an equivalence for all $X \in \mathcal{D}^{\mathrm{p}}(A)_S$. For $m = \pm\infty$ this follows from the fact that $p_!$ is a Poincaré functor and an equivalence on underlying ∞-categories, as this evidently implies that $(p, \eta)_!$ induces an equivalence on bilinear parts. We are thus reduced to considering the linear parts for finite m. Using the adjunction $p_! \vdash p^*$, we have to show that for every S-torsion perfect complex of A-modules X, the map

$$\hom_A(X, \tau_{\geq m}(M^{\mathrm{tC}_2})) \longrightarrow \hom_A(X, p^*\tau_{\geq m}(N^{\mathrm{tC}_2}))$$

induced by η is an equivalence. Since the ∞-category $\mathcal{D}^{\mathrm{p}}_S(A)$ in generated under finite colimits and desuspensions by objects of the form $A /\!\!/ s = \mathrm{cof}(A \xrightarrow{\cdot s} A)$ one can equivalently show that every element $s \in S$ acts invertibly on $F_m = \mathrm{cof}\left(\tau_{\geq m}(M^{\mathrm{tC}_2}) \to f^*\tau_{\geq m}(N^{\mathrm{tC}_2})\right)$, i.e. that the canonical map

$$F_m \longrightarrow F_m[S^{-1}]$$

is an equivalence. We note that $F_m \to F_{-\infty}$ induces an isomorphism on homology groups in degrees larger than m, and that there is an exact sequence

$$0 \longrightarrow \mathrm{H}_m(F_m) \longrightarrow \mathrm{H}_m(F_{-\infty}) \longrightarrow K \longrightarrow 0,$$

where

$$K = \ker\left(\widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; M) \to \widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; N)\right).$$

From the fact that the bilinear parts of the two functors agree, we find that S acts invertibly on $F_{-\infty}$. Hence it remains to show that S acts invertibly on $\mathrm{H}_m(F_m)$. The above short exact sequence maps into its localisation at S. Since this localisation is an exact functor, the snake lemma implies that it suffices to check that the map $K \to K[S^{-1}]$ is injective. Writing $M[S^{-1}]$ as $(R[S^{-1}] \otimes R[S^{-1}]) \otimes_{R\otimes R} M$ and likewise for N, using assumption ii), we find that

$$K[S^{-1}] = \ker\left(\widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; M[S^{-1}]) \to \widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; N[p(S^{-1})])\right),$$

since Tate cohomology commutes with filtered colimits in the coefficients (see the discussion in the proof of Proposition 1.4.8). The kernel of $K \to K[S^{-1}]$ therefore canonically identifies with the kernel of

$$\widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; M) \longrightarrow \widehat{\mathrm{H}}^{-m+1}(\mathrm{C}_2; N \oplus M[S^{-1}])$$

which vanishes by assumption v). □

As the simplest non-trivial special case, we for example obtain:

4.4.21. **Corollary.** *Let R be a discrete commutative ring, M an invertible R-module with an R-linear involution, $f, g \in R$ elements spanning the unit ideal and $c \subseteq \mathrm{K}_0(R)$ closed under the involution associated to M. Then the square*

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{D}^{\mathrm{c}}(R), \Ϙ^{\geq m}_M) \arrow[r] \arrow[d] & \mathrm{GW}(\mathcal{D}^{\mathrm{im(c)}}(R[\frac{1}{f}]), \Ϙ^{\geq m}_{M[1/f]}) \arrow[d] \\
\mathrm{GW}(\mathcal{D}^{\mathrm{im(c)}}(R[\frac{1}{g}]), \Ϙ^{\geq m}_{M[1/g]}) \arrow[r] & \mathrm{GW}(\mathcal{D}^{\mathrm{im(c)}}(R[\frac{1}{fg}]), \Ϙ^{\geq m}_{M[1/fg]})
\end{tikzcd}$$

and the analogous squares in K *and* L*-theory are cartesian.*

Proof. We verify conditions i) through v) of the previous proposition. The first and fourth are obvious and the second is implied by the two R-module structures on M agreeing. For the third one simply notes that g acts invertibly on $R /\!\!/ f$, since with f and g also any powers thereof span the unit ideal. To verify the final condition, recall that Tate cohomology groups over C_2 with coefficients in M are 2-periodic with values alternating between the kernels of the norm maps $\mathrm{id}_M \pm \sigma \colon M_{\mathrm{C}_2} \to M^{\mathrm{C}_2}$. Thus, we may check that $M \to M[1/f] \oplus M[1/g]$ induces injections on both these kernels. But taking coinvariants commutes with localisation at both f and g, so the map in question is injective on the entire coinvariants. □

In completely analogous fashion one can treat the inversion of some prime l in $R \to R^\wedge_l$, leading to a localisation-completion square, see Proposition [III].2.3.6, and also the case of localisation of rings with involution at elements invariant under the involution, but let us refrain from spelling this out here.

4.5. **The real algebraic K-theory spectrum and Karoubi-Ranicki periodicity.** Just as in §3.7, the fundamental fibre square can be cleanly encapsulated as the isotropy separation square of a genuine C_2-spectrum:

4.5.1. **Definition.** We define the *real algebraic* K*-theory spectrum* $\mathrm{KR}(\mathcal{C}, \Omega)$ of a Poincaré ∞-category $(\mathcal{C}, \Omega)$ to be the genuine C_2-spectrum $\mathrm{GW}^{\mathrm{ghyp}}(\mathcal{C}, \Omega)$.

In particular, from Corollary 3.7.4, we obtain:

4.5.2. **Corollary.** *The real algebraic* K*-theory spectra define an additive functor*

$$\mathrm{KR}: \mathrm{Cat}^{\mathrm{p}}_{\infty} \longrightarrow \mathcal{S}p^{\mathrm{gC}_2},$$

such that

$$u\,\mathrm{KR} \simeq \mathrm{K}, \quad \mathrm{KR}^{\mathrm{gC}_2} \simeq \mathrm{GW} \quad \textit{and} \quad \mathrm{KR}^{\varphi\mathrm{C}_2} \simeq \mathrm{L},$$

where $u: \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p^{\mathrm{hC}_2}$ *denotes the functor extracting the underlying* C_2*-spectrum, and* $(-)^{\mathrm{gC}_2}$ *and* $(-)^{\varphi\mathrm{C}_2}: \mathcal{S}p^{\mathrm{gC}_2} \to \mathcal{S}p$ *denote the genuine and geometric fixed points, respectively. Furthermore, the isotropy separation square associated to* $\mathrm{KR}(\mathcal{C}, \Omega)$ *is naturally equivalent to the fundamental fibre square of* $(\mathcal{C}, \Omega)$.

And from Theorem 3.7.7, we succinctly find:

4.5.3. **Corollary.** *There are canonical equivalences*

$$\mathrm{KR}(\mathcal{C}, \Omega^{[1]}) \simeq \mathbb{S}^{1-\sigma} \otimes \mathrm{KR}(\mathcal{C}, \Omega)$$

natural in the Poincaré ∞*-category* $(\mathcal{C}, \Omega)$*. In particular, any equivalence* $(\mathcal{C}, \Omega) \to (\mathcal{C}, \Omega^{[n]})$ *induces a periodicity equivalence*

$$\mathrm{KR}(\mathcal{C}, \Omega) \simeq \mathbb{S}^{n-n\sigma} \otimes \mathrm{KR}(\mathcal{C}, \Omega).$$

Recall from [I].7.4.19 and the discussion preceding 4.3.4 that, generally,

$$(\mathcal{C}, \Omega^{[2n]}) \simeq (\mathcal{C}, (\mathbb{S}^{n-n\sigma} \otimes \widetilde{\Omega})^{\mathrm{gC}_2})$$

via the n-fold shift in $\mathcal{C}$, and that there are many examples where Ω and $(\mathbb{S}^{n-n\sigma} \otimes \widetilde{\Omega})^{\mathrm{gC}_2})$ are closely related, if not equal for some n. The simplest such situation is:

4.5.4. **Corollary.** *Let* R *be an* E_1*-algebra over an* E_∞*-ring* k *equipped with an* $n\sigma$*-orientation,* M *an invertible* k*-module with involution over* R *and* $\mathrm{c} \subseteq \mathrm{K}_0(R)$ *a subgroup closed under the involution induced by* M*. Then there are canonical equivalences*

$$\mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{s}}_M) \simeq \mathbb{S}^{2n-2n\sigma} \otimes \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{s}}_M) \quad \textit{and} \quad \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{q}}_M) \simeq \mathbb{S}^{2n-2n\sigma} \otimes \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{q}}_M).$$

If $n = 2$*, e.g. if* k *is complex oriented or* $2 \in \pi_0(k)^\times$*, this refines to*

$$\mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{s}}_{-M}) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{s}}_M) \quad \textit{and} \quad \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{q}}_{-M}) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\mathrm{q}}_M),$$

and if R *is furthermore connective we also have*

$$\mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\geq m+1}_{-M}) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathrm{Mod}^{\mathrm{c}}_R, \Omega^{\geq m}_M).$$

In particular, we obtain the following periodicity result:

4.5.5. **Corollary** (Karoubi-Ranicki periodicity)**.** *Let* R *be a discrete ring,* M *a discrete invertible module with involution over* R *and* $\mathrm{c} \subseteq \mathrm{K}_0(R)$ *a subgroup closed under the involution induced by* M*. Then the genuine* C_2*-spectra*

$$\mathrm{KR}(\mathcal{D}^{\mathrm{c}} R, \Omega^{\mathrm{s}}_M) \quad \textit{and} \quad \mathrm{KR}(\mathcal{D}^{\mathrm{c}} R, \Omega^{\mathrm{q}}_M)$$

are $(4 - 4\sigma)$*-periodic, and even* $(2 - 2\sigma)$*-periodic if* R *has characteristic* 2*. Furthermore, we have*

$$\mathbb{S}^{2\sigma-2} \otimes \mathrm{KR}(\mathcal{D}^{\mathrm{c}} R, \Omega^{\mathrm{gq}}_M) \simeq \mathrm{KR}(\mathcal{D}^{\mathrm{c}} R, \Omega^{\mathrm{ge}}_{-M}) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^{\mathrm{c}} R, \Omega^{\mathrm{gs}}_M).$$

Passing to geometric fixed points in 4.5.4 extends Ranicki's classical periodicity results for L-groups from the case of discrete rings:

4.5.6. **Corollary.** *Let R be an E_1-algebra over an E_∞-ring k equipped with an $n\sigma$-orientation, M an invertible k-module with involution over R and $\mathrm{c} \subseteq \mathrm{K}_0(R)$ a subgroup closed under the involution induced by M. Then there are canonical equivalences*

$$\mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_M) \simeq \mathbb{S}^{2n} \otimes \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_M) \quad \textit{and} \qquad \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_M) \simeq \mathbb{S}^{2n} \otimes \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_M).$$

If $n = 2$, e.g. if k is complex oriented or $2 \in \pi_0(k)^\times$, this refines to

$$\mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_{-M}) \simeq \mathbb{S}^{n} \otimes \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{s}}_M) \quad \textit{and} \qquad \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_{-M}) \simeq \mathbb{S}^{n} \otimes \mathrm{L}(\mathrm{Mod}^{\mathrm{c}}_R, \Qoppa^{\mathrm{q}}_M).$$

Of course, this corollary can also easily be obtained straight from the shifting behaviour of bordism invariant functors. Let us also recall briefly from [I].3.5.14 that there are also many examples with $n\sigma$-orientations for $n > 2$: For example, any Spin-orientation induces a 4σ-orientation and a String-orientation an 8σ-orientation. We thus find that for example that

$$\mathrm{KR}(\mathrm{Mod}^{\omega}_{\mathrm{ko}}, \Qoppa^{\mathrm{s}}) \quad \text{and} \quad \mathrm{KR}(\mathrm{Mod}^{\omega}_{\mathrm{tmf}}, \Qoppa^{\mathrm{s}})$$

are $(8 - 8\sigma)$- and $(16 - 16\sigma)$-periodic, respectively.

Let us finally discuss the case of form parameters as recalled at the end of §4.3. The equivalence

$$\big(\mathcal{D}^{\mathrm{p}}(R), (\Qoppa^{\mathrm{g}\lambda}_M)^{[2]}\big) \longrightarrow \big(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{g}\check{\lambda}}_{-M}\big).$$

associated to a form parameter

$$M_{\mathrm{C}_2} \xrightarrow{\tau} Q \xrightarrow{\rho} M^{\mathrm{C}_2}$$

with second map injective and dual

$$(-M)_{\mathrm{C}_2} \longrightarrow M/Q \longrightarrow (-M)^{\mathrm{C}_2}$$

gives:

4.5.7. **Corollary.** *For a discrete ring R, a discrete invertible module with involution M over R, a subgroup $c \subseteq \mathrm{K}_0(R)$ closed under the involution induced by M and a form parameter $\lambda = (Q, \tau, \rho)$ on M with ρ injective, there is a canonical equivalence*

$$\mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{g}\check{\lambda}}_{-M}) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{KR}(\mathcal{D}^c(R), \Qoppa^{\mathrm{g}\lambda}_M).$$

In particular, we find

$$\mathrm{L}(\mathcal{D}^c(R), \Qoppa^{\mathrm{g}\check{\lambda}}_{-M}) \simeq \mathbb{S}^{2} \otimes \mathrm{L}(\mathcal{D}^c(R), \Qoppa^{\mathrm{g}\lambda}_M)$$

by passing to geometric fixed points (or directly from bordism invariance of L-theory). Note also, that this corollary can be applied twice whenever the dual form parameter $(-M)_{\mathrm{C}_2} \to M/Q \to (-M)^{\mathrm{C}_2}$ again has its second map injective. This is the case if and only if $\rho\colon Q \to M^{\mathrm{C}_2}$ is an isomorphism, in which case $\Qoppa^{\mathrm{g}\lambda}_M = \Qoppa^{\mathrm{gs}}_M$ per construction. We thus find equivalences

$$\mathbb{S}^4 \otimes \mathrm{L}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gq}}_M) \simeq \mathbb{S}^2 \otimes \mathrm{L}(\mathcal{D}^c(R), \Qoppa^{\mathrm{ge}}_{-M}) \simeq \mathrm{L}(\mathcal{D}^c(R), \Qoppa^{\mathrm{gs}}_M),$$

as claimed in the introduction.

4.5.8. **Remark.**
- i) The genuine L-spectra really are not periodic in general, as we will show in Paper [III] of this series by explicit computation of $\mathrm{L}(\mathcal{D}^{\mathrm{p}}\mathbb{Z}, \Qoppa^{\mathrm{gs}})$.
- ii) The higher periodicities for the KR- and L-spectra of ring spectra such as ko and tmf, or their periodic versions, seem to be a new phenomenon; we do not, however, currently understand these spectra well-enough to actually rule out lower periodicity in any of the cases. To illustrate that this can actually occur, let us mention that the algebraic π-π-Theorem of Weiss and Williams from [WW89], see also [III].1.2.33, gives an equivalence

$$\mathrm{L}(\mathrm{Mod}^{\omega}_R, \Qoppa^{\mathrm{q}}) \longrightarrow \mathrm{L}(\mathcal{D}^{\mathrm{p}}(\pi_0 R), \Qoppa^{\mathrm{q}})$$

  for all connective E_1-rings R, so that these spectra are always 4-periodic.
- iii) It follows from [WW14, Theorem 4.5], that $\mathrm{L}(\mathrm{Mod}^{\omega}_{\mathbb{S}}, \Qoppa^{\mathrm{s}})$ is not periodic at all; we will explain this in Remark 4.6.5 below. Consequently, some assumption like the existence of a symmetry, or more precisely a Thom isomorphism for the vector bundle $k\sigma$, i.e. $\gamma_1^{\oplus k} \to \mathrm{BC}_2$, for some k, is a definite requirement for a periodicity statement even for the symmetric Poincaré structure.

4.6. LA**-theory after Weiss and Williams.** In this final subsection, we would like to relate the fundamental fibre square to the LA-spectra arising in the work of Weiss and Williams [WW14]. We start by comparing the map $\Xi\colon \mathrm{L}\to\mathrm{K}^{\mathrm{tC}_2}$ appearing in Corollary 4.4.14 with the map $\mathrm{L}\to\mathrm{K}^{\mathrm{tC}_2}$ constructed by Weiss and Williams in [WW98, Section 9]. Translated to our set-up, they consider the map

$$\mathcal{L}(\mathcal{C},Ϙ^{\mathrm{s}}) = |\mathrm{Cr}\,\rho(\mathcal{C},Ϙ^{\mathrm{s}})^{\mathrm{hC}_2}| \longrightarrow \Omega|\mathrm{Cr}\,\mathrm{Q}\,\rho(\mathcal{C},(Ϙ^{\mathrm{s}})^{[1]})^{\mathrm{hC}_2}| \longrightarrow \Omega^\infty\mathrm{ad}(\mathrm{K}^{\mathrm{hC}_2})(\mathcal{C},Ϙ^{\mathrm{s}}),$$

where the second map is a colimit-limit interchange and the first is the realisation (in the ρ-direction) of the structure maps for the group completions of the additive functor $\mathrm{Cr}^{\mathrm{hC}_2}$; here $Ϙ^{\mathrm{s}}$ denotes the symmetrisation of a hermitian structure $Ϙ$ on $\mathcal{C}$, given by $Ϙ^{\mathrm{s}}(X)=\mathrm{B}_Ϙ(X,X)^{\mathrm{hC}_2}$ as in Example [I].1.1.17. For the reader filling in the details of the translation, we also note, that Weiss and Williams use $\Omega^\sigma|\mathrm{Cr}\mathrm{S}^e\rho(\mathcal{C},Ϙ)|$ in place of $\Omega|\mathrm{Cr}\,\mathrm{Q}\,\rho(\mathcal{C},(Ϙ^{\mathrm{s}})^{[1]})^{\mathrm{hC}_2}|$, where S^e is the edgewise subdivision of Segal's S-construction. We shall discuss the comparison between the Q- and S-constructions in some detail in Appendix B.1, and the equivalence between these terms is then an instance of Karoubi-periodicity, for example it is obtained by applying Ω^∞ to the equivalence of 4.5.3 above.

Precomposing with the composite

$$\mathrm{Pn}(\mathcal{C},Ϙ)\longrightarrow\mathcal{L}(\mathcal{C},Ϙ)\xrightarrow{\mathrm{fgt}}\mathcal{L}(\mathcal{C},Ϙ^{\mathrm{s}})$$

and unwinding definitions this is the same as

$$\mathrm{Pn}(\mathcal{C},Ϙ)\longrightarrow\mathcal{GW}(\mathcal{C},Ϙ)\longrightarrow|\,\mathcal{GW}\,\rho(\mathcal{C},Ϙ)|\xrightarrow{\mathrm{fgt}}|\,\mathcal{K}\,\rho(\mathcal{C},Ϙ^{\mathrm{s}})^{\mathrm{hC}_2}|\longrightarrow\Omega^\infty\mathrm{ad}(\mathrm{K}^{\mathrm{hC}_2})(\mathcal{C},Ϙ^{\mathrm{s}}).$$

The latter part of this composite can in turn be rewritten as

$$\mathcal{GW}(\mathcal{C},Ϙ)\simeq\Omega^\infty\,\mathrm{GW}(\mathcal{C},Ϙ)\longrightarrow\Omega^\infty\mathrm{ad}\,\mathrm{GW}(\mathcal{C},Ϙ)\xrightarrow{\mathrm{fgt}}\Omega^\infty\mathrm{ad}(\mathrm{K}^{\mathrm{hC}_2})(\mathcal{C},Ϙ^{\mathrm{s}}).$$

Now, the canonical map $\mathrm{ad}(\mathrm{K}^{\mathrm{hC}_2})(\mathcal{C},Ϙ)\to\mathrm{ad}(\mathrm{K}^{\mathrm{hC}_2})(\mathcal{C},Ϙ^{\mathrm{s}})$ is an equivalence, so the forgetful map is nothing but $\Omega^\infty\Xi\colon\Omega^\infty\,\mathcal{L}(\mathcal{C},Ϙ)\to\Omega^\infty\,\mathrm{K}(\mathcal{C},Ϙ)^{\mathrm{tC}_2}$ under the identifications of Corollary 3.6.13 and Theorem 4.4.12. By the universal property of L-theory in Theorem 4.4.12, we conclude that the Weiss-Williams map $\mathrm{L}\Rightarrow\mathrm{K}^{\mathrm{tC}_2}$ agrees with ours.

4.6.1. **Corollary.** *For a space $B\in\mathcal{S}$ and a stable spherical fibration ξ over B the spectrum $\mathrm{GW}(\mathcal{S}p^\omega_B,Ϙ^r_\xi)$, identifies with Weiss' and Williams' $\mathrm{LA}^r(B,\xi)$, where $r\in\{\mathrm{s},\mathrm{v},\mathrm{q}\}$, i.e. either of symmetric, visible or quadratic. In particular, we find equivalences*

$$\Omega^{\infty-1}\mathrm{LA}^r(B,\xi)\simeq|\mathrm{Cob}(\mathcal{S}p^\omega_B,Ϙ^r_\xi)|.$$

We think of the displayed equivalence as a cycle model for the left hand object, which seems to be new. In particular, specialising to $B=*$ we find that the (-1)-st infinite loop spaces of

$$\mathrm{GW}(\mathcal{S}p^\omega,Ϙ^{\mathrm{s}})\simeq\mathrm{LA}^{\mathrm{s}}(*)\quad\text{and}\quad\mathrm{GW}(\mathcal{S}p^\omega,Ϙ^{\mathrm{u}})\simeq\mathrm{LA}^{\mathrm{v}}(*),$$

where $Ϙ^{\mathrm{u}}\colon(\mathcal{S}p^\omega)^{\mathrm{op}}\to\mathcal{S}p$ is the universal hermitian structure of §[I].4.1, are the homotopy types of the cobordism ∞-categories of Spanier-Whitehead selfdual spectra, and selfdual spectra equipped with a lift along

$$\mathrm{D}_{\mathbb{S}}X\to(\mathrm{D}_{\mathbb{S}}X)^\wedge_2\simeq\hom_{\mathcal{S}p}(X,\mathrm{D}_{\mathbb{S}}X)^{\mathrm{tC}_2}$$

of the image of the selfduality map, respectively.

4.6.2. **Remark.** Here we applied a naming scheme similar to Lurie's suggestion of writing $\mathrm{L}^{\mathrm{q}}(R)$ and $\mathrm{L}^{\mathrm{s}}(R)$ instead of Ranicki's $\mathrm{L}_\bullet(R)$ and $\mathrm{L}^\bullet(R)$ for what we would systematically call $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R),Ϙ^{\mathrm{q}}_R)$ and $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R),Ϙ^{\mathrm{s}}_R)$.

In [WW14] the spectra $\mathrm{LA}^{\mathrm{q}}(B,\xi),\mathrm{LA}^{\mathrm{v}}(B,\xi)$ and $\mathrm{LA}^{\mathrm{s}}(B,\xi)$ are called $\mathrm{LA}_\bullet(B,\xi\otimes\mathbb{S}^d,d)$, $\mathrm{VLA}^\bullet(B,\xi\otimes\mathbb{S}^d,d)$ and $\mathrm{LA}^\bullet(B,\xi\otimes\mathbb{S}^d,d)$, where d is the dimension of ξ.

Proof. The spectra $\mathrm{LA}^r(B,\xi)$ are defined by certain pullbacks [WW14, Definition 9.5]

$$\begin{tikzcd}
\mathrm{LA}^r(B,\xi) \arrow[r] \arrow[d] & \mathrm{L}^r(B,\xi) \arrow[d, "\Xi"] \\
\mathrm{A}(B,\xi)^{\mathrm{hC}_2} \arrow[r] & \mathrm{A}(B,\xi)^{\mathrm{tC}_2},
\end{tikzcd}$$

which we claim correspond precisely to our fundamental fibre square Corollary 4.4.14 for $(\mathcal{S}p^\omega_B,Ϙ^r_\xi)$.

As we detailed in Section [I].4.4, the sets of quadratic, symmetric and visible Poincaré objects that Weiss and Williams consider canonically map to $\mathrm{Pn}(\mathcal{S}p_B^\omega, \Qoppa_\xi^r)$ for the appropriate value of r. Since they define their L-spaces by a point-set implementation of the ρ-construction, and then deloop them by shifting the duality, see [WW98, Sections 10 & 11], there result comparison maps between the L-spectra, that are equivalences by [I].4.4.12 and [I].4.4.14. As we identified the map Ξ occurring in the definition of the LA-spectra with ours above, we obtain the claim from the well-known equivalence $\mathrm{A}(B) \simeq \mathrm{K}(\mathcal{S}p_B^\omega)$.

For completeness' sake let us give a reader's digest of the comparison of Poincaré objects. Weiss and Williams work in the dual set-up, i.e. they describe Poincaré objects via their coforms, rather than forms. Translated to modern language they consider the functor

$$\mathcal{S}p_B^\omega \longrightarrow \mathcal{S}p^{\mathrm{hC}_2}, \quad E \longmapsto \mathrm{M}(E \otimes_B E \otimes_B \xi),$$

where $\mathrm{M}\colon \mathcal{S}p_B \to \mathcal{S}p$ is the Thom spectrum functor (corresponding to $\mathrm{colim}\colon \mathrm{Fun}(B, \mathcal{S}p) \to \mathcal{S}p$), and define symmetric and quadratic coforms objects on some $E \in \mathcal{S}p_B^\omega$ by forming homotopy fixed points and orbits, respectively. The translation to our language is achieved via the Costenoble-Waner duality equivalence

$$\mathrm{D}_B\colon (\mathcal{S}p_B^\omega)^{\mathrm{op}} \to \mathcal{S}p_B^\omega,$$

i.e. the duality associated to $\Qoppa^{\mathrm{s}}_{\mathbb{S}_B}$, see Corollary [I].4.4.3: Indeed, one calculates that

$$\mathrm{B}_{\Qoppa_\xi}(\mathrm{D}_B E, \mathrm{D}_B F) \simeq \mathrm{M}(E \otimes_B F \otimes_B \xi),$$

for perfect $E, F \in \mathcal{S}p_B$. To define visible coforms, they promote $\mathrm{M}(E \otimes_B E \otimes_B \xi)$ to a genuine C_2-spectrum, i.e. a relative version of the Hill-Hopkins-Ravenel norm (though for perfect objects it is much easier to construct and analyse). They then take a visible coform on some $E \in \mathcal{S}p_B^\omega$ to be an element of $\Omega^\infty \mathrm{M}(E \otimes_B E \otimes_B \xi)^{\mathrm{gC}_2}$. Just as in the absolute setting, the geometric fixed points of this genuine refinement are given by $\mathrm{M}(E \otimes_B \xi)$ and the natural map to $\mathrm{M}(E \otimes_B E \otimes_B \xi)^{\mathrm{tC}_2}$ is induced by the Tate diagonal, see [I].4.4.13. Now one readily calculates

$$\Lambda_{\Qoppa_\xi^{\mathrm{v}}}(\mathrm{D}_B E) \simeq \mathrm{M}(E \otimes_B \xi),$$

whence we obtain an identification between the isotropy separation square of $\mathrm{M}(E \otimes_B E \otimes_B \xi)$ and

$$\begin{tikzcd}
\Qoppa_\xi^{\mathrm{v}}(\mathrm{D}_B E) \arrow[r] \arrow[d] & \Lambda_{\Qoppa_\xi^{\mathrm{v}}}(\mathrm{D}_B E) \arrow[d] \\
\mathrm{B}_{\Qoppa_\xi^{\mathrm{v}}}(\mathrm{D}_B E, \mathrm{D}_B E)^{\mathrm{hC}_2} \arrow[r] & \mathrm{B}_{\Qoppa_\xi^{\mathrm{v}}}(\mathrm{D}_B E, \mathrm{D}_B E)^{\mathrm{tC}_2},
\end{tikzcd}$$

which gives the claim also in this final case. □

4.6.3. **Remark.** In subsequent work, we will construct for ξ a stable $(-d)$-dimensional vector bundle over B a functor

$$\mathrm{Cob}_d^\xi \to \mathrm{Cob}(\mathcal{S}p_B^\omega, \Qoppa_\xi^{\mathrm{v}})$$

from the geometric, normally-ξ oriented cobordism category into the algebraic cobordism ∞-category of parametrised spectra over B. Through the equivalence

$$\Omega^\infty \mathrm{LA}^{\mathrm{v}}(B, \xi) \simeq \Omega|\mathrm{Cob}(\mathcal{S}p_B^\omega, \Qoppa_\xi^{\mathrm{v}})|$$

this provides a factorisation of the Weiss-Williams index map

$$\mathrm{BTop}^\xi(M) \longrightarrow \Omega^\infty \mathrm{LA}^{\mathrm{v}}(B, \xi)$$

whenever M is a closed ξ-oriented manifold with stable normal bundle ν_M, through the (topological) cobordism category Cob_d^ξ; here $\mathrm{Top}^\xi(M)$ denotes the E_1-group of ξ-oriented homeomorphisms of M. Now the homotopy type of the cobordism category is excisive in the bundle data by the main result of [GLK22], which was exploited by Raptis and the ninth author in the K-theoretic context for a new proof of the Dwyer-Weiss-Williams index theorem [RS17]. One can now follow their strategy so as to provide a canonical lift of the map $\Omega|\mathrm{Cob}_d^\xi| \to \Omega^\infty \mathrm{LA}^{\mathrm{v}}(B, \xi)$, into the source of the assembly map of LA^{v}. Inserting $\xi = \nu_M$, one obtains a lift of the Weiss-Williams index, resulting in a new perspective on substantial parts of [WW14] and by compatibility with the classical comparison between block homeomorphism and L-spaces also on Waldhausen's map

$$\widetilde{\mathrm{Top}}(M)/\mathrm{Top}(M) \longrightarrow \mathrm{Wh}(M)_{\mathrm{hC}_2},$$

into the (topological) Whitehead spectrum of M, by investing the fundamental fibre sequence.

We offer one application of the identification $\mathrm{GW}(\mathcal{S}p^{\omega}, \Qoppa^{\mathrm{u}}) \simeq \mathrm{LA}^{\mathrm{v}}(*)$. To this end, recall that the functors GW_0 and L_0 and $\mathrm{K}_0 : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{A}b$ are compatibly lax symmetric monoidal for the tensor product of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ and that $(\mathcal{S}p^{\omega}, \Qoppa^{\mathrm{u}})$ is the unit of the tensor product on $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. Hence, there are rings maps

$$\mathrm{K}_0(\mathcal{S}p^{\omega}) \xleftarrow{\mathrm{fgt}} \mathrm{GW}_0(\mathcal{S}p^{\omega}, \Qoppa^{\mathrm{u}}) \xrightarrow{\mathrm{bord}} \mathrm{L}_0(\mathcal{S}p^{\omega}, \Qoppa^{\mathrm{u}}).$$

Abbreviating the underlying spectra to $\mathrm{K}(\mathbb{S})$, $\mathrm{GW}^{\mathrm{u}}(\mathbb{S})$ and $\mathrm{L}^{\mathrm{u}}(\mathbb{S})$, and similarly their homotopy groups, we have:

4.6.4. **Proposition.** *There is a commutative diagram with vertical maps isomorphisms*

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] & \mathbb{Z}[e,h]/I \arrow[l, "8 \leftarrow\!\shortmid e,\ 2 \leftarrow\!\shortmid h"'] \arrow[r, "e \mapsto e,\ h \mapsto 0"] \arrow[d] & \mathbb{Z}[e]/(e^2 - 8e) \arrow[d] \\
\mathrm{K}_0(\mathbb{S}) & \mathrm{GW}^{\mathrm{u}}_0(\mathbb{S}) \arrow[l] \arrow[r] & \mathrm{L}^{\mathrm{u}}_0(\mathbb{S})
\end{tikzcd}$$

where e and h denote the classes of the spherical E_8-lattice and $\mathrm{hyp}(\mathbb{S})$*, respectively, and I is the ideal generated by $e^2 - 8e$, $h^2 - 2h$ and $eh - 8h$.*

Furthermore, there are canonical isomorphisms

$$\mathrm{GW}^{\mathrm{u}}_{-i}(\mathbb{S}) \cong \mathrm{L}^{\mathrm{u}}_{-i}(\mathbb{S}) \cong \mathrm{L}^{\mathrm{q}}_{-i}(\mathbb{Z})$$

for $i > 2$ induced by the comparison maps with quadratic L*-theory of the sphere spectrum, whereas* $\mathrm{GW}^{\mathrm{u}}_{-1}(\mathbb{S})$ *and* $\mathrm{GW}^{\mathrm{u}}_{-2}(\mathbb{S})$ *both vanish.*

The calculation of $\mathrm{K}_0(\mathbb{S}) = \pi_0\mathrm{A}(*)$ is of course due to Waldhausen and the calculation of $\mathrm{L}^{\mathrm{u}}_0(\mathbb{S})$ is due to Weiss-Williams (due to the identification $\mathrm{L}^{\mathrm{u}}(\mathbb{S}) = \mathrm{L}(\mathcal{S}p^{\omega}, \Qoppa^{\mathrm{v}}_{\mathbb{S}})$). The spherical lift of the E_8-lattice comes from the canonical map $\pi_0\Qoppa^{\mathrm{q}}(\mathbb{S}^{\oplus i}) \to \pi_0\Qoppa^{\mathrm{q}}(\mathbb{Z}^i)$ being an isomorphism, which in fact implies that every quadratic form over $\mathbb{Z}$ lifts to $\mathbb{S}$ uniquely up to homotopy.

Without multiplicative structures the result says that $\mathrm{GW}^{\mathrm{u}}_0(\mathbb{S})$ is free of rank 3 generated by the Poincaré spectra $\mathrm{hyp}(\mathbb{S})$, $(\mathbb{S}, \mathrm{id}_{\mathbb{S}})$ and the spherical lift of the E_8-lattice. In particular, as already observed by Weiss and Williams, the equality $[E_8] = 8[\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}]$ in the symmetric (Grothendieck-)Witt-group of the integers (a consequence of the classification of indefinite forms over $\mathbb{Z}$ through rank, parity and signature) does not lift to the sphere spectrum.

Proof. We first identify the underlying abelian groups in all cases. From Corollary 4.4.14, we have a fibre sequence

$$\mathrm{K}(\mathbb{S})_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}^{\mathrm{u}}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}(\mathbb{S}),$$

which we identified with

$$\mathrm{A}(*)_{\mathrm{hC}_2} \longrightarrow \mathrm{LA}^{\mathrm{v}}(*) \longrightarrow \mathrm{L}^{\mathrm{v}}(*)$$

above. Using the former naming, Weiss and Williams constructed a fibre sequence

$$\mathrm{L}^{\mathrm{q}}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}(\mathbb{S}) \longrightarrow \mathbb{S} \oplus \mathrm{MTO}(1),$$

by identifying the latter term with visible, normal (or hyperquadratic) L-theory of the sphere in [WW14, Theorem 4.3]. By the algebraic π-π-theorem the base change map

$$\mathrm{L}^{\mathrm{q}}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{q}}(\mathbb{Z})$$

is an equivalence; this appears for example as [WW89, Proposition 6.2], a proof in the present language is given in [Lur11, Lecture 14] and we will also derive it in the third instalment of this series, see Corollary [III].1.2.33. We thus obtain an exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{u}}_1(\mathbb{S}) \longrightarrow \pi_1(\mathbb{S} \oplus \mathrm{MTO}(1)) \longrightarrow \mathbb{Z} \longrightarrow \mathrm{L}^{\mathrm{u}}_0(\mathbb{S}) \longrightarrow \pi_0(\mathbb{S} \oplus \mathrm{MTO}(1)) \longrightarrow 0,$$

since the odd quadratic L-groups of the integers vanish, whereas $\mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) = \mathbb{Z}$, spanned by the E_8-lattice. Thus we also find that $\mathrm{L}^{\mathrm{q}}_0(\mathbb{S})$ is spanned by a spherical lift of E_8. Now to obtain the homotopy groups of MTO(1), recall from [GTMW09, Section 3] the fibre sequence

$$\mathrm{MTO}(1) \longrightarrow \mathbb{S}[\mathrm{BO}(1)] \longrightarrow \mathrm{MTO}(0),$$

the latter term being equivalent to the sphere $\mathbb{S}$. Now the first nine (reduced) homotopy groups of $\mathbb{S}[\mathrm{BO}(1)]$ were computed by Liulevicius in [Liu63, Theorem II.6], and the map $\mathbb{S}[\mathrm{BO}(1)] \to \mathbb{S}$ is easily checked to be the transfer map for the canonical double cover of BO(1). Therefore it is 2-locally surjective on positive homotopy groups by the Kahn-Priddy theorem [KP78] and given by multiplication by 2 on π_0. We obtain

$$\pi_i(\mathbb{S} \oplus \mathrm{MTO}(1)) = \begin{cases} 0 & i < -1 \\ \mathbb{Z}/2 & i = -1 \\ \mathbb{Z} & i = 0 \\ (\mathbb{Z}/2)^2 & i = 1 \end{cases}$$

Thus we find $\mathrm{L}^{\mathrm{u}}_0(\mathbb{S}) \cong \mathbb{Z}^2$ generated by the spherical E_8-lattice and $(\mathbb{S}, \mathrm{id}_{\mathbb{S}})$, compare the discussion following [WW14, Theorem 4.3]. Furthermore, we also find $\mathrm{L}^{\mathrm{u}}_1(\mathbb{S}) = (\mathbb{Z}/2)^2$, so obtain an exact sequence

$$0 \longrightarrow \mathrm{K}_0(\mathbb{S})_{\mathrm{C}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}^{\mathrm{u}}_0(\mathbb{S}) \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

because the first term is torsion free, since the involution $\mathrm{D}_{Ϙ^{\mathrm{u}}}$ evidently acts trivially on $\mathrm{K}_0(\mathbb{S}) \cong \mathbb{Z}$. This gives the first claim.

The second claim also follows, as $\mathbb{S} \oplus \mathrm{MTO}(1)$ is (-2)-connected, so the maps

$$\mathrm{GW}^{\mathrm{u}}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}(\mathbb{S}) \longleftarrow \mathrm{L}^{\mathrm{q}}(\mathbb{S})$$

are isomorphisms on homotopy groups from degree -3 on. For degrees -1 and -2 we find an exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{u}}_{-1}(\mathbb{S}) \longrightarrow \pi_{-1}(\mathbb{S} \oplus \mathrm{MTO}(1)) \longrightarrow \mathrm{L}^{\mathrm{q}}_{-2}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}_{-2}(\mathbb{S}) \longrightarrow 0$$

with both middle terms isomorphic to $\mathbb{Z}/2$. We now claim that the right map vanishes, forcing the middle one to be an isomorphism completing the computation of the additive structure (see also Corollary [III].1.2.33 iv) for a more direct proof that the outer terms vanish).

For this, we first note that the canonical map $\mathrm{L}^{\mathrm{q}}_{-2}(\mathbb{Z}) \to \mathrm{L}^{\mathrm{s}}_{-2}(\mathbb{Z})$ vanishes; indeed, the source is spanned by the standard unimodular skew-quadratic form of Arf-invariant 1 on $\mathbb{Z}^2$ (regarded as a chain complex concentrated in degree 1), given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

whose underlying anti-symmetric bilinear form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

admits the Lagrangian $\mathbb{Z} \oplus 0$. But then $\mathrm{L}^{\mathrm{q}}_{-2}(\mathbb{S}) = \pi_0 \mathrm{L}(\mathcal{S}p^{\mathrm{p}}, Ϙ^{\mathrm{q}[2]})$ is spanned by the lift of this quadratic form to $\mathbb{S}^1 \oplus \mathbb{S}^1$ and we claim that the Lagrangian lifts as well: Decoding this is implied by

$$0 = \pi_0(Ϙ^{\mathrm{u}})^{[2]}(\mathbb{S}^1) = \pi_{-2}Ϙ^{\mathrm{u}}(\mathbb{S}^1),$$

which gives the vanishing of the underlying form $q \in \Omega^\infty(Ϙ^{\mathrm{u}})^{[2]}(\mathbb{S}^1 \oplus \mathbb{S}^1)$ restricted to one of the summands; the resulting object of $\mathrm{Fm}(\mathrm{Met}(\mathcal{S}p^{\mathrm{p}}, (Ϙ^{\mathrm{u}})^{[2]}))$ is automatically Poincaré, as this can be checked after base change to the integers by Whitehead's theorem, where it reduces to the computation above.

To see the vanishing, consider the square

$$\begin{tikzcd}
Ϙ^{\mathrm{u}}(\mathbb{S}^1) \arrow[r] \arrow[d] & \hom_{\mathbb{S}}(\mathbb{S}^1, \mathbb{S}) \arrow[d] \\
\hom_{\mathbb{S}}(\mathbb{S}^1 \otimes \mathbb{S}^1, \mathbb{S})^{\mathrm{hC}_2} \arrow[r] & \hom_{\mathbb{S}}(\mathbb{S}^1 \otimes \mathbb{S}^1, \mathbb{S})^{\mathrm{tC}_2}
\end{tikzcd}$$

from the definition of $Ϙ^{\mathrm{u}}$ (with the C_2-action flipping the $\mathbb{S}^1$-factors). Since $\hom_{\mathbb{S}}(\mathbb{S}^1 \otimes \mathbb{S}^1, \mathbb{S}) \simeq \mathbb{S}^{-1-\sigma}$ it gives rise to a diagram

$$\begin{tikzcd}
\pi_{-1}\mathbb{S}^{-1} \arrow[r] \arrow[d] & \pi_{-2}\mathbb{S}^{-1-\sigma}_{\mathrm{hC}_2} \arrow[r] \arrow[d, equal] & \pi_{-2}Ϙ^{\mathrm{u}}(\mathbb{S}^1) \arrow[r] \arrow[d] & \pi_{-2}\mathbb{S}^{-1} \arrow[d] \\
\pi_{-1}(\mathbb{S}^{-1-\sigma})^{\mathrm{tC}_2} \arrow[r] & \pi_{-2}\mathbb{S}^{-1-\sigma}_{\mathrm{hC}_2} \arrow[r] & \pi_{-2}(\mathbb{S}^{-1-\sigma})^{\mathrm{hC}_2} \arrow[r] & \pi_{-2}(\mathbb{S}^{-1-\sigma})^{\mathrm{tC}_2}.
\end{tikzcd}$$

with exact rows. Now, the top right corner vanishes and the homotopy orbit terms evaluate to $\mathbb{Z}/2$. Thus we will be done if we show that the homotopy fixed point term vanishes as well, since then the lower left horizontal map is surjective and by Lin's theorem [Lin80] (identifying the left vertical map as $\mathbb{Z} \to \mathbb{Z}_2^\wedge$), so is the upper left horizontal map. But dualising the fibre sequence $\mathbb{S} \otimes \mathrm{C}_2 \to \mathbb{S} \to \mathbb{S}^\sigma$ and applying homotopy fixed points yields a fibre sequence

$$(\mathbb{S}^{-\sigma})^{\mathrm{hC}_2} \longrightarrow \mathbb{S}^{\mathrm{hC}_2} \longrightarrow \mathbb{S}$$

with the right hand map the forgetful one. This map is split, since the C_2-action on $\mathbb{S}$ is trivial, and so we find that the negative homotopy groups of the left and middle term agree. But in the exact sequence

$$\pi_{-1}\mathbb{S}_{\mathrm{hC}_2} \longrightarrow \pi_{-1}\mathbb{S}^{\mathrm{hC}_2} \longrightarrow \pi_{-1}\mathbb{S}^{\mathrm{tC}_2}$$

both outer terms vanish (by connectivity on the left, and Lin's theorem on the right). The claim follows.

We are left to calculate the ring structures on $\mathrm{GW}_0^{\mathrm{u}}(\mathbb{S})$ and $\mathrm{L}_0^{\mathrm{u}}(\mathbb{S})$. We start with the latter. By Example [I].5.4.11, the map

$$\mathrm{L}_0^{\mathrm{q}}(\mathbb{S}) \longrightarrow \mathrm{L}_0^{\mathrm{u}}(\mathbb{S})$$

is an $\mathrm{L}_0^{\mathrm{u}}(\mathbb{S})$-module map, so $[E_8]^2 = n[E_8]$ for some $n \in \mathbb{Z}$. Mapping to the integers shows that $n = 8$, giving the claim. For the ring structure of $\mathrm{GW}_0^{\mathrm{u}}(\mathbb{S})$, we similarly observe that the exact sequence

$$\mathrm{K}_0(\mathbb{S}) \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0^{\mathrm{u}}(\mathbb{S}) \longrightarrow \mathrm{L}_0^{\mathrm{u}}(\mathbb{S}),$$

consists of $\mathrm{GW}_0^{\mathrm{u}}(\mathbb{S})$-modules by Corollary [I].7.5.13. This immediately gives $eh = 8h$ and $h^2 = 2h$, and also that $e^2 = 8e + kh$ for some $k \in \mathbb{Z}$. But then we find

$$64h = 8he = he^2 = h(8e + kh) = 64h + 2kh$$

which forces $k = 0$. □

4.6.5. **Remark.** Similar to the sequence used in the previous proof, Weiss and Williams produce a fibre sequence

$$\mathrm{L}^{\mathrm{q}}(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{s}}(\mathbb{S}) \longrightarrow (\mathbb{S}_2^\wedge \otimes \mathbb{S}_2^\wedge) \oplus \mathrm{MTO}(1),$$

in [WW14, Theorem 4.5], which rules out any sort of periodicity for $\mathrm{L}^{\mathrm{s}}(\mathbb{S})$.

Finally, we use Proposition 4.6.4 to determine the automorphisms of the Grothendieck-Witt and L-theory functors. Yoneda's lemma, the universal property of the Grothendieck-Witt spectrum and Proposition [I].4.1.3 provide equivalences

$$\mathrm{Nat}(\mathrm{GW}, \mathrm{GW}) \simeq \mathrm{Nat}(\mathrm{Pn}, \Omega^\infty \mathrm{GW}) \simeq \mathrm{Nat}(\mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{p}}}((\mathcal{S}p^{\mathrm{p}}, Ϙ^{\mathrm{u}}), -), \mathcal{GW}) \simeq \mathcal{GW}^{\mathrm{u}}(\mathbb{S}).$$

Similarly,

$$\mathrm{Nat}(\mathrm{L}, \mathrm{L}) \simeq \mathcal{L}^{\mathrm{u}}(\mathbb{S}),$$

while bordification induces a map

$$\mathrm{Nat}(\mathrm{GW}, \mathrm{GW}) \longrightarrow \mathrm{Nat}(\mathrm{L}, \mathrm{L}),$$

which identifies with

$$\mathcal{GW}^{\mathrm{u}}(\mathbb{S}) \xrightarrow{\mathrm{bord}} \mathcal{L}^{\mathrm{u}}(\mathbb{S}),$$

giving in particular E_1-structures to these spaces.

4.6.6. **Corollary.** *These identifications provide isomorphisms*

$$\pi_0 \operatorname{Nat}(\mathrm{GW}, \mathrm{GW}) \cong \mathbb{Z}[e,h]/I \quad \textit{and} \quad \pi_0 \operatorname{Nat}(\mathrm{L}, \mathrm{L}) \cong \mathbb{Z}[e]/(e^2 - 8e)$$

with $I = (e^2 - 8e, he - 8h, h^2 - 2h)$ *as before.*

In particular, we have

$$\pi_0 \operatorname{Aut}(\mathrm{GW}) = \{\pm \mathrm{id}_{\mathrm{GW}}, \pm(\mathrm{id}_{\mathrm{GW}} - \mathrm{hyp})\} \cong (\mathrm{C}_2)^2 \quad \textit{and} \quad \pi_0 \operatorname{Aut}(\mathrm{L}) = \{\pm \mathrm{id}_{\mathrm{L}}\} \cong \mathrm{C}_2.$$

Proof. It only remains to show that the identifications

$$\mathcal{GW}^{\mathrm{u}}(\mathbb{S}) \simeq \operatorname{Nat}(\mathrm{GW}, \mathrm{GW})$$

are compatible with the multiplicative structures present on their 0-th homotopy groups, and similarly in L-theory. This will immediately follow from our work in Paper [IV], where we show that both GW and L carry lax symmetric monoidal structures. But we can also argue more directly:

The spaces $\operatorname{Nat}(\mathrm{GW}, \mathrm{GW})$ and $\operatorname{Nat}(\mathrm{L}, \mathrm{L})$ receive compatible E_1-maps from $\operatorname{Nat}(\mathrm{Pn}, \mathrm{Pn})$ and from Yoneda's lemma we find

$$\operatorname{Nat}(\mathrm{Pn}, \mathrm{Pn}) \simeq \operatorname{Hom}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{S}p^\omega, \Omega^{\mathrm{u}}), (\mathcal{S}p^{\mathrm{p}}, \Omega^{\mathrm{u}})) \simeq \mathrm{Pn}(\mathcal{S}p^\omega, \Omega^{\mathrm{u}}).$$

Since $(\mathcal{S}p^\omega, \Omega^{\mathrm{u}})$ is the unit of the symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{p}}_\infty$, the functor $\mathrm{Pn} \simeq \operatorname{Hom}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{S}p^\omega, \Omega^{\mathrm{u}}), -)$ inherits a lax symmetric monoidal structure. The left hand equivalence is then a map of E_1-spaces using the composition, and the right hand map refines to one of E_∞-space for the multiplication induced by the tensor product of Poincaré ∞-categories. But on the middle term this E_∞-structure restricts to the composition product by naturality. In total then, we obtain an E_1-refinement of the canonical map

$$\mathrm{Pn}(\mathcal{S}p^\omega, \Omega^{\mathrm{u}}) \longrightarrow \mathcal{GW}(\mathcal{S}p^\omega, \Omega^{\mathrm{u}}) \simeq \operatorname{Nat}(\mathrm{GW}, \mathrm{GW}).$$

Since the map $\pi_0 \mathrm{Pn}(\mathcal{S}p^\omega, \Omega^{\mathrm{u}}) \to \pi_0\, \mathcal{GW}(\mathcal{S}p^\omega, \Omega^{\mathrm{u}}) = \mathrm{GW}^{\mathrm{u}}_0(\mathbb{S})$ is surjective, this shows that the isomorphism

$$\mathrm{GW}^{\mathrm{u}}_0(\mathbb{S}) \simeq \pi_0 \operatorname{Nat}(\mathrm{GW}, \mathrm{GW})$$

is multiplicative and similarly in L-theory. The claims then follow from Proposition 4.6.4 and a quick calculation of the units in the displayed rings. □

A. VERDIER SEQUENCES, KAROUBI SEQUENCES AND STABLE RECOLLEMENTS

In this appendix, we investigate in detail the ∞-categorical variants of the notion of Verdier sequences, i.e. fibre-cofibre sequences in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ and the same notion up to idempotent completion, called Karoubi sequences. The results are mostly well-known and various parts can be found in the literature, but we do not know of a coherent account at the level of detail we need. In the hope that it can serve as a general reference for this material, we have kept this appendix self-contained.

To this end, let us briefly recall that Cat_∞ denotes the ∞-category of (small) ∞-categories, and $\mathrm{Cat}^{\mathrm{ex}}_\infty$ its (non-full) subcategory spanned by stable ∞-categories and exact functors.

Remark. For the reader familiar with [BGT13], here is a comparison of terminology: A Karoubi sequence is called an *exact sequence* in [BGT13], while our notion of a Verdier sequence corresponds to that of a *strict-exact* sequence in [BGT13]; this follows from Proposition A.1.6, Proposition A.1.9 and Proposition A.3.7. Our notion of a split Verdier sequence is however stricter than the corresponding notion of *split-exact* sequence in [BGT13], since we require the projection to have both adjoints (in which case these adjoints are automatically fully faithful, and the injection has both adjoints as well, see Proposition A.2.11), while in the corresponding notion in [BGT13] only the right adjoints are assumed to exist.

A.1. **Verdier sequences.**

We start out by analysing in detail the notion of a Verdier sequence. We recall the definition:

A.1.1. **Definition.** Let

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \tag{70}$$

be a sequence in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ with vanishing composite. We will say that (70) is a *Verdier sequence* if it is both a fibre and a cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_\infty$. In this case we will refer to f as a *Verdier inclusion* and to p as a *Verdier projection*.

A.1.2. **Remark.** The condition that the composite of the sequence (70) vanishes simply means that it sends every object of $\mathcal{C}$ to a zero object in $\mathcal{E}$. Equivalently, the exact functor $p \circ f : \mathcal{C} \to \mathcal{E}$ is a zero object in the stable ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$. Since the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ spanned by zero objects is contractible we may identify $p \circ f$ in this case with a composite functor of the form $\mathcal{C} \to \{0\} \subseteq \mathcal{E}$ in an essentially unique manner. Thus, (70) refines to a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\ \{0\} \arrow[r] & \mathcal{E} \end{tikzcd}$$

in an essentially unique manner, and the condition of being a fibre or cofibre sequence refers to this diagram being cartesian or cocartesian, respectively.

Let us recall how to compute fibres and cofibres in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$: The fibre of an exact functor $f : \mathcal{C} \to \mathcal{D}$ is computed in Cat_{∞} and given by the kernel $\ker(f)$, which is the full subcategory of $\mathcal{C}$ on all objects mapping to a zero object in $\mathcal{D}$. Cofibres, in turn, are described by Verdier quotients:

A.1.3. **Definition.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. We say that a map in $\mathcal{D}$ is an *equivalence modulo* $\mathcal{C}$ if its fibre (equivalently, its cofibre) lies in the smallest stable subcategory spanned by the essential image of f. We write $\mathcal{D}/\mathcal{C}$ for the localisation of $\mathcal{D}$ with respect to the collection W of equivalences modulo $\mathcal{C}$ and refer to $\mathcal{D}/\mathcal{C}$ as the *Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$.

A.1.4. **Remark.** Let us stress that we differ in our use of the term *localisation* from Lurie's: For us, the localisation of an ∞-category $\mathcal{D}$ at a set W of morphisms is the essentially unique functor $\mathcal{D} \to \mathcal{D}[W^{-1}]$ such that for any ∞-category $\mathcal{D}'$, the pull-back functor

$$\mathrm{Fun}(\mathcal{D}[W^{-1}], \mathcal{D}') \to \mathrm{Fun}(\mathcal{D}, \mathcal{D}')$$

is fully faithful with essential image the functors sending the morphisms from W to equivalences. We refer to localisations which are left or right adjoints as left and right *Bousfield localisations*, respectively; see Lemmata A.2.1 and A.2.2 below for the precise relation between the two notions.

The following result is proven in [NS18, Theorem I.3.3(i)] (at least in the case where f is fully faithful, but the general case follows at once).

A.1.5. **Proposition.** *Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. Then:*

i) The ∞-category $\mathcal{D}/\mathcal{C}$ is stable and the localisation functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.

ii) For every stable ∞-category $\mathcal{E}$, the restriction functor $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$ is fully faithful, and its essential image is spanned by those functors which vanish after precomposition with f. In particular, the sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$.

A.1.6. **Proposition.** *Let $p : \mathcal{D} \to \mathcal{E}$ be an exact functor between stable ∞-categories. Then the following are equivalent:*

i) p is a Verdier projection.

ii) p is the canonical map into a Verdier quotient of $\mathcal{D}$.

iii) p is a localisation (at the maps it takes to equivalences).

Proof. If p is a Verdier projection, then it is a cofibre in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$. So i)⇒ii) follows from Proposition A.1.5; and ii)⇒iii) holds by definition of Verdier quotient. Finally, assume that iii) holds. Since p is exact, a morphism in $\mathcal{D}$ maps to an equivalence in $\mathcal{E}$ if and only if its cofibre lies in the kernel of p. Therefore p is indeed the localisation at the class of equivalences modulo $\ker(p)$, and therefore the cofibre of the inclusion $\ker(p) \to \mathcal{D}$. Thus, the sequence $\ker(p) \to \mathcal{C} \to \mathcal{D}$ is both a fibre sequence and a cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$, so that i) holds. ☐

A.1.7. **Corollary.** *Every Verdier projection is essentially surjective.*

We now examine the notion of a Verdier inclusion. For this, we need the following result:

A.1.8. **Lemma.** *The kernel of the canonical map $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ consists of all objects of $\mathcal{D}$ which are retracts of objects in $\mathcal{C}$.*

Proof. Clearly, any retract of an object in $\mathcal{C}$ lies in the kernel of p. For the converse inclusion, let x be an object of $\ker(p)$. We note that by Proposition A.1.5, every exact functor $\mathcal{D} \to \mathcal{S}p$ that vanishes on $\mathcal{C}$ also vanishes on $\ker(p)$. In particular, we may consider the exact functor $\varphi_x \colon \mathcal{D} \to \mathcal{S}p$ given by the formula

$$\varphi_x(y) = \operatorname*{colim}_{[\beta\colon z \to y] \in \mathcal{C}_{/y}} \hom_{\mathcal{D}}(x, \operatorname{cof}(\beta))$$

where $\mathcal{C}_{/y} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}$ is the associated comma ∞-category. Then φ_x vanishes on $\mathcal{C}$: indeed, for $y \in \mathcal{C}$, we have that $\mathcal{C}_{/y}$ has a final object given by the identity $\mathrm{id}\colon y \to y$, and $\hom_{\mathcal{D}}(x, \operatorname{cof}(\mathrm{id})) = 0$, which means that $\varphi_x(y) = 0$.

By the above, we then get that φ_x vanishes on $\ker(p)$. In particular, φ_x vanishes on $x \in \ker(p)$ itself, which implies the existence of a map $\beta\colon z \to x$ for some $z \in \mathcal{C}$ such that $\mathrm{id}\colon x \to x$ is in the kernel of the composite map $\pi_0 \hom_{\mathcal{D}}(x, \operatorname{cof}(0 \to x)) \to \pi_0 \hom_{\mathcal{D}}(x, \operatorname{cof}(\beta))$. We may then conclude that $\mathrm{id}\colon x \to x$ factors through z and hence x is retract of z, as desired. □

A.1.9. **Proposition.** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. Then the following are equivalent:*

i) f is a Verdier inclusion.
ii) f is fully faithful and its essential image is closed under retracts in $\mathcal{D}$.

Proof. If f is a Verdier inclusion, then it is a kernel so that ii) holds. On the other hand, if ii) holds, then f extends to a cofibre sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$, and by Lemma A.1.8 this is also a fibre sequence. □

Summarising our discussion, we obtain:

A.1.10. **Corollary.** *For a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ with vanishing composite, the following are equivalent:*

i) The sequence is a Verdier sequence.
ii) f is fully faithful with essential image closed under retracts in $\mathcal{D}$, and p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.
iii) p is a localisation, and f exhibits $\mathcal{C}$ as the kernel of p.

Finally, we record:

A.1.11. **Lemma.** *Any pullback of a Verdier projection is again a Verdier projection.*

Proof. Consider a cartesian diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "k"] \arrow[d, "p"] & \mathcal{D}' \arrow[d, "p'"] \\
\mathcal{E} \arrow[r, "l"] & \mathcal{E}'
\end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ with p' a Verdier projection and $\mathcal{C}$ the common vertical fibre. We claim that the canonical map $\overline{p}\colon \mathcal{D}/\mathcal{C} \to \mathcal{E}$ is an equivalence, which gives the statement. Since p' is essentially surjective by Corollary A.1.7, so is $\overline{p}$ by inspection, so we are left to check full faithfulness of $\overline{p}$. Using [NS18, Theorem I.3.3 (ii)] twice, we find

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(d, d') &\simeq \operatorname*{colim}_{c \in \mathcal{C}_{/d'}} \mathrm{Hom}_{\mathcal{D}}(d, \operatorname{cof}(c \to d)) \\
&\simeq \operatorname*{colim}_{c \in \mathcal{C}_{/d'}} \mathrm{Hom}_{\mathcal{D}'}(k(d), k(\operatorname{cof}(c \to d'))) \times_{\mathrm{Hom}_{\mathcal{E}'}(lp(d), lp(\operatorname{cof}(c \to d')))} \mathrm{Hom}_{\mathcal{E}}(p(d), p(\operatorname{cof}(c \to d'))) \\
&\simeq \operatorname*{colim}_{c \in \mathcal{C}_{/d'}} \mathrm{Hom}_{\mathcal{D}'}(k(d), \operatorname{cof}(c \to k(d'))) \times_{\mathrm{Hom}_{\mathcal{E}'}(lp(d), lp(d'))} \mathrm{Hom}_{\mathcal{E}}(p(d), p(d')) \\
&\simeq \operatorname*{colim}_{c \in \mathcal{C}_{/k(d')}} \mathrm{Hom}_{\mathcal{D}'}(k(d), \operatorname{cof}(c \to k(d'))) \times_{\mathrm{Hom}_{\mathcal{D}'/\mathcal{C}}(k(d), k(d'))} \mathrm{Hom}_{\mathcal{E}}(p(d), p(d')) \\
&\simeq \mathrm{Hom}_{\mathcal{E}}(p(d), p(d')),
\end{aligned}$$

where we have invested $\mathcal{C}_{/d'} \simeq \mathcal{C}_{/k(d')}$ into the fourth step; this equivalence is immediate by regarding $\mathcal{C}_{/d'}$ as the pullback of $\mathcal{C} \times \{d'\} \to \mathcal{D} \times \mathcal{D} \leftarrow \mathrm{Ar}(\mathcal{D})$, and then commuting the pullback defining $\mathcal{D}$ out. □

Finally, let us mention that we proved various preservation properties for Poincaré-Verdier sequences as part of our analysis in Section 1.4, collected in Propositions 1.4.10 to 1.4.15. Upon dropping hermitian structures, the proofs show analogous results for Verdier sequences.

A.2. **Split Verdier sequences, Bousfield localisations and stable recollements.** We now discuss the existence of adjoints to the inclusion and projection in a Verdier sequence. It leads to the central theme of this section, the notion of *split Verdier sequence* (Definition A.2.4), and its relationship with stable recollements (Definition A.2.10 and Proposition A.2.11).

To obtain criteria similar to Propositions A.1.6 and A.1.9 for exact functors fitting into split Verdier sequences, we first recall the relationship between two notions of localisation: the universal one we have used so far and the notion of Bousfield localisation, compare Remark A.1.4.

A.2.1. **Lemma.** *Let $\mathcal{C}$ be a small ∞-category and W a collection of morphisms in $\mathcal{C}$. Then the localisation $p\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ admits a left or right adjoint, if and only if for every $X \in \mathcal{C}$ there exists a $Y \in \mathcal{C}$ and an equivalence $pX \to pY$ in $\mathcal{C}[W^{-1}]$, such that the functors*

$$\mathrm{Hom}_{\mathcal{C}}(Y, -) \quad \textit{or} \quad \mathrm{Hom}_{\mathcal{C}}(-, Y),$$

send all morphisms in W to equivalences in $\mathcal{S}$, respectively.

In either case, the Yoneda lemma assembles such choices of objects Y for all $X \in \mathcal{C}$ into the requisite adjoint to the localisation functor, which is automatically fully faithful, and therefore renders p into a right or left Bousfield localisation, respectively.

A.2.2. **Lemma.** *If a functor $p\colon \mathcal{C} \to \mathcal{D}$ admits a fully faithful left adjoint L, i.e. p is a right Bousfield localisation, then it is a localisation at those maps $X \to Y$ in $\mathcal{C}$, for which the induced map*

$$\mathrm{Hom}_{\mathcal{C}}(L-, X) \to \mathrm{Hom}_{\mathcal{C}}(L-, Y)$$

is natural equivalence of functors $\mathcal{D} \to \mathcal{S}$.

The same of course holds mutatis mutandis for left Bousfield localisations.

Proof of Lemma A.2.1. We prove the left adjoint variant. Since $p\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ is essentially surjective, Yoneda's lemma implies that p admits a left adjoint if and only if, for each $X \in \mathcal{C}$ the functor

$$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pX, p-)\colon \mathcal{C} \to \mathcal{S}$$

is representable. We claim that a representing object is precisely an object $Y \in \mathcal{C}$ as in the statement.

To see this, let us note generally, that for any $Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ inverts the morphisms in W, p provides a natural equivalence

$$\mathrm{Hom}_{\mathcal{C}}(Y, -) \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pY, p-).$$

To see this, descend $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ to a functor $F_Y\colon \mathcal{C}[W^{-1}] \to \mathcal{S}$ and compute

$$\begin{aligned} \mathrm{Nat}(F_Y, G) &\simeq \mathrm{Nat}(F_Y p, Gp) \\ &\simeq \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(Y, -), Gp) \\ &\simeq G(pY) \\ &\simeq \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pY, -), G) \end{aligned}$$

for an arbitrary $G\colon \mathcal{C}[W^{-1}] \to \mathcal{S}$; the first equivalence arising from the definition of localisations. But then Yoneda's lemma implies that $F_Y \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pY, -)$ and precomposing with p gives the claim. Therefore a $Y \in \mathcal{C}$ as in the statement represents the functor $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pX, p-)$.

If, on the other hand, p admits a left adjoint L, and $X \in \mathcal{C}$, then one can take LpX for Y: By adjunction

$$\mathrm{Hom}_{\mathcal{C}}(LpX, -) \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pX, p-)$$

inverts the morphisms in W, and by the previous consideration, we then find

$$\mathrm{Hom}_{\mathcal{C}}(LpX, -) \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(pLpX, p-),$$

which gives $pLpX \simeq pX$ via the adjunction unit, since p is essentially surjective.

The adjunction unit being an equivalence also implies that L is automatically fully faithful. □

Proof of Lemma A.2.2. The proof that Bousfield localisations are indeed localisations in our sense is [Lur09a, Proposition 5.2.7.12], and the characterisation of the morphisms that are inverted is immediate from Yoneda's lemma. □

Let us apply this to give a criterion to recognise Verdier projections with a one-sided adjoint. In what follows, given a stable ∞-category $\mathcal{D}$ and a full subcategory $\mathcal{C} \subseteq \mathcal{D}$, let us say that an object $y \in \mathcal{D}$ is *right orthogonal* to $\mathcal{C}$ if $\hom_{\mathcal{D}}(x, y) \simeq 0$ for every $x \in \mathcal{C}$ and that y is *left orthogonal* to $\mathcal{C}$ if $\hom_{\mathcal{D}}(y, x) \simeq 0$ for every $x \in \mathcal{C}$.

Let us write $\mathcal{C}^r$ and $\mathcal{C}^l$ for the subcategories spanned by these objects, respectively.

A.2.3. **Lemma.** *Let $p\colon \mathcal{D} \to \mathcal{E}$ be an exact functor of stable ∞-categories. Then the following are equivalent:*

- *i) p is a Verdier projection and admits a right (or left) adjoint.*
- *ii) p is a localisation, and $\ker(p)^r$ (or $\ker(p)^l$) projects essentially surjectively to $\mathcal{E}$ via p.*
- *iii) p is a localisation, and its restriction to $\ker(p)^r$ (or $\ker(p)^l$) is an equivalence.*
- *iv) p admits a fully-faithful right (left) adjoint, i.e. is a left (or right) Bousfield localisation.*

In this situation, $\ker(p)^r$ (or $\ker(p)^l$) agrees with the essential image of the right (or left) adjoint of p.

Proof of Lemma A.2.3. Let us treat the non-parenthesised variants. Recalling from Proposition A.1.6 that Verdier projections are localisations, the implications between i) and iv) are proven in Lemmas A.2.1 and A.2.2.

Now suppose that p admits a fully faithful right adjoint R, then p and R determine mutually inverse equivalences between $\mathcal{E}$ and the essential image of R, and it follows from Lemma A.2.1 that this essential image of R agrees with $\mathcal{C}$. Together with Lemma A.2.2, this proves the implication iv)⇒ iii) and the last claim. The implication iii)⇒ ii) is trivial. Finally, if ii) holds, then preimages under $p\colon \ker(p)^r \to \mathcal{E}$ yield exactly the desired objects to obtain a right adjoint via Lemma A.2.1. □

A.2.4. **Definition.** A Verdier sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

is *split* if p admits both a left and a right adjoint.

In this definition, we might just as well require that f admit both adjoints, by the following result:

A.2.5. **Lemma.** *Let*

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \tag{71}$$

be a sequence in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ with vanishing composite. Then the following are equivalent:

- *i) (71) is a fibre sequence, and p admits a fully-faithful left (right) adjoint q.*
- *ii) (71) is a cofibre sequence, and f is fully-faithful and admits a left (right) adjoint g.*

Furthermore, if i) and ii) hold, then both the original sequence and that formed by the left (right) adjoints,

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \quad \textit{and} \quad \mathcal{E} \xrightarrow{q} \mathcal{D} \xrightarrow{g} \mathcal{C},$$

are Verdier sequences.

Explicitly, in the case of left adjoints, g is described as the cofibre of the counit $qp \to \mathrm{id}_{\mathcal{D}}$, thought of as a functor $\mathcal{D} \to \mathcal{D}$ that vanishes after projection to $\mathcal{E}$ and therefore uniquely lifts to $\mathcal{C}$. Similarly, the adjoint q is described as the fibre of the unit $\mathrm{id}_{\mathcal{D}} \to fg$, thought of as a functor $\mathcal{D} \to \mathcal{D}$ that vanishes after restriction to $\mathcal{C}$ and therefore uniquely factors through $\mathcal{E}$.

A.2.6. **Corollary.** *An exact functor $p\colon \mathcal{D} \to \mathcal{E}$ is a split Verdier projection if and only if it admits fully faithful left and right adjoints. An exact functor $f\colon \mathcal{C} \to \mathcal{D}$ is a split Verdier inclusion if and only if it is fully faithful and admits left and right adjoints.*

Proof of Lemma A.2.5. We prove the claim for left adjoints. The claim for right adjoints follows by the dual argument (or by replacing all ∞-categories by their opposites).

Suppose first that i) holds. Then we obtain a left adjoint g of f by considering the exact functor

$$\tilde{g} = \mathrm{cof}[qp \to \mathrm{id}]\colon \mathcal{D} \to \mathcal{D}$$

given by the cofibre of the counit. Since q is fully-faithful, the unit map $\mathrm{id} \to qp$ is an equivalence, from which we can conclude that $p \circ \tilde{g}$ vanishes. Thus, $\tilde{g}$ factors uniquely through f, giving rise to a functor $g\colon \mathcal{D} \to \mathcal{C}$. We now claim that the canonical transformation $\mathrm{id} \to \tilde{g} = f \circ g$ acts as a unit exhibiting g as left adjoint to f. Given objects $x \in \mathcal{D}$ and $y \in \mathcal{C}$, it will suffice to check that the composite map

$$\hom_{\mathcal{C}}(g(x), y) \to \hom_{\mathcal{D}}(fg(x), f(y)) \to \hom_{\mathcal{D}}(x, f(y))$$

is an equivalence of spectra. Indeed, the first map is an equivalence since f is fully-faithful and the second map is an equivalence because its cofibre is $\hom_{\mathcal{D}}(qp(x), f(y)) \simeq \hom_{\mathcal{E}}(p(x), pf(y)) \simeq 0$.

In this situation, p is a localisation by Lemma A.2.2, so the sequence formed by f and p is a Verdier sequence by Corollary A.1.10, in particular a cofibre sequence. Also, the kernel of g consists, by the adjunction rule, of those objects that are left orthogonal to $\mathcal{C}$, and by Lemma A.2.3, this agrees with the essential image of q. So, the sequence formed by the adjoints satisfies i) (in the version with right adjoints), and is therefore also a Verdier sequence by what we have just shown.

On the other hand, suppose that ii) holds. Then g is a localisation by Lemma A.2.1 and thus the essential image of f is given by the right orthogonal of $\ker(g)$. It is therefore, in particular, closed under retracts in $\mathcal{D}$. But according to Proposition A.1.6, p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by this image so it equals $\ker(p)$ by Lemma A.1.8. This shows that (71) is a fibre sequence. To see that p admits a left adjoint, we can appeal to Lemma A.2.1: For $x \in \mathcal{D}$ the fibre of the unit map $x \to fg(x)$ clearly projects to $p(x)$ under p, and for $c \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{D}}\left(\mathrm{fib}(x \to fg(x)), f(c)\right) \simeq \mathrm{cof}\left[\mathrm{Hom}_{\mathcal{D}}(x, f(c)) \to \mathrm{Hom}_{\mathcal{D}}(fg(x), f(c))\right]$$

and since f is fully faithful the latter term is also given by $\mathrm{Hom}_{\mathcal{C}}(g(x), c)$, which identifies the map on the right as the adjunction equivalence. □

As a straight-forward consequence of Corollary A.2.6, we record:

A.2.7. **Corollary.** *A pullback of a split Verdier projection is again a split Verdier projection.*

Proof. Using the universal property of the pullback, one readily constructs the requisite functors from the original adjoints (using the fact that these are fully faithful, and therefore sections of the original Verdier projection). That these are again fully faithful adjoints follows immediately from the description of mapping spaces in pullbacks of ∞-categories as pullbacks of mapping spaces. □

We call Verdier sequences as in Lemma A.2.5 *left-split* and *right-split*, respectively, and take this opportunity to frame the following corollary, which shows that they can be recognised in several ways (we make use of this in Section 3.2). For the statement, recall that we denote by $\mathcal{C}^r$ or $\mathcal{C}^l$ the left or right orthogonal to a full subcategory $\mathcal{C} \subseteq \mathcal{D}$, respectively.

A.2.8. **Corollary.** *Let $\mathcal{D}$ be a stable ∞-category and $\mathcal{C}, \mathcal{E} \subseteq \mathcal{D}$ two full stable subcategories such that $\hom_{\mathcal{D}}(x, y) \simeq 0$ for every $x \in \mathcal{C}, y \in \mathcal{E}$. Then the following are equivalent:*

i) $\mathcal{C} \subseteq \mathcal{D}$ admits a right adjoint $p\colon \mathcal{D} \to \mathcal{C}$ and the inclusion $\mathcal{E} \subseteq \mathcal{C}^r$ is an equivalence.
ii) $\mathcal{E} \subseteq \mathcal{D}$ is a Verdier inclusion and the projection $\mathcal{C} \to \mathcal{D}/\mathcal{E}$ is an equivalence.
iii) $\mathcal{E} \subseteq \mathcal{D}$ admits a left adjoint $q\colon \mathcal{D} \to \mathcal{E}$ and the inclusion $\mathcal{C} \subseteq \mathcal{E}^l$ is an equivalence.
iv) $\mathcal{C} \subseteq \mathcal{D}$ is a Verdier inclusion and the projection $\mathcal{E} \to \mathcal{D}/\mathcal{C}$ is an equivalence.

Furthermore, when either of these equivalent conditions holds, the resulting sequences

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E} \quad \textit{and} \quad \mathcal{E} \to \mathcal{D} \to \mathcal{C}$$

formed by the inclusions and their adjoints are right-split and left-split Verdier sequences, respectively.

Proof. The implications i) ⇒ ii) and iii) ⇒ iv) are dual to each other, and the same for the implications ii) ⇒ iii) and iv) ⇒ i). It will hence suffice to show i) ⇒ ii) ⇒ iii), along with the last claim.

To prove the first of these implications, suppose that $i\colon \mathcal{C} \subseteq \mathcal{D}$ admits a right adjoint $p\colon \mathcal{D} \to \mathcal{C}$ and that $\mathcal{E} \subseteq \mathcal{C}^r$ is an equivalence. By the adjunction rule, $\mathcal{C}^r$ agrees with the kernel of p so we have a right-split Verdier sequence

$$\mathcal{E} \to \mathcal{D} \xrightarrow{p} \mathcal{C}$$

from which we conclude that the map $\mathcal{D}/\mathcal{E} \to \mathcal{C}$ induced by p is an equivalence. The projection $\mathcal{C} \to \mathcal{D}/\mathcal{E}$ is a one-sided inverse and therefore also an equivalence.

On the other hand, if ii) holds then by Lemma A.2.3, the projection $\mathcal{D} \to \mathcal{D}/\mathcal{E}$ has a left adjoint, and the inclusion of $\mathcal{C}$ into $\mathcal{E}^l$ is an equivalence (since both project to $\mathcal{D}/\mathcal{E}$ by an equivalence); the existence of the left adjoint q follows from Lemma A.2.5.

□

A.2.9. **Remark.** There is another characterisation of split Verdier projections, that we want to mention: An exact functor $\mathcal{C} \to \mathcal{D}$ is a right split Verdier projection if and only if it is a cartesian fibration, and similarly for left split Verdier projections and cocartesian fibrations.

One implication we proved in Lemma 2.6.1 above, and the other is recorded in [HHLN22, Example 5.6].

We now come back to the notion of a split Verdier sequence and show that it is essentially equivalent to that of a recollement in the sense of [Lur17, Section A.8] in the setting of stable ∞-categories. Specialising the definition to this case, we have:

A.2.10. **Definition.** A stable ∞-category $\mathcal{D}$ is a *stable recollement* of a pair of stable subcategories $\mathcal{C}$ and $\mathcal{E}$ if

i) the inclusions of both $\mathcal{C}$ and $\mathcal{E}$ admit left adjoints $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$,
ii) the composite $\mathcal{C} \to \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$ vanishes, and
iii) $L_{\mathcal{E}}$ and $L_{\mathcal{C}}$ are jointly conservative.

A.2.11. **Proposition.** *If $\mathcal{D}$ is a stable recollement of $\mathcal{C}$ and $\mathcal{E}$, then the sequence $\mathcal{C} \to \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$ is a split Verdier sequence.*

Conversely, if $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a split Verdier sequence, then $\mathcal{D}$ is a stable recollement of the essential images $f(\mathcal{C})$ and $q(\mathcal{E})$, where q denotes the right adjoint of p.

Proof. Consider the first statement. We claim that the sequence under consideration is a fibre sequence, so that it is split Verdier by Lemma A.2.5. Since the composite is zero by assumption, we are left to show that every object x in $\ker(L_{\mathcal{E}})$ already belongs to the essential image of $\mathcal{C}$. Denoting by $L_{\mathcal{C}}$ the left adjoint of the inclusion of $\mathcal{C}$, then the unit $x \to L_{\mathcal{C}}(x)$ is mapped to an equivalence under both $L_{\mathcal{E}}$ and $L_{\mathcal{C}}$. By assumption $L_{\mathcal{E}}$ and $L_{\mathcal{C}}$ are jointly conservative, so the unit $x \to L_{\mathcal{C}}(x)$ is an equivalence and therefore x indeed lies in the essential image of $\mathcal{C}$.

For the second statement, f admits a left adjoint g by Lemma A.2.5 , since p does and it remains to see that p and g are jointly conservative. Since we are in the stable setting it will suffice to show that the functors p and g together detect zero objects. Indeed, if $x \in \mathcal{D}$ is such that $p(x) \simeq 0$ then x belongs to the essential image of f. In this case, if $g(x)$ is zero as well, then $x \simeq 0$ because the counit of $g \dashv f$ is an equivalence. □

In pictures, a stable recollement is given by

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{D} \arrow[l, bend right=50, "L_{\mathcal{C}}"'] \arrow[r, "L_{\mathcal{E}}"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E}. \arrow[l, bend left=50]
\end{tikzcd}$$

We defined a split Verdier sequence as on the left of the following diagram and in [BG16] Barwick and Glasman considered diagrams as on the right:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E} \arrow[l, bend right=50] \arrow[l, bend left=50]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{D} \arrow[l, bend right=50] \arrow[l, bend left=50] \arrow[r, "p"] & \mathcal{E}.
\end{tikzcd}$$

Here, the non-curved maps form a Verdier sequence and left adjoints are on top. Our results above show that all of these types of diagrams can be completed to the full

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{D} \arrow[l, bend right=50, "g"'] \arrow[l, bend left=50, "g'"] \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E}, \arrow[l, bend right=50, "q"'] \arrow[l, bend left=50, "q'"]
\end{tikzcd}$$

in which both the top and the bottom left pointing maps also form Verdier sequences, and whose maps are related by the bifibre sequences

$$\begin{tikzcd}
 & qp \arrow[d, Rightarrow] & \\
fg' \arrow[r, Rightarrow] & \mathrm{id}_{\mathcal{D}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & q'p. \\
 & fg &
\end{tikzcd}$$

From this data, one obtains a canonical transformation $g' \Rightarrow g$, whose (co)fibre descends to a functor $\mathcal{E} \to \mathcal{C}$, and another transformation $q \Rightarrow q'$, whose (co)fibre also lifts to a functor $\mathcal{E} \to \mathcal{C}$. We then have

$$gq' \simeq \mathrm{cof}(q \Rightarrow q') \simeq \mathrm{cof}(g' \Rightarrow g) \simeq \Sigma_{\mathcal{C}} g'q,$$

where the middle equivalence comes from the cofibre sequence describing the cofibre of a composition in terms of the cofibres of the constituents. The functor $c\colon \mathcal{E} \to \mathcal{C}$ specified by any of the formulae above is said to classify the recollement, as it participates in the following result:

A.2.12. **Proposition.** *Given a split Verdier sequence in the notation above, the diagram*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "g \to cp"] \arrow[d, "p"] & \mathrm{Ar}(\mathcal{C}) \arrow[d, "\mathrm{t}"] \\
\mathcal{E} \arrow[r, "c"] & \mathcal{C}
\end{tikzcd}$$

is cartesian, where t *is the target projection. Moreover, for any object $x \in \mathcal{D}$, there is a cartesian diagram*

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & fg(x) \arrow[d] \\
q'p(x) \arrow[r] & fgq'p(x)
\end{tikzcd}$$

with all maps induced by the units of the respective adjunctions.

We shall refer to the lower square as the *fracture* or *Tate square* of X with respect to the recollement. Let us also remark that the sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{r}"] & \mathrm{Ar}(\mathcal{C}) \arrow[l, bend right, "\mathrm{s}"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, "\mathrm{t}"] & \mathcal{C} \arrow[l, bend right, "\mathrm{q}"'] \arrow[l, bend left, "\delta"]
\end{tikzcd}$$

is indeed a split Verdier sequence, where $\mathrm{r}(x) = (x \to 0)$, with left and right adjoints being $\mathrm{s}(x \to y) = x$ and $\mathrm{fib}(x \to y)$, respectively, while $\mathrm{t}(x \to y) = y$ with left and right adjoints $\mathrm{q}(y) = (0 \to y)$ and $\delta(y) = \mathrm{id}_y$, respectively. It underlies the metabolic sequence of Example 1.2.5, which plays a fundamental role in our results.

Proof. The inverse functor from the pullback to $\mathcal{D}$ is given by sending a pair $(e, a \to c(e))$ to the pullback $q'(e) \times_{fc(e)} f(a)$, with the left structure map coming from the definition of c. That the composite on the pullback $\mathcal{E} \times_{\mathcal{C}} \mathrm{Ar}(\mathcal{C})$ is equivalent to the identity follows from unwinding the definitions, whereas for the composite on $\mathcal{D}$ it is precisely the cartesianness of the diagram from the statement. But the induced map on its vertical fibres is the unit map of $fg'(x) \to fgfg'(x)$ of the adjunction fg which is an equivalence since f is fully faithful, together with the triangle identity. □

A.2.13. **Remark.** A monoidal refinement of this result was recently given in [QS19, Section 1].

Finally, we characterise the horizontal maps appearing in Proposition A.2.12. To this end, consider a square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, "p"] & \mathcal{D}' \arrow[d, "p'"] \\
\mathcal{E} \arrow[r] & \mathcal{E}'
\end{tikzcd}$$

with vertical split Verdier projections. Such a diagram gives rise to two new lax diagrams of shape

$$\begin{tikzcd}
\mathcal{D} \arrow[r] & \mathcal{D}' \\
\mathcal{E} \arrow[r] \arrow[u] & \mathcal{E}' \arrow[u]
\end{tikzcd}$$

by passing to either left or right adjoints in the vertical direction. The original square is called *adjointable* if both squares of adjoints do in fact commute, i.e. if the Beck-Chevalley transformations connecting the composites are equivalences, see [Lur09a, Section 7.3.1] for details. It is readily checked that cartesian squares as above are adjointable.

A.2.14. **Proposition.** *Given a split Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ and another stable ∞-category $\mathcal{C}'$, the full subcategory of* $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathrm{Ar}(\mathcal{C}'))$ *spanned by the functors φ that give rise to adjointable squares*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\varphi"] \arrow[d] & \mathrm{Ar}(\mathcal{C}') \arrow[d, "t"] \\
\mathcal{E} \arrow[r, "\overline{\varphi}"] & \mathcal{C}'
\end{tikzcd}$$

is equivalent to $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}')$ *via restriction to horizontal fibres.*

In particular, the classifying functor in Proposition A.2.12 is uniquely determined by yielding a cartesian diagram and inducing the identity on fibres, so $t\colon\ \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ really is the universal split Verdier projection with fibre $\mathcal{C}$. Similarly, we find that for a cartesian square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, "p"] & \mathcal{D}' \arrow[d, "p'"] \\
\mathcal{E} \arrow[r] & \mathcal{E}'
\end{tikzcd}$$

with common fibre $\mathcal{C}$ the classifying functor $\mathcal{E} \to \mathcal{C}$ of p is the composite of that for p' and the given map $\mathcal{E} \to \mathcal{E}'$. We make use of the functoriality of the classifying map in adjointable (and not just cartesian) squares arising from Proposition A.2.14 in Lemma 1.5.3.

Proof. Using the fibre sequences connecting the various adjoints, one readily checks that generally adjointability of the two squares

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\varphi"] \arrow[d, "p"] & \mathcal{D}' \arrow[d, "p'"] \\
\mathcal{E} \arrow[r] & \mathcal{E}'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "f"] & \mathcal{C}' \arrow[d, "f'"] \\
\mathcal{D} \arrow[r, "\varphi"] & \mathcal{D}'
\end{tikzcd}$$

are equivalent conditions for two (vertical) Verdier sequences. We will use the latter description in the case at hand to see that the restriction functor in the statement is fully faithful: Rewriting then $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathrm{Ar}(\mathcal{C}')) = \mathrm{Ar}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{C}'))$, we compute for $\varphi, \psi\colon\ \mathcal{D} \to \mathrm{Ar}(\mathcal{C}')$ that

$$\mathrm{nat}(\varphi, \psi) \simeq \mathrm{nat}(s\varphi, s\psi) \times_{\mathrm{nat}(s\varphi, t\psi)} \mathrm{nat}(t\varphi, t\psi).$$

Using the fact that $s, \mathrm{fib}\colon\ \mathrm{Ar}(\mathcal{C}') \to \mathcal{C}$ are the left and right adjoint to the Verdier inclusion $\mathcal{C}' \to \mathrm{Ar}(\mathcal{C}')$, we find $s\varphi \simeq \varphi_{|F} \circ g$ and $t\varphi \simeq \varphi_{|F} \circ cp$ from adjointability of φ and similarly for ψ. Thus, the above can be rewritten as

$$\mathrm{nat}(\varphi_{|F} \circ g, \psi_{|F} \circ g) \times_{\mathrm{nat}(\varphi_{|F} \circ g, \psi_{|F} \circ cp)} \mathrm{nat}(\varphi_{|F} \circ cp, \psi_{|F} \circ cp).$$

But $g\colon\ \mathcal{D} \to \mathcal{C}$ is a localisation (since it has f as a fully faithful right adjoint) so

$$\mathrm{nat}(\varphi_{|F} \circ g, \psi_{|F} \circ g) \simeq \mathrm{nat}(\varphi_{|F}, \psi_{|F})$$

and we claim that the restriction map

$$\mathrm{nat}(\varphi_{|F} \circ cp, \psi_{|F} \circ cp) \longrightarrow \mathrm{nat}(\varphi_{|F} \circ g, \psi_{|F} \circ cp)$$

is an equivalence, which gives full faithfulness. To see this, consider its fibre $\mathrm{nat}(\varphi_{|F} \circ g', \psi_{|F} \circ cp)$ and recall that

$$(g')^*\colon\ \mathrm{Fun}(\mathcal{C}, \mathcal{C}') \underset{\perp}{\rightleftarrows} \mathrm{Fun}(\mathcal{D}, \mathcal{C}')\ : f^*$$

is also an adjunction, so

$$\mathrm{nat}(\varphi_{|F}\circ g', \psi_{|F}\circ cp) \simeq \mathrm{nat}(\varphi_{|F}, \psi_{|F}\circ cpf) \simeq 0$$

as desired.

We are left to show that the restriction functor is essentially surjective, but this is obvious by following the classification arrow from Proposition A.2.12 with the one induced by the given functor $\mathcal{C} \to \mathcal{C}'$ on arrow categories. ☐

A.2.15. **Remark.** Proposition A.2.12 and the entire discussion preceding it apply equally well to stable ∞-categories that are not small, and for example recover the observation of Barwick and Glasman [BG16, Proposition 7] that the left and right orthogonal to the inclusion of $\mathcal{C}$ in a stable recollement are canonically equivalent.

One example we established in Paper [I] is given by

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p) \arrow[r] & \mathrm{Fun}^{\mathrm{q}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p) \arrow[l, bend right, "\Lambda"', "\perp"] \arrow[l, bend left, "\perp"'] \arrow[r] & \mathrm{Fun}^{\mathrm{s}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p), \arrow[l, bend right, "(-)^{\mathrm{q}}"', "\perp"] \arrow[l, bend left, "(-)^{\mathrm{s}}", "\perp"'] \end{tikzcd}$$

whose fracture square gives exactly the classification of quadratic functors in Corollary [I].1.3.12.

Another standard example is the case where $\mathcal{D} = \mathcal{S}p$ and f is the inclusion of those spectra on which a prime l acts invertibly:

$$\begin{tikzcd} \mathcal{S}p[\frac{1}{l}] \arrow[r] & \mathcal{S}p \arrow[l, bend right, "(-)[\frac{1}{l}]"', "\perp"] \arrow[l, bend left, "\mathrm{div}_l", "\perp"'] \arrow[r] & \mathcal{S}p[\text{l-adic equiv's}^{-1}], \arrow[l, bend right, "(-)[l^\infty]"', "\perp"] \arrow[l, bend left, "(-)^\wedge_l", "\perp"'] \end{tikzcd}$$

where $\mathrm{div}_l(X) = \lim_{-\cdot l} X$ is the l-divisible part of X, together with the fibre sequences

$$\mathrm{div}_l(X) \longrightarrow X \longrightarrow X^\wedge_l \quad \text{and} \quad X[l^\infty] \longrightarrow X \longrightarrow X[\tfrac{1}{l}],$$

classifying functor

$$X \mapsto X^\wedge_l[\tfrac{1}{l}] \simeq \Omega\mathrm{div}_l(X[l^\infty])$$

and fracture square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X[\frac{1}{l}] \arrow[d] \\ X^\wedge_l \arrow[r] & X^\wedge_l[\frac{1}{l}]. \end{tikzcd}$$

A.3. **Karoubi sequences.** We now move to the more general notion of Karoubi sequences, which are a version of Verdier sequences invariant under the addition of direct summands in the ∞-categories at hand.

Let us briefly record some basic statements:

A.3.1. **Definition.** We call an exact functor $\mathcal{C} \to \mathcal{D}$ between stable ∞-categories a *Karoubi equivalence* if it is fully faithful and has dense image, in the sense that every object of $\mathcal{D}$ is a retract of an object in the essential image.

The most important example of Karoubi equivalences are of course idempotent completions $\mathcal{C} \to \mathcal{C}^\natural$. When fixing the target, Karoubi equivalences can be entirely classified, see [Tho97, Theorem 2.1]:

A.3.2. **Theorem** (Thomason)**.** *Karoubi equivalences induce injections on* $\mathrm{K}_0 : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{A}b$*, and Karoubi equivalences to a fixed small stable ∞-category $\mathcal{C}$ (up to equivalence over $\mathcal{C}$) are in bijection with subgroups of* $\mathrm{K}_0(\mathcal{C})$ *by taking the image of their induced map.*

Note that the statement in [Tho97] is for triangulated categories, but the proof works verbatim in the setting of stable ∞-categories.

A.3.3. **Proposition.** *The localisation of* $\mathrm{Cat}^{\mathrm{ex}}_\infty$ *at the Karoubi equivalences is both a left and a right Bousfield localisation. The right adjoint is given by* $\mathcal{C} \mapsto \mathcal{C}^\natural$*, and the left adjoint takes* $\mathcal{C}$ *to* $\mathcal{C}^{\min}$*, the full subcategory spanned by the objects* $x \in \mathcal{C}$ *with* $0 = [x] \in \mathrm{K}_0(\mathcal{C})$.

Furthermore, an exact functor is a Karoubi equivalence if and only if it induces an equivalence on minimalisations or equivalently idempotent completions.

Denoting by $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ the full subcategory of $\mathrm{Cat}^{\mathrm{ex}}_\infty$ spanned by the small, idempotent complete stable ∞-categories, we in particular find that $(-)^\natural : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ preserves both limits and colimits.

A.3.4. **Definition.** Small stable ∞-categories $\mathcal{C}$ with the property that $\mathrm{K}_0(\mathcal{C})$ vanishes we will call *minimal* and refer to the assignment $\mathcal{C} \mapsto \mathcal{C}^{\min}$ as *minimalisation*.

Proof of Proposition A.3.3. It is an exercise in pasting retract diagrams to check that Karoubi equivalences are closed under 2-out-of-3. The characterisation in the last statement then follows immediately from the fact that both inclusions $\mathcal{C}^{\min} \subseteq \mathcal{C} \subseteq \mathcal{C}^\natural$ are Karoubi equivalences, the former since every $X \in \mathcal{C}$ is a retract of $X \oplus \Sigma X \in \mathcal{C}^{\min}$. Furthermore, [Lur09a, Lemma 5.1.4.7] then implies that given a Karoubi equivalence $i : \mathcal{C} \to \mathcal{D}$ and a functor $f : \mathcal{D} \to \mathcal{E}$, the exactness of f is equivalent to that of fi.

The statement about the adjoints now follows from Lemma A.2.1: That idempotent completion satisfies the requisite conditions is [Lur09a, Proposition 5.1.4.9] and that minimalisations do is immediate from the functoriality of K_0. □

Let us now define our main object of study in this section.

A.3.5. **Definition.** A sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

of exact functors with vanishing composite is a *Karoubi sequence* if the sequence

$$\mathcal{C}^\natural \to \mathcal{D}^\natural \to \mathcal{E}^\natural$$

is both a fibre and cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$. In this case, we refer to f as a *Karoubi inclusion* and to p as a *Karoubi projection*.

A.3.6. **Remark.** Equivalently, by Proposition A.3.3, we might ask the sequence

$$\mathcal{C}^{\min} \to \mathcal{D}^{\min} \to \mathcal{E}^{\min}$$

to be both a fibre and a cofibre sequence in the full subcategory of $\mathrm{Cat}^{\mathrm{ex}}_\infty$ spanned by the minimal stable ∞-categories, or more symmetrically that the original sequence give a fibre and cofibre sequence in the localisation of $\mathrm{Cat}^{\mathrm{ex}}_\infty$ at the Karoubi equivalences.

We have chosen the present formulation as the idempotent completion plays a disproportionally more important role, both in the detection of Karoubi sequences and in applications.

We also have a concrete characterisation of Karoubi sequences, analogous to the one for Verdier sequences of Corollary A.1.10.

A.3.7. **Proposition.** *Let* $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *be a sequence of exact functors between small stable ∞-categories with vanishing composite. Then*

i) the sequence $\mathcal{C}^\natural \xrightarrow{f^\natural} \mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural$*is a fibre sequence in* $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ *if and only if* f *becomes a Karoubi equivalence when regarded as a functor* $\mathcal{C} \to \ker(p)$.

ii) the sequence $\mathcal{C}^\natural \xrightarrow{f^\natural} \mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural$ *is a cofibre sequence in* $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ *if and only if the induced functor from the Verdier quotient of* $\mathcal{D}$ *by the stable subcategory generated by the image of* f *is a Karoubi equivalence to* $\mathcal{E}$.

iii) the sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *is a Karoubi sequence if and only if* f *is fully-faithful and the induced map* $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ *is a Karoubi equivalence.*

In particular, every Verdier sequence is a Karoubi sequence.

Let us explicitly warn the reader, however, that the Verdier quotient of two idempotent complete, stable ∞-categories need not be idempotent complete.

Proof. By Proposition A.3.3, the functor $(-)^\natural \colon \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ preserves both limits and colimits, and $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ is closed under limits in $\mathrm{Cat}^{\mathrm{ex}}_\infty$. This yields an equivalence

$$\ker(p^\natural) \simeq \ker(p)^\natural,$$

which proves i).

Similarly, ii) follows from the description of cofibres in $\mathrm{Cat}^{\mathrm{ex}}_\infty$ as Verdier quotients together with the preservation of cofibres under idempotent completion.

Finally, the forward direction of iii) follows directly from the previous two statements. On the other hand, if f is fully faithful, and $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is a Karoubi equivalence, then the kernel of p agrees with the kernel of the projection $q \colon \mathcal{D} \to \mathcal{D}/\mathcal{C}$. Thus, by Lemma A.1.8, the map $f \colon \mathcal{C} \to \ker(q)$ has dense essential image and therefore is a Karoubi equivalence. The reverse claim thus also follows from the first two statements. □

A.3.8. **Corollary.** *An exact functor $f \colon \mathcal{C} \to \mathcal{D}$ is a Karoubi inclusion if and only if it is fully-faithful. It is a Karoubi projection if and only if it has dense essential image $f(\mathcal{C}) \subseteq \mathcal{D}$, and the induced functor $f \colon \mathcal{C} \to f(\mathcal{C})$ is Verdier projection.*

Combining this statement with Thomason's result above, we find:

A.3.9. **Corollary.** *Let $p \colon \mathcal{D} \to \mathcal{E}$ be a Karoubi projection. Then the following are equivalent:*

i) p is a Verdier projection.
ii) p is essentially surjective.
iii) The induced group homomorphism $\mathrm{K}_0(\mathcal{D}) \to \mathrm{K}_0(\mathcal{E})$ is surjective.

We also note:

A.3.10. **Lemma.** *Any pullback of a Karoubi projection is again a Karoubi projection.*

Proof. Given Lemma A.1.11 and the characterisation of Karoubi projections in Corollary A.3.8, it suffices to show that the pullback $\mathcal{D} \to \mathcal{D}'$ of a Karoubi equivalence $\mathcal{E} \to \mathcal{E}'$ along $i \colon \mathcal{D}' \to \mathcal{E}'$ is again one such. But one readily checks that this pullback is given by the full subcategory $\{x \in \mathcal{E}' \mid i[x] \in \mathrm{K}_0(\mathcal{E})\}$ of $\mathcal{E}'$, whence Thomason's theorem A.3.2 gives the claim. □

Next, we record the following detection criterion for Karoubi-sequences, often called the Thomason-Neeman localisation theorem in the context of triangulated categories, see [Nee92, Theorem 2.1]. To state it, we need to extend the notion of Verdier sequences to non-small stable ∞-categories. This is achieved for example by Corollary A.1.10, which does not require any smallness assumption.

A.3.11. **Theorem.** *A sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of small stable ∞-categories and exact functors with vanishing composite is a Karoubi sequence if and only if the induced sequence*

$$\mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{D}) \longrightarrow \mathrm{Ind}(\mathcal{E})$$

is a Verdier sequence (of not necessarily small ∞-categories).

Here, Ind denotes the inductive completion of a small ∞-category, characterised for example as the smallest subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ stable under filtered colimits and containing all representable functors.

Proof. First of all, note that inductive completion preserves both stability of ∞-categories and exactness of functors, for example as a consequence of [Lur09a, Proposition 5.3.5.10]: The colimit preserving extension of suspension is suspension, and the extension of loops is its inverse. Furthermore, it preserves full faithfulness by [Lur09a, 5.3.5.11], commutes with Verdier quotients by [NS18, Proposition I.3.5] and by [Lur09a, Lemma 5.4.2.4] the compact objects in $\mathrm{Ind}(\mathcal{C})$ form an idempotent completion of $\mathcal{C}$. Combining these statements, it follows that an exact functor is a Karoubi equivalence if and only if it induces an equivalence on inductive completions: The backwards direction is immediate, and given a Karoubi equivalence $\mathcal{C} \to \mathcal{D}$, we find $\mathrm{Ind}(\mathcal{C})$ the kernel of $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{D}/\mathcal{C}) \simeq 0$ by Lemma A.1.8, since cocomplete ∞-categories are in particular idempotent complete by [Lur09a, Corollary 4.4.5.16].

Reusing the three statements, the claim now follows from our characterisation of Verdier and Karoubi sequences, Corollary A.1.10 and Proposition A.3.7. □

In fact, given a Karoubi sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ the sequence $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ consists of the left adjoints in a stable recollement (note the order reversal)

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{E}) \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Ind}(\mathcal{D}) \arrow[l, bend right, "\mathrm{Ind}(p)"'] \arrow[l, bend left] \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Ind}(\mathcal{C}) \arrow[l, bend right, "\mathrm{Ind}(i)"'] \arrow[l, bend left]
\end{tikzcd}$$

It follows immediately from [NS18, Proposition I.3.5], that $\mathrm{Ind}(p)$ admits a fully faithful right adjoint which preserves colimits. By [Lur09a, Corollary 5.5.2.9], it then follows that this functor has a further right adjoint, whence the results of the previous section give the adjoint to $\mathrm{Ind}(i)$ and its adjoint.

The other functors in this recollement do not, however, in general preserve compact objects, so one cannot pass to these to obtain further Karoubi sequences.

A.3.12. **Remark.** By [NS18, Theorem I.3.3], the adjoint on inductive completions may be explicitly described as taking $p(x)$ to $\mathrm{colim}_{z \in \mathcal{C}_{/x}} \mathrm{cof}(z \to x)$, and dually the right adjoint on projective completions is given by taking x to $\lim_{z \in \mathcal{C}_{x/}} \mathrm{fib}(x \to y)$. As adjoints to localisations, these are fully faithful and via the inclusions $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and $\mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$, they give a concrete way of constructing the Verdier quotient.

A.3.13. **Remark.** Let us also warn the reader of the following well-known asymmetry: Suppose given compactly generated stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ (i.e. cocomplete, stable ∞-categories that admit a set of compact objects which jointly detect equivalences) and a functor $F\colon \mathcal{C} \to \mathcal{D}$ which preserves colimits and compact objects.

If such F is a Verdier inclusion (of non-small ∞-categories) its restriction f to compact objects is automatically a Karoubi inclusion, since full faithfulness is clearly retained. In fact, such an F is automatically of the form $\mathrm{Ind}(f)$ by [Lur09a, Propositions 5.4.2.17 & 5.4.2.19], and another application of [NS18, Proposition I.3.5] exhibits the Verdier quotient of F as the inductive completion of that of f.

In the other direction, however, if F is a Verdier projection (of non-small ∞-categories), it need not follow that its restriction to compact objects is a Karoubi projection, as the kernel of F may fail to be compactly generated; in fact $\ker(F)$ need not have any non-trivial compact at all. The first example of such a situation was exhibited by Keller in [Kel94], we recall it in Example A.4.7 below.

The fibre of a Verdier projection between compactly generated ∞-categories is, however, automatically dualisable in the symmetric monoidal ∞-category of stable presentable ∞-categories. In as of now unpublished work, Efimov constructed an extension of any localising invariant $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}} \to \mathcal{S}p$, such as nonconnective K-theory, to such dualisable ∞-categories. This allows one to circumvent the difficulties for localisation sequences caused by the failure of compact generation, see [Hoy18] or [Efi18] for an account.

Finally, we extend the classification result Proposition A.2.12 for split Verdier sequences to the non-split case. To this end consider a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$. Recalling $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ we obtain from Theorem A.3.11 and the discussion thereafter a split Verdier sequence

$$\begin{tikzcd}
\mathrm{Pro\,Ind}(\mathcal{C}) \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Pro\,Ind}(\mathcal{D}) \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Pro\,Ind}(\mathcal{E}) \arrow[l, bend right, "q"'] \arrow[l, bend left, "q'"]
\end{tikzcd}$$

of large ∞-categories, together with a classifying functor $c\colon \mathrm{Pro\,Ind}(\mathcal{E}) \to \mathrm{Pro\,Ind}(\mathcal{C})$. Now consider the ∞-categories $\mathrm{Tate}(\mathcal{C})$ and $\mathrm{Latt}(\mathcal{C})$ of (elementary) Tate objects and their lattices from [Hen17] (though we warn the reader that Hennion denotes by $\mathrm{Tate}(\mathcal{C})$ the idempotent completion of the ∞-category we consider here): $\mathrm{Tate}(\mathcal{C})$ is the smallest stable subcategory of $\mathrm{Pro\,Ind}(\mathcal{C})$ spanned by its full subcategories $\mathrm{Pro}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{C})$, and $\mathrm{Latt}(\mathcal{C})$ is the full subcategory of $\mathrm{Ar}(\mathrm{Pro\,Ind}(\mathcal{C}))$ spanned by the arrows with source in $\mathrm{Ind}(\mathcal{C})$ and target in $\mathrm{Pro}(\mathcal{C})$. We obtain a square

(72)
$$\begin{tikzcd}
\mathrm{Pro\,Ind}(\mathcal{D}) \arrow[r, "g' \to g"] \arrow[d] & \mathrm{Ar}(\mathrm{Pro\,Ind}(\mathcal{C})) \arrow[d, "\mathrm{cof}"] \\
\mathrm{Pro\,Ind}(\mathcal{E}) \arrow[r, "c"] & \mathrm{Pro\,Ind}(\mathcal{C}),
\end{tikzcd}$$

which is a pullback by (a rotation in the top right corner of) Proposition A.2.12. By direct inspection it restricts to

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "g' \to g"] \arrow[d] & \mathrm{Latt}(\mathcal{C}) \arrow[d, "\mathrm{cof}"] \\
\mathcal{E} \arrow[r, "c"] & \mathrm{Tate}(\mathcal{C})
\end{tikzcd}$$

and we find:

A.3.14. **Proposition.** *For any stable ∞-category $\mathcal{C}$, the map* $\mathrm{cof}\colon \mathrm{Latt}(\mathcal{C}) \to \mathrm{Tate}(\mathcal{C})$ *is a Verdier projection with fibre $\mathcal{C}^\natural$ and for a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ with $\mathcal{C}$ idempotent complete, the diagram above is cartesian.*

The first part of this result is a special case of Clausen's discussion of cone categories in [Cla17, Section 3.1], particularly [Cla17, Remark 3.23], whereas the second part along with the uniqueness statement in Proposition A.3.15 below was first observed by the eighth author in [Nik20], which also discusses a monoidal version.

Combined, they imply that the functor $\mathrm{cof}\colon \mathrm{Latt}(\mathcal{C}) \to \mathrm{Tate}(\mathcal{C})$ is the universal Verdier projection with fibre $\mathcal{C}$ though it does not run between small ∞-categories. One also readily checks that a Verdier projection $\mathcal{D} \to \mathcal{E}$ is right or left split if and only if the functor $\mathcal{E} \to \mathrm{Tate}(\mathcal{C})$ takes values in $\mathrm{Pro}(\mathcal{C})$ or $\mathrm{Ind}(\mathcal{C})$, respectively, so that the pullbacks to these ∞-categories give the universal right or left split Verdier sequences. For consistency, also note that $\mathrm{Ind}(\mathcal{C}) \cap \mathrm{Pro}(\mathcal{C}) = \mathcal{C}^\natural$: Write $X \in \mathrm{Ind}(\mathcal{C}) \cap \mathrm{Pro}(\mathcal{C})$ both as the limit of a projective system P_i and as the colimit of an inductive system I_j in $\mathcal{C}$. Then, by the computation of mapping spaces in Ind- and Pro-categories in [Lur09a, Section 5.3], the identity of X factors as

$$X \to P_i \to I_j \to X$$

for some i and j, making X a retract of either object. Thus Proposition A.3.14 specialises back to the split case Proposition A.2.12 (under the additional assumption that $\mathcal{C}$ be idempotent complete). Similarly, $\mathrm{Latt}(\mathcal{C}) \to \mathrm{Tate}(\mathcal{C})^\natural$ is the universal Karoubi projection with fibre $\mathcal{C}^\natural$, and one readily checks that Verdier projections are characterised among Karoubi projections by the property that the classifying functor factors through $\mathrm{Tate}(\mathcal{C}) \subseteq \mathrm{Tate}(\mathcal{C})^\natural$.

Proof. We start with the first claim: Evidently, the kernel of the functor cof consists exactly of the equivalences from an inductive to a projective object in $\mathcal{C}$. This forces both to be constant (by the argument we gave before the proof), whence the kernel is the full subcategory of $\mathrm{Ar}(\mathcal{C}^\natural)$ spanned by the equivalences, which is equivalent to $\mathcal{C}^\natural$ itself. Consider then the natural functor

$$\mathrm{Latt}(\mathcal{C})/\mathcal{C}^\natural \to \mathrm{Tate}(\mathcal{C})$$

which we have to show is an equivalence. We start with full faithfulness. On the one hand, using $\Omega\,\mathrm{cof} \simeq \mathrm{fib}$ the space $\mathrm{Hom}_{\mathrm{Tate}(\mathcal{C})}(\mathrm{cof}(i \to p), \mathrm{cof}(i' \to p'))$ can be described as $\Omega^{\infty-1}$ of the total fibre of the square

$$\begin{tikzcd}
\hom_{\mathrm{Tate}(\mathcal{C})}(p, i') \arrow[r] \arrow[d] & \hom_{\mathrm{Tate}(\mathcal{C})}(i, i') \arrow[d] \\
\hom_{\mathrm{Tate}(\mathcal{C})}(p, p') \arrow[r] & \hom_{\mathrm{Tate}(\mathcal{C})}(i, p')
\end{tikzcd}$$

using the evident maps. On the other hand, using [NS18, Theorem I.3.3 (ii)] we have

$$\begin{aligned}
\hom_{\mathrm{Latt}(\mathcal{C})/\mathcal{C}^\natural}(i \to p, i' \to p') &\simeq \operatorname*{colim}_{c \in \mathcal{C}^\natural_{/i'}} \hom_{\mathrm{Latt}(\mathcal{C})}(i \to p, \mathrm{cof}(c \to i') \to \mathrm{cof}(c \to p')) \\
&\simeq \operatorname*{colim}_{c \in \mathcal{C}^\natural_{/i'}} \hom_{\mathrm{Tate}(\mathcal{C})}(i, \mathrm{cof}(c \to i')) \times_{\hom_{\mathrm{Tate}(\mathcal{C})}(i, \mathrm{cof}(c \to p'))} \hom_{\mathrm{Tate}(\mathcal{C})}(p, \mathrm{cof}(c \to p'))
\end{aligned}$$

Now, the total fibre above is invariant under replacing i' and p' by $\mathrm{cof}(c \to i')$ and $\mathrm{cof}(c \to p')$, respectively, so straight from the definition of total fibres, we find the fibre of

$$\hom_{\mathrm{Latt}(\mathcal{C})/\mathcal{C}^\natural}(i \to p, i' \to p') \longrightarrow \hom_{\mathrm{Tate}(\mathcal{C})}(\mathrm{cof}(i \to p), \mathrm{cof}(i' \to p')).$$

given by

$$\operatorname*{colim}_{c\in\mathcal{C}^\natural_{/i'}} \hom_{\mathrm{Tate}(\mathcal{C})}(p, \mathrm{cof}(c \to i')).$$

We claim that this term vanishes. For, writing $p = \lim_{k\in K} p_k$ for some $K \to \mathcal{C}$, we find from the computation of mapping spaces in ∞-categories of projective systems in [Lur09a, Section 5.3], that

$$\begin{aligned}\operatorname*{colim}_{c\in\mathcal{C}^\natural_{/i'}} \hom_{\mathrm{Tate}(\mathcal{C})}(p, \mathrm{cof}(c \to i')) &\simeq \operatorname*{colim}_{c\in\mathcal{C}^\natural_{/i'}} \operatorname*{colim}_{k\in K} \hom_{\mathrm{Ind}(\mathcal{C})}(p_k, \mathrm{cof}(c \to i'))\\ &\simeq \operatorname*{colim}_{k\in K} \hom_{\mathrm{Ind}(\mathcal{C})/\mathcal{C}^\natural}(p_k, i')\end{aligned}$$

and the last term clearly vanishes.

Finally, we note that the image of the functor $\mathrm{Latt}(\mathcal{C})/\mathcal{C}^\natural$ in $\mathrm{Tate}(\mathcal{C})$ is a stable subcategory containing both $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ so is essentially surjective by definition of $\mathrm{Tate}(\mathcal{C})$.

We thus turn to the cartesianness of the square involving the Verdier projection $\mathcal{D} \to \mathcal{E}$. We will reduce the statement to the split case by means of the embedding into (72), see [Nik20] for a more direct argument. Let P denote the pullback of $\mathcal{E} \to \mathrm{Tate}(\mathcal{C}) \leftarrow \mathrm{Latt}(\mathcal{C})$. Then the induced functor $\mathcal{D} \to P$ is fully faithful, since the square in question fully faithfully embeds into the right hand square before the proposition, which is cartesian. It remains to check that $\mathcal{D} \to P$ is essentially surjective. But by Proposition A.2.12 any $e \in \mathcal{E}$, together with $i \to p \in \mathrm{Latt}(\mathcal{C})$ and an equivalence $\mathrm{cof}(i \to p) \simeq c(e)$, determines an essentially unique object $d \in \mathrm{Pro}\,\mathrm{Ind}(\mathcal{D})$, namely $q'(e) \times_{c(e)} p$. This object lies in $\mathcal{D}^\natural = \mathrm{Pro}(\mathcal{D}) \cap \mathrm{Ind}(\mathcal{D}) \subseteq \mathrm{Pro}\,\mathrm{Ind}(\mathcal{D})$, since by construction there are fibre sequences

$$q(e) \to d \to p \quad \text{and} \quad i \to d \to q'(e)$$

as $c(e) \simeq \mathrm{cof}(q(e) \to q'(e))$ and the outer terms on the left are projective systems, whereas those on the right are inductive ones. We claim that

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, "p"] & \mathcal{D}^\natural \arrow[d, "p^\natural"] \\ \mathcal{E} \arrow[r] & \mathcal{E}^\natural \end{tikzcd}$$

is cartesian, whence d actually defines an object of $\mathcal{D}$, which one readily checks to be a preimage of the desired sort. For this final claim, it is clearly necessary that $\mathcal{C}$ be idempotent complete, but this also suffices: The functor from the pullback P of the remaining diagram (with $\mathcal{D}$ removed) to $\mathcal{D}^\natural$ is clearly fully faithful, thus so is $\mathcal{D} \to P$. It remains to show that this functor is essentially surjective. Pick then a $d \in \mathcal{D}^\natural$ with $p^\natural(d) \in \mathcal{E}$ and a witnessing retract diagram

$$d \longrightarrow d' \longrightarrow d$$

with $d' \in \mathcal{D}$. By yet another application of [NS18, Theorem I.3.3 (ii)], we can find an $x \in \mathcal{D}$ together with a map $d' \to x$ covering the projection $p(d') \to p^\natural(d)$. But then the fibre of the composite $d \to d' \to x$ lies in $\mathcal{C}^\natural = \mathcal{C} \subseteq \mathcal{D}$, and thus so does $d \in \mathcal{D}$ as desired. □

Regarding the uniqueness of the classifying map in Proposition A.3.14, we extend the notion of adjointability to diagrams

$$\begin{tikzcd} \mathcal{D} \arrow[r, "i"] \arrow[d, "p"] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "j"] & \mathcal{E}' \end{tikzcd}$$

with vertical Verdier projections by requiring their inductive and projective completions to be right and left adjointable, respectively. Then we find:

A.3.15. **Proposition.** *Given a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ and another idempotent complete stable ∞-category $\mathcal{C}'$, the full subcategory of* $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathrm{Latt}(\mathcal{C}'))$ *spanned by the functors φ that give rise to* Ind/Pro*-adjointable squares*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\varphi"] \arrow[d] & \mathrm{Latt}(\mathcal{C}') \arrow[d, "\mathrm{cof}"] \\ \mathcal{E} \arrow[r, "\overline{\varphi}"] & \mathrm{Tate}(\mathcal{C}') \end{tikzcd}$$

in the sense just described is equivalent to $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}')$ *via restriction to vertical fibres. Furthermore, any cartesian square whose vertical maps are Verdier projections is* Ind/Pro*-adjointable.*

Proof. The first part follows from Proposition A.2.14 by unwinding definitions. The argument that inductive completions of cartesian squares are adjointable is, however, more subtle (the case of the projective completion is dual): To see that

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{D}) \arrow[r, "\varphi_!"] & \mathrm{Ind}(\mathcal{D}') \\ \mathrm{Ind}(\mathcal{E}) \arrow[r, "\overline{\varphi}_!"] \arrow[u, "p^*"] & \mathrm{Ind}(\mathcal{E}') \arrow[u, "(p')^*"'] \end{tikzcd}$$

commutes, note first that by the universal property of inductive completions it suffices to check this after restriction to $\mathcal{E} \subseteq \mathrm{Ind}(\mathcal{E})$. Next, note that the statement becomes true after postcomposition with $p'_! \colon \mathrm{Ind}(\mathcal{D}') \to \mathrm{Ind}(\mathcal{E}')$, since

$$p'_!\varphi_!p^* \simeq \overline{\varphi}_!p_!p^* \simeq \overline{\varphi}_! \simeq p'_!(p')^*\overline{\varphi}_!$$

via the canonical maps, since p^* and $(p')^*$ are fully faithful by assumption. It therefore only remains to check that the composite $\varphi_!p^*$ takes values in the image of $(p')^*$, since $p'_!$ restricts to an equivalence on this part on account of being a localisation. Using the equivalence $\mathrm{Ind}(\mathcal{D}') \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}')^{\mathrm{op}}, \mathcal{S}p)$ stemming from the universal property of $\mathcal{S}p$ (combine [Lur09a, Corollary 5.3.5.4] and [Lur17, Proposition 1.4.2.22]), this image unwinds to exactly those functors $(\mathcal{D}')^{\mathrm{op}} \to \mathcal{S}p$ that vanish on $\mathcal{C}'$, the kernel of p'. Under this embedding $\varphi_!p^*(e)$ unwinds to the left Kan extension of $\hom_{\mathcal{E}}(p-, e) \colon \mathcal{D}^{\mathrm{op}} \to \mathcal{S}p$ along $\varphi^{\mathrm{op}} \colon \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$. Evaluating at some $c' \in \mathcal{C}'$ using the pointwise formula yields

$$[\varphi_!p^*(e)](c') \simeq \operatorname*{colim}_{d \in \mathcal{D}_{/c'}} \hom_{\mathcal{E}}(p(d), e).$$

But since we started with a cartesian square, picking a preimage $c \in \mathcal{C} = \ker(p)$ of c' yields an equivalence $\mathcal{D}_{/c} \to \mathcal{D}_{/c'}$, which shows that $(c, \varphi(c) \to c')$ is a terminal object in $\mathcal{D}_{/c'}$, so

$$[\varphi_!p^*(e)](c') \simeq \hom_{\mathcal{E}}(p(c), e) \simeq 0$$

as desired. □

A.3.16. **Example.** Let $\mathcal{C}$ and $\mathcal{E}$ be stable ∞-categories, with $\mathcal{C}$ idempotent complete, and let $\mathrm{B} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}p$ be a bilinear functor. Interpreting B as a functor $\mathcal{E} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p) \simeq \mathrm{Ind}(\mathcal{C}) \subset \mathrm{Tate}(\mathcal{C})$, we can pull back the universal Verdier sequence with fibre $\mathcal{C}$ along B as to obtain a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ (which automatically has a left adjoint, since the restriction of the universal Verdier sequence to $\mathrm{Ind}(\mathcal{C}) \subset \mathrm{Tate}(\mathcal{C})$ does). Then, this Verdier sequence is the sequence

$$\mathcal{C} \xrightarrow{f} \mathrm{Pair}(\mathcal{C}, \mathcal{E}, \mathrm{B}) \xrightarrow{p} \mathcal{E}$$

obtained from the pairings construction from §[I].7.1, where the first map includes $\mathcal{C}$ as objects of the form $(c, 0, 0)$ and the second map projects (c, e, β) to e: Indeed, the classifying map of the latter sequence is given by the suspension of the composite

$$\mathcal{E} \xrightarrow{q} \mathrm{Pair}(\mathcal{C}, \mathcal{E}, \mathrm{B}) \xrightarrow{g'} \mathrm{Ind}(\mathcal{C}) \subset \mathrm{Tate}(\mathcal{C}),$$

where q is the left adjoint of p (given by the inclusion as objects of the form $(0, e, 0)$), and g' is the right adjoint of $\mathrm{Ind}(f)$. Identifying $\mathrm{Ind}(\mathcal{C})$ with $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p)$, this right adjoint is given by the formula

$$X \mapsto \mathrm{Hom}_{\mathrm{Pair}(\mathcal{C},\mathcal{E},\mathrm{B})}(f(-), X),$$

so the above composite corresponds to the bilinear functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}p, \quad (c, e) \mapsto \mathrm{Hom}_{\mathrm{Pair}(\mathcal{C},\mathcal{E},\mathrm{B})}(f(c), q(e)),$$

whose suspension agrees with $\mathrm{B}(c, e)$ by the formula for mapping spaces in pairing ∞-categories, [I].(174).

A.3.17. **Example.** There is also a description of the ∞-category $\mathrm{Latt}(\mathcal{C})$ in terms of the pairings ∞-category from Section [I].7.1, namely we claim that $\mathrm{Latt}(\mathcal{C}) = \mathrm{Pair}(\mathrm{Ind}(\mathcal{C}), \mathrm{Pro}(\mathcal{C}), \mathrm{Hom}_{\mathrm{Pro\,Ind}(\mathcal{C})})$. This follows from the general fact that $\mathrm{Ar}(\mathcal{D}) = \mathrm{Pair}(\mathcal{D}, \mathcal{D}, \mathrm{Hom}_{\mathcal{D}})$, see [HMS22, Corollary A.2.5] or [HHLN22, Example 6.15], by isolating the appropriate subcategories. The embedding of $\mathcal{C} \to \mathrm{Latt}(\mathcal{C})$ is then given on objects by $c \mapsto (c, c, \mathrm{id}_c)$.

A.4. **Verdier and Karoubi sequences among module categories.** Let $\phi \colon A \to B$ be a map of E_1-ring spectra. Extension of scalars induces an exact functor

$$\phi_! \colon \mathrm{Mod}_A \to \mathrm{Mod}_B, \quad M \mapsto B \otimes_A M$$

on the ∞-categories of (left) modules, which is left adjoint to the restriction of scalars functor $\phi^* \colon \mathrm{Mod}(B) \to \mathrm{Mod}(A)$. Extension of scalars restricts to functors

$$\phi_! \colon \mathrm{Mod}^{\omega}_A \to \mathrm{Mod}^{\omega}_B \quad \text{and} \quad \phi_! \colon \mathrm{Mod}^{\mathrm{c}}_A \to \mathrm{Mod}^{\phi(\mathrm{c})}_B,$$

where $\mathrm{c} \subseteq \mathrm{K}_0(A)$ is a subgroup and $\mathrm{Mod}^{\mathrm{c}}_A$ the full subcategory of Mod^{ω}_A spanned by those A-modules X with $[X] \in \mathrm{c} \subseteq \mathrm{K}_0(A)$. The most important special case of the latter construction is the case where c is the image of the canonical map $\mathbb{Z} \to \mathrm{K}_0(A)$, $1 \mapsto A$, in which case $\mathrm{Mod}^{\mathrm{c}}_A = \mathrm{Mod}^{\mathrm{f}}_A$ is the stable subcategory of Mod^{ω}_A generated by A. In this section we analyse when these functors are Verdier or Karoubi projections.

Remark. We remind the reader mainly interested in the classical case of discrete rings of the following dictionary: The Eilenberg-Mac Lane spectrum of a discrete ring A is an E_1-ring spectrum, which we denote by $\mathrm{H}A$. The ∞-category $\mathrm{Mod}(\mathrm{H}A)$ of $\mathrm{H}A$-module spectra is then equivalent to the (unbounded) derived ∞-category of A, that is, the ∞-categorical localisation of the category of A-chain complexes at the class of homology equivalences, see [Lur17, Remark 7.1.1.16].

The reader should be aware that under this equivalence, $\mathrm{H}M \otimes_{\mathrm{H}A} \mathrm{H}N$ corresponds to the derived tensor product $M \otimes^{\mathbb{L}}_A N$ of M and N which may be non-discrete, even if M and N are discrete; in this case the derived tensor product is connective and we have

$$\pi_i(\mathrm{H}M \otimes_{\mathrm{H}A} \mathrm{H}N) \cong \mathrm{Tor}^A_i(M, N), \quad (i \geq 0).$$

Now let $\mathrm{Mod}(A)_B \subseteq \mathrm{Mod}(A)$ denote the kernel of the functor $\phi_! \colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ extending scalars.

A.4.1. **Lemma.** *Let $\phi \colon A \to B$ be a map of E_1-ring spectra and denote by I the fibre of ϕ, considered as an A-bimodule. Then the following are equivalent:*

- *i) The multiplication $B \otimes_A \mathrm{B} \to B$ is an equivalence.*
- *ii) We have $B \otimes_A I \simeq 0$.*
- *iii) The diagram*

$$\begin{tikzcd}
\mathrm{Mod}(B) \arrow[r] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & \mathrm{Mod}(A) \arrow[l, bend right, "\phi_!"'] \arrow[l, bend left, "\phi_*"] \arrow[r] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & \mathrm{Mod}(A)_B \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\mathrm{hom}_A(I,-)"]
\end{tikzcd}$$

with the right pointing arrows given by ϕ^* and $I \otimes_A -$, respectively, is a stable recollement.

Note that iii) in particular contains the statement that $I \otimes_A - \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ has image in $\mathrm{Mod}(A)_B$ as indicated. Of course, one may as well replace $\mathrm{Mod}(A)_B$ in the statement by the kernel of ϕ_* and extrapolating from the example $B = A[s^{-1}]$ one might call such modules ϕ-complete: In this case the lower adjoint becomes the inclusion, the right pointing map $\mathrm{hom}_A(I, -)$ and the top adjoint $I \otimes_A -$.

Proof. For the equivalence between the first two items, simply note that

$$B \simeq B \otimes_A A \xrightarrow{\mathrm{id} \otimes \phi} B \otimes_A B$$

is always a right inverse to the multiplication map of B. So the latter is an equivalence if the fibre of the former vanishes. The statement of iii) contains ii), since $I = I \otimes_A A \in \mathrm{Mod}(A)_B$. Finally, assuming the first two items, we first find that

$$B \otimes_A I \otimes_A X \simeq 0$$

so that $I \otimes_A X \in \mathrm{Mod}(A)_B$ for all $X \in \mathrm{Mod}(A)$ and the diagram in iii) is well-defined. Furthermore, it follows that ϕ^* is fully faithful: For this one needs to check that the counit transformation $B \otimes_A Y \to Y$ is an equivalence for every B-module Y. But as both sides preserve colimits and B generates $\mathrm{Mod}(B)$ under colimits and shifts, it suffices to check this for $Y = B$, where we have assumed it. It then follows from the discussion after Proposition A.2.11 that the diagram

$$\begin{tikzcd}
\mathrm{Mod}(B) \arrow[r, "\perp", "\perp"'] & \mathrm{Mod}(A) \arrow[l, bend right=40, "\phi_!"'] \arrow[l, bend left=40, "\phi_*"]
\end{tikzcd}$$

can be completed to a stable recollement and the fibre sequences connecting the various adjoints are easily checked to give the formulae from the statement. ☐

A.4.2. **Definition.** We will call a map $\phi\colon A \to B$ of E_1-ring spectra satisfying the equivalent conditions of the previous lemma a *localisation*.

A map $R \to S$ between discrete rings is called a *derived localisation* if the associated map $\mathrm{H}R \to \mathrm{H}S$ is a localisation in the sense above.

A.4.3. **Remark.**

i) We warn the reader that it is not true that a localisation of discrete rings $A \to B$ is generally a derived localisation in the sense of Definition A.4.2. The latter condition additionally entails that $\mathrm{Tor}_i^A(B, B) = 0$ for all $i > 0$. This is automatic if A and B are commutative or more generally if the localisation satisfies an Ore condition, see Corollary A.4.5 below, but can fail in general.

ii) As examples that are not necessarily localisations at elements, let us mention that any for any open immersion of affine schemes $X \to Y$ the restriction map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is a derived localisation and under certain finiteness assumptions this in fact characterises open immersions among affines by [TV07, Lemma 2.1.4].

iii) The discrete counterpart of Definition A.4.2 for ordinary rings and ordinary tensor products was studied by Bousfield and Kan in [BK72], where they have classified all commutative rings R whose multiplication $R \otimes_{\mathbb{Z}} R \to R$ is an isomorphism, a property which is called *solidity* in [BK72]. We note that for a map of connective E_1-rings $A \to B$, being a localisation implies the solidity of $\pi_0 A \to \pi_0 B$ but even for discrete A and B, the converse is not true. In fact, the kernel of $\pi_0 A \to \pi_0 B$ is necessarily an idempotent ideal in this case, see Theorem A.4.8 below.

iv) Every flat and idempotent ideal $J \subseteq R$ in an ordinary ring R gives rise to a derived localisation $R \to R/J$: For these conditions imply that J satisfies $J \otimes_R^{\mathbb{L}} J \simeq J$, which is clearly equivalent to the conditions of Lemma A.4.1.

    This example places the basic setup of Falting's almost mathematics into the context of our discussion. We mention that, in this language, an object $M \in \mathcal{D}(R)$ is said to be almost zero with respect to J if and only if its homology groups are annihilated by J, which is equivalent to $J \otimes_R^{\mathbb{L}} M \simeq 0$, so that the Verdier quotient of $\mathcal{D}(R)$ by the almost zero objects is equivalent to $\mathrm{Mod}(\mathrm{H}R)_{\mathrm{H}(R/J)}$. See the discussion after Example A.4.7 below for a much more general assertion, that entirely avoids the flatness assumption on J.

    Also note that idempotent ideals J in commutative rings are rather trivial if finitely generated: By Nakayama's lemma J is then principal on an idempotent element, and thus a direct factor in a ring decomposition $R \cong J \times J'$. In particular, the projection $R \to R/J = J'$ is a ring-theoretic localisation in this case; Example A.4.7 below shows that this is not usually the case in general.

v) If $\phi\colon A \to B$ is a localisation of E_1-rings and A carries an E_n-structure for $n \geq 2$, then B and the structure map inherit an E_n-structure as well: Via the fully faithful functor $\phi^*\colon \mathrm{Mod}(B) \to \mathrm{Mod}(A)$, the ∞-category of modules over B inherits an E_{n-1}-monoidal structure with $\phi^*(M \otimes_B N) \simeq \phi^* M \otimes_A \phi^* N$ and unit B from $\mathrm{Mod}(A)$ (so that ϕ^* is lax but not quite strong E_{n-1}-monoidal). Now ϕ identifies (as an E_1-map) with the map induced by the functor $\phi_!\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ on the endomorphisms of the units. But $\phi_!$ inherits a (strong) E_{n-1}-monoidal structure from the lax one on ϕ^*, so this map upgrades to an $\mathrm{E}_1 \otimes \mathrm{E}_{n-1} \simeq \mathrm{E}_n$-monoidal one, and per construction the induced E_n-structure on A is the given one.

Now, to state the main result of this section, we need a bit of terminology. By Lemma A.4.1, a map $\phi\colon A \to B$ is a localisation if and only if $I = \mathrm{fib}(A \to B) \in \mathrm{Mod}(A)_B$. Given a full subcategory $\mathcal{C} \subseteq \mathrm{Mod}(A)$, let us write $\mathcal{C}_B = \mathcal{C} \cap \mathrm{Mod}(A)_B$ and say that ϕ has perfectly generated fibre if I lies in the smallest subcategory of $\mathrm{Mod}(A)_B$ containing $(\mathrm{Mod}^\omega_A)_B$ and closed under colimits.

A.4.4. **Proposition.** *Let $\phi\colon A \to B$ be a localisation of E_1-rings with perfectly generated fibre. Then*

$$(\mathrm{Mod}^\omega_A)_B \longrightarrow \mathrm{Mod}^\omega_A \xrightarrow{\phi_!} \mathrm{Mod}^\omega_B$$

is a Karoubi sequence and

$$(\mathrm{Mod}^{\mathrm{c}}_A)_B \longrightarrow \mathrm{Mod}^{\mathrm{c}}_A \xrightarrow{\phi_!} \mathrm{Mod}^{\phi(\mathrm{c})}_B$$

is a Verdier sequence for every $\mathrm{c} \subseteq \mathrm{K}_0(A)$.

Proof. Combining Theorem A.3.11 and Lemma A.4.1, it only remains to show that $\mathrm{Ind}((\mathrm{Mod}^\omega_A)_B) \simeq \mathrm{Mod}(A)_B)$ to obtain the first claim. But by [Lur09a, Proposition 5.3.5.11], the former term is equivalent to the smallest subcategory of $\mathrm{Mod}(A)_B$ containing $(\mathrm{Mod}^\omega_A)_B$ closed under colimits, so by assumption it contains I. But the smallest stable subcategory of $\mathrm{Mod}(A)$ containing I and closed under colimits is $\mathrm{Mod}(A)_B$, as follows immediately from the stable recollement of iii) (since A generates $\mathrm{Mod}(A)$ under colimits and shifts).

Since the inclusion $\mathrm{Mod}^{\mathrm{c}}_A \to \mathrm{Mod}^\omega_A$ is a Karoubi equivalence and similarly for B, it follows that $\phi_!\colon \mathrm{Mod}^{\mathrm{c}}_A \to \mathrm{Mod}^{\phi(\mathrm{c})}_B$ is a Karoubi projection as well. But the essential image of this functor is then the Verdier quotient by its kernel, see Corollary A.3.8, and therefore a dense stable subcategory of Mod^ω_B. The second claim follows from the classification of dense subcategories A.3.2 and Proposition A.3.9. □

A.4.5. **Corollary.** *Given an E_1-ring A and a subset $S \in \pi_*(A)$ of homogeneous elements satisfying the left Ore condition, for example $\pi_*(A)$ could be (skew-)commutative, then*

$$(\mathrm{Mod}^\omega_A)_S \longrightarrow \mathrm{Mod}^\omega_A \xrightarrow{-[S^{-1}]} \mathrm{Mod}^\omega_{A[S^{-1}]}$$

is a Karoubi sequence and for every $c \subseteq \mathrm{K}_0(A)$

$$(\mathrm{Mod}^{\mathrm{c}}_A)_S \longrightarrow \mathrm{Mod}^{\mathrm{c}}_A \xrightarrow{-[S^{-1}]} \mathrm{Mod}^{\mathrm{im(c)}}_{A[S^{-1}]}$$

is a Verdier sequence.

Here we have abbreviated $(\mathrm{Mod}^\omega_A)_{A[S^{-1}]}$ to $(\mathrm{Mod}^\omega_A)_S$ and similarly in the case of finitely presented module spectra.

Proof. Under the Ore condition, the fibre $\mathrm{Mod}(A)_S$ is generated under colimits by the perfect modules $A/s = \mathrm{cof}[A \xrightarrow{s} A]$ for $s \in S$, see [Lur17, Lemma 7.2.3.13]. Thus, Proposition A.4.4 applies. □

A.4.6. **Example.** In the situation of Proposition A.4.4, the map $\mathrm{Mod}^\omega_A \to \mathrm{Mod}^\omega_B$ can easily fail to be a Verdier projection. For example, suppose that k is a field and $R = k[x,y]/y^2 - x^3 - x^2$ is the (ordinary) commutative ring over k whose spectrum is a reduced and geometrically irreducible affine curve with a unique a singular point which is a node. Set $A = R[t]$ and $B = R[t,t^{-1}]$. Then by [Wei01, Lemma 2.3], one has $\mathrm{K}_{-1}(R) \cong \mathbb{Z}$. The exact sequence of Bass

$$0 \longrightarrow \mathrm{K}_i(R) \longrightarrow \mathrm{K}_i(R[t]) \oplus \mathrm{K}_i(R[t^{-1}]) \longrightarrow \mathrm{K}_i(R[t,t^{-1}]) \longrightarrow \mathrm{K}_{i-1}(R) \longrightarrow 0$$

then implies that $\mathrm{coker}(\mathrm{K}_0(A) \to \mathrm{K}_0(B)) \neq 0$. In particular, the functor

$$\mathcal{D}^{\mathrm{p}}(A) \to \mathcal{D}^{\mathrm{p}}(B)$$

is not essentially surjective and hence not a Verdier projection, see Corollary A.1.7, while still a Karoubi projection by Proposition A.4.4.

While the theorem of Thomason-Trobaugh for example implies that for an open embedding $X \to Y$ of affine schemes the map $\mathcal{O}(Y) \to \mathcal{O}(X)$ has perfectly generated fibre, this condition is unfortunately not automatic for a general localisation, and we do not know of a reformulation in purely ring theoretic terms, even when all constituents rings are discrete. The following counter-example is due to Keller [Kel94]; we thank Akhil Mathew for pointing it out to us:

A.4.7. **Example.** Let k be a field and $A = k[T^{1/2^\infty}] = k[t, t^{1/2}, t^{1/4}, ...]$ the commutative k-algebra obtained from the polynomial algebra $k[t]$ by adding a formal 2^i-order root $t^{1/2^i}$ of t for every $i \geq 1$. Let $I \subseteq A$ be the ideal generated by the $t^{1/2^i}$ for $i \geq 0$ and $\phi \colon A \to A/I = k$ the quotient map. We then claim that ϕ is a derived localisation. To see this, note first that by Lemma A.4.1 it will suffice to show that $I \otimes^{\mathbb{L}}_A k \simeq 0$. Now, the ascending filtration of I by the free cyclic submodules $t^{1/2^i}A \subseteq I$ gives a presentation of I as a filtered colimit

$$I = \operatorname{colim}[A \xrightarrow{t^{1/2}} A \xrightarrow{t^{1/4}} A \xrightarrow{t^{1/8}} ...]$$

so I is flat and $I \otimes^{\mathbb{L}}_A k \simeq I \otimes_A k \simeq \operatorname{colim}[k \xrightarrow{0} k \xrightarrow{0} ...] \simeq 0$, cf. Wodzicki [Wod89, Example 4.7(3)].

But the fibre of ϕ is not perfectly generated. To see this, let $S \subseteq A$ be the multiplicative set of all elements which are not in I and let $A[S^{-1}]$ be the localisation of A at S, so that $A[S^{-1}]$ is a local k-algebra with maximal ideal $I[S^{-1}]$. By (a derived version of) Nakayama's lemma every perfect A-module M such that $M \otimes^{\mathbb{L}}_A k \simeq 0$ will also satisfy $M[S^{-1}] \simeq 0$. It then follows that also the colimit closure of $(\mathcal{D}^{\mathrm{p}}(A))_{A/I}$ is contained in $\mathcal{D}(A)_S$. But $I_S \neq 0$ and so these do not contain I.

We finish this section by showing that the above example is part of a systematic story, which one may regard as a fully derived version of the basic set-up of Faltings' almost mathematics. We thank Peter Scholze for explaining this to us.

A.4.8. **Theorem.** *For a connective* E_k*-ring* A *with* $1 \leq k \leq \infty$*, consider the full subcategory* Alm_A *of* $\mathrm{Alg}_{\mathrm{E}_k}(\mathrm{Sp})_{A/}$ *spanned by those maps* $\phi \colon A \to B$ *such that*

i) ϕ *is a localisation,*
ii) B *is connective, and*
iii) $\pi_0(\phi) \colon \pi_0 A \to \pi_0 B$ *is surjective.*

Then the functor

$$\mathrm{Alm}_A \longrightarrow \{I \subseteq \pi_0 A \mid I^2 = I\}, \quad \phi \longmapsto \ker(\pi_0\phi)$$

is an equivalence of ∞*-categories, where we regard the target as a poset via the inclusion ordering. The inverse of some* I *is given by* A/I^∞*, where*

$$I^\infty = \lim_{n \in \mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}$$

with $J_I \to A$ *the fibre of the canonical map* $A \to \mathrm{H}(\pi_0(A)/I)$ *and this inverse system stabilises on* π_i *for* $n > i + 1$.

Furthermore, the image of the Verdier-inclusion $\phi^* \colon \mathrm{Mod}(A/I^\infty) \to \mathrm{Mod}(A)$ *consists exactly of those* A*-modules* M *with* $I \cdot \pi_n(M) = 0$ *for all* $n \in \mathbb{Z}$*.*

The kernel of the map $\phi_! \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A/I^\infty)$ can therefore also be described as the Verdier quotient of $\mathrm{Mod}(A)$ by its full subcategory $\mathrm{Ann}_I(A)$ of those modules, whose homotopy is annihilated by I. This quotient is the ∞-category of *almost A-modules* with respect to I and we denote it by $\mathrm{aMod}_I(A)$. Using this description, the split Verdier sequence associated to $A \to A/I^\infty$ takes the form

$$\begin{tikzcd} \mathrm{Ann}_I(A) \arrow[r] & \mathrm{Mod}(A) \arrow[l, bend right, "A/I^\infty \otimes_A -"'] \arrow[l, bend left, "\hom_A(A/I^\infty{,}-)"] \arrow[r] & \mathrm{aMod}_I(A) \arrow[l, bend right, "I^\infty \otimes_A -"'] \arrow[l, bend left, "\hom_A(I^\infty{,}-)"] \end{tikzcd}$$

Proof of Theorem A.4.8. We start with the observation that Alm_A is indeed equivalent to a poset, i.e. its mapping spaces are either empty or contractible, by the characterisation of localisations as $\otimes_A$-idempotent objects. Let us also immediately verify that $\ker(\pi_0\phi)$ is indeed idempotent for ϕ a localisation among connective ring spectra. Tensoring the fibre sequence $F \to A \to B$ with F gives

$$F \otimes_A F \longrightarrow F \longrightarrow F \otimes_A B$$

and the right hand term vanishes by A.4.1. But the map $\pi_0(F) \to \ker \pi_0\phi$ is surjective, whence a chase in the diagram

$$\begin{tikzcd}
\ker(\pi_0\phi) \otimes_{\pi_0 A} \ker(\pi_0\phi) \arrow[rr] & & \ker(\pi_0\phi) \\
\pi_0 F \otimes_{\pi_0 A} \pi_0 F \arrow[r, "\sim"] \arrow[u] & \pi_0(F \otimes_A F) \arrow[r, "\sim"] & \pi_0 F \arrow[u]
\end{tikzcd}$$

shows that the multiplication $\ker(\pi_0\phi) \otimes_{\pi_0 A} \ker(\pi_0\phi) \to \ker(\pi_0\phi)$ is surjective as desired.

Next, we verify the second to last claim from the statement, in fact we show slightly more, namely that the cofibre $A/J_I \otimes_A J_I^{\otimes_A n}$ of the canonical map $J_I^{\otimes_A n+1} \to J_I^{\otimes_A n}$ is n-connective. Since $A/J_I = \mathrm{H}(\pi_0(A)/I)$ is an E_k-ring annihilated by I, we immediately deduce that the homotopy groups of this cofibre are annihilated by I (from the left, but the same argument works on the right-most factor, so this is also true for the I-action from the right).

Now, for $n = 0$, the connectivity claim is clear, and if we inductively assume that $A/J_I \otimes_A J_I^{\otimes_A n}$ is n-connective, then

$$A/J_I \otimes_A J_I^{\otimes_A n+1} = \left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_A J_I$$

is clearly also n-connective and its nth homotopy group is $\pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0 A} I$. Since the left hand term is annihilated by I, we compute

$$\begin{aligned}
\pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0 A} I &= \pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0(A)/I} \pi_0(A)/I \otimes_{\pi_0 A} I \\
&= \pi_n\left(A/J_I \otimes_A J_I^{\otimes_A n}\right) \otimes_{\pi_0(A)/I} I^2/I = 0.
\end{aligned}$$

As the next step, we show that the tautological map $M = A \otimes_A M \to A/I^\infty \otimes_A M$ is an equivalence whenever the homotopy of M is annihilated by I, or in other words that $I^\infty \otimes_A M \simeq 0$. We start with the simplest case $M = A/J_I$, where the claim is equivalent to the multiplication map

$$\left(\lim_{n \in \mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}\right) \otimes_A J_I \longrightarrow \lim_{n \in \mathbb{N}^{\mathrm{op}}} J_I^{\otimes_A n}$$

being an equivalence. But since the limit stabilises degreewise and J_I is connective, we can move the limit out of the tensor product (the cofibre of the interchange map is a limit of terms with growing connectivity), and then the statement follows from finality.

For an arbitrary A-module M concentrated in degree 0 and killed by the action of I, choose a free resolution of $\pi_0 M$ by $\pi_0(A)/I$-modules, which by the Dold-Kan theorem yields a diagram $F : \Delta^{\mathrm{op}} \to \mathcal{D}(\pi_0(A)/I)$ with each F_n concentrated in degree 0, $\pi_0(F_n)$ free and $\operatorname{colim}_{\Delta^{\mathrm{op}}} F \simeq (\pi_0 M)[0]$, so that $\operatorname{colim}_{\Delta^{\mathrm{op}}} \iota F \simeq M$, where ι is the composite $\mathcal{D}(\pi_0(A)/I) \simeq \mathrm{Mod}(A/J_I) \to \mathrm{Mod}(A)$. But then

$$I^\infty \otimes_A M \simeq \operatorname*{colim}_{k \in \Delta^{\mathrm{op}}} I^\infty \otimes_A \iota F_k \simeq 0$$

since each ιF_k is a direct sum of A/J_I. By exactness of $I^\infty \otimes_A (-)$, the claim then follows for each bounded A-module M whose homotopy is annihilated by I using the Postnikov tower of M, and then for bounded below M, we have

$$I^\infty \otimes_A M \simeq I^\infty \otimes_A \left(\lim_{k \in \mathbb{N}^{\mathrm{op}}} \tau_{\leq k} M\right) \simeq \lim_{k \in \mathbb{N}^{\mathrm{op}}} I^\infty \otimes_A \tau_{\leq k} M \simeq 0$$

by commuting the limit out using the same argument as above. Finally, for arbitrary M whose homotopy is killed by I, we find

$$I^\infty \otimes_A M \simeq I^\infty \otimes_A \left(\operatorname*{colim}_{k \in \mathbb{N}} \tau_{\geq -k} M\right) \simeq \operatorname*{colim}_{k \in \mathbb{N}} I^\infty \otimes_A \tau_{\geq -k} M \simeq 0$$

Now, since $A \to A/J_I = \mathrm{H}(\pi_0(A)/I)$ is a map of E_k-rings, it follows that $J_I \to A$ is an E_k-Smith-ideal in A. It then formally follows that so is $J_I^{\otimes_A n} \to A$, whence $A/J_I^{\otimes_A n}$ and thus A/I^∞ are E_k-rings; for completeness' sake we briefly outline the argument at the end of this section since we are unaware of a reference. Since $\pi_0(A/I^\infty) = \pi_0(A)/I$, all homotopy groups of A/I^∞ are annihilated by I, and so the canonical map $A \to A/I^\infty$ induces an equivalence $A/I^\infty \to A/I^\infty \otimes_A A/I^\infty$, which shows that $A \to$

A/I^∞ is a localisation. Furthermore, it implies that the homotopy of every A/I^∞-module is a $\pi_0(A)/I$-module, so combined with the previous point, we learn that the image of the fully faithful restriction functor $\mathrm{Mod}(A/I^\infty) \to \mathrm{Mod}(A)$ consists exactly of those modules whose homotopy is killed by I, as desired.

Finally, we are ready to verify that the construction $I \mapsto (A \to A/I^\infty)$ induces an inverse to taking kernels. The composition starting with an ideal is clearly the identity. So we are left to show that for every $\phi\colon A \to B$ in Alm_A the canonical map $\psi\colon A_{\ker(\pi_0\phi)} \to B$, arising from the homotopy of B being annihilated by $\ker(\pi_0\phi)$, is an equivalence. Per construction it induces an equivalence on π_0. By the lemma below, the functor $\psi_!\colon \mathrm{Mod}(A/\ker(\pi_0\phi)^\infty) \to \mathrm{Mod}(B)$ is thus conservative when restricted to bounded below modules. But the map

$$B \simeq \psi_!\left(A/\ker(\pi_0\phi)^\infty\right) \xrightarrow{\psi_!(\phi)} \phi_!(B) = B \otimes_{A/\ker(\pi_0\phi)^\infty} B \simeq B \otimes_A B$$

is induced by the unit and thus an equivalence since ϕ is a localisation. □

A.4.9. **Lemma.** *If $\psi\colon A \to B$ is a map of connective E_1-rings which is an isomorphism on π_0, then*

$$\psi_!\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$$

is conservative when restricted to bounded below A-modules.

Proof. If $M \in \mathrm{Mod}(A)$ with $\pi_i(M) = 0$ for $i < n$, then $\pi_n(B \otimes_A M) = \pi_0(B) \otimes_{\pi_0(A)} \pi_n(M) = \pi_n(M)$, so if M is bounded below with $B \otimes_A M \simeq 0$ then also $M \simeq 0$. Considering cofibres of morphisms, this implies the statement. □

A.4.10. **Remark.** i) A different way of phrasing Theorem A.4.8 is that there is a one-to-one correspondence between idempotent ideals in $\pi_0(A)$ and idempotent Smith-ideals in A for every connective E_k-algebra A, which makes $I \subseteq \pi_0(A)$ and $I^\infty \to A$ correspond.

ii) If R is a discrete ring with an ideal I that is flat as a left or right R-module and satisfies $I^2 = I$, then $\mathrm{H}I^{\otimes_{\mathrm{H}R} n} = \mathrm{H}(I^{\otimes_R n}) = \mathrm{H}(I^n) = \mathrm{H}I$, so $I^\infty = \mathrm{H}(I)$ and $(\mathrm{H}R)/I^\infty = \mathrm{H}(R/I)$.

A commutative ring R together with an idempotent, flat ideal $I \subseteq R$ is indeed one of the standard set-ups for almost mathematics, see e.g. [GR18, Bha], and in this case $\mathrm{aMod}_I(\mathrm{H}R)$ is the derived category of the ordinary category of almost R-modules. In the latter reference, it is also explained that whenever $(K, |\cdot|)$ is a perfectoid field, then $\mathfrak{m} = \{x \in K \mid |x| < 1\}$ is a flat and idempotent ideal in the valuation ring $\mathcal{O} = \{x \in K \mid |x| \leq 1\}$; the example of Keller above is of very similar spirit.

iii) Whenever $I \otimes_R I$ is flat over R, one still has $(\mathrm{H}R)/I^\infty = \mathrm{H}(R /\!\!/ I^{\otimes_R 2})$, where $/\!\!/$ denotes the cofibre in $\mathcal{D}(R)$, i.e.

$$\pi_i((\mathrm{H}R)/I^\infty) = \begin{cases} R/I & i = 0 \\ \ker(I \otimes_R I \to I) & i = 1 \\ 0 & i \geq 2. \end{cases}$$

For the multiplication map $I^{\otimes^{\mathbb{L}}_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \to I^{\otimes^{\mathbb{L}}_R n}$ factors as

$$I^{\otimes^{\mathbb{L}}_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \longrightarrow I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n} \longrightarrow I^{\otimes^{\mathbb{L}}_R n}$$

so the limit computing I^∞ can be replaced by that over the terms $\mathrm{H}(I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I^{\otimes^{\mathbb{L}}_R n})$. But this system is constant, as can be seen inductively from the fibre sequence

$$I^{\otimes_R 2} \otimes^{\mathbb{L}}_R I \longrightarrow I^{\otimes_R 2} \longrightarrow I^{\otimes_R 2} \otimes^{\mathbb{L}}_R R/I$$

whose last term is $I^{\otimes_R 2} \otimes_R R/I = I/I^2 \otimes_R I = 0$.

Note also that $I^{\otimes_R n} \cong I \otimes_R I$ for all $n \geq 2$, e.g. by the stability assertion of Theorem A.4.8.

iv) The ring A/I^∞ can also be constructed as the limit of the Amitsur (or cobar) complex of the map $A \to \mathrm{H}(\pi_0(A)/I)$, see e.g. [MNN17, Section 2.1]; this is a cosimplicial object with $[n] \mapsto \mathrm{H}(\pi_0(A)/I)^{\otimes_A n+1}$ and face and degeneracy maps induced by the unit and multiplication, respectively. For example, this allows one to bypass the discussion of Smith ideals we used in the proof and shows that the evident analogue of Theorem A.4.8 for animated commutative rings (in place of connective E_∞-rings) holds as well.

In particular, for a discrete commutative ring R and idempotent $I \subseteq R$, the E_∞-ring $\mathrm{H}R/I^\infty$ underlies a canonical animated commutative ring R/I^∞.

v) The condition $I \cdot \pi_n M = 0$ of $M \in \mathrm{Mod}(A)$ being almost zero is in fact equivalent to the a priori stronger condition that $I \otimes_{\pi_0 A} \pi_n M = 0$: For the former condition makes $\pi_n M$ into an $\pi_0(A)/I$-module so that

$$I \otimes_{\pi_0 A} \pi_n(M) = I \otimes_{\pi_0 A} \pi_0(A)/I \otimes_{\pi_0(A)/I} \pi_n M = I/I^2 \otimes_{\pi_0(A)/I} \pi_n M = 0.$$

In contrast to this, it need not be true, however, that $J_I \otimes_A M \simeq 0$ for M an almost zero A-module: For example, take $A = \mathrm{H}R$ with $R = K[T^{1/2^\infty}]/T$, where $K[T^{1/2^\infty}]$ is the ring from Example A.4.7 and $I = (T^{1/2^\infty})$ is the ideal generated by all the 2-power roots of T (despite the notation it is not principal). Then $R/I = K$ is clearly almost 0, but $I \otimes^{\mathbb{L}}_R K \simeq \bigoplus_{i \geq 1} K[2i-1]$ does not vanish: Writing $R(n) = K[T^{1/2^n}]/T$ and $I(n) = (T^{1/2^n})$ for the principal ideal therein, so that $R = \mathrm{colim}_n R(n)$ and $I = \mathrm{colim}_n I(n)$ and so $I \otimes^{\mathbb{L}}_R K = \mathrm{colim}_n I(n) \otimes^{\mathbb{L}}_{R(n)} K$, we can freely resolve the inclusion $I(n) \to I(n+1)$ by the periodic

$$\begin{tikzcd}
\dots \arrow[r] & R(n) \arrow[r, "\cdot T^{(2^n-1)/2^n}"] \arrow[d, "\mathrm{incl}"] & R(n) \arrow[r, "\cdot T^{1/2^n}"] \arrow[d, "\cdot T^{1/2^{n+1}}"] & R(n) \arrow[r, "\cdot T^{(2^n-1)/2^n}"] \arrow[d, "\mathrm{incl}"] & R(n) \arrow[d, "T^{1/2^{n+1}}"] \\
\dots \arrow[r] & R(n+1) \arrow[r, "\cdot T^{(2^{n+1}-1)/2^{n+1}}"] & R(n+1) \arrow[r, "\cdot T^{1/2^{n+1}}"] & R(n+1) \arrow[r, "\cdot T^{(2^{n+1}-1)/2^{n+1}}"] & R(n+1).
\end{tikzcd}$$

After tensoring with K each term is K with horizontal maps vanishing and vertical maps alternating between 0 and id_K. This gives the claim upon taking vertical colimits.

vi) The algebra R and ideal I from the previous point form a typical example for which $I \otimes_R I$, but not I itself, is flat: In fact, over $K[T^{1/2^\infty}]$ the ideal $(T^{1/2^\infty})$ is flat (see Example A.4.7 above), and one easily checks that

$$I \otimes_R I \simeq R \otimes_{K[T^{1/2^\infty}]} (T^{1/2^\infty}) \otimes_{K[T^{1/2^\infty}]} (T^{1/2^\infty}),$$

whereas I itself is not the base change of $(T^{1/2^\infty})$. From the fibre sequence $I \otimes^{\mathbb{L}}_R I \to I \to I \otimes^{\mathbb{L}}_R K$ and the calculation in the previous point, one then reads off that $\ker(I \otimes_R I \to I) \cong K$ which gives

$$\pi_*(R/I^\infty) = \Lambda^*_K[K[1]],$$

an exterior algebra on one generator in degree 1. For

$$R_n = R^{\otimes_K n} = K[T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty}]/(T_1, \dots, T_n)$$

and $I_n = (T_1^{1/2^\infty}, \dots, T_n^{1/2^\infty})$, we then have $R_n/I_n^\infty = (R/I^\infty)^{\otimes^{\mathbb{L}}_K n}$ (e.g. immediately from the description in terms of the Amitsur complex), so

$$\pi_*(R_n/I_n^\infty) = \Lambda^*_K[K^n[1]].$$

This in particular shows that even for discrete rings R, the animated rings R/I^∞ can have arbitrarily high non-trivial homotopy in the absence of any flatness assumption on I.

Finally, we briefly sketch the background for our use of Smith-ideals for lack of a reference. See however [Hov14] for a treatment of the E_1-version in model categorical language. Consider then an E_k-monoidally cocomplete stable ∞-category $\mathcal{C}$, and give $\mathrm{Ar}(\mathcal{C})$ the induced E_k-monoidal Day convolution structure with respect to taking minima on $[1]$, which makes the evaluation functor $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ strongly monoidal. A Smith-ideal in an E_k-algebra A in $\mathcal{C}$ is an E_k-algebra $J \to A$ in $\mathrm{Ar}(\mathcal{C})$ lifting the E_k-structure on A. Such objects correspond in a one-to-one fashion to E_k-ring maps out of A by taking (co)fibres: To see this consider $\mathrm{Fun}([1]^2, \mathcal{C})$ equipped with Day convolution with respect to taking minima in the first, and maxima in the second component of $[1]^2$ (note that Day convolution with respect to taking maxima in $[1]$ is just the pointwise monoidal structure on $\mathrm{Ar}(\mathcal{C})$). Now, the full subcategory of $\mathrm{Alg}_{\mathrm{E}_k}(\mathrm{Fun}([1]^2, \mathcal{C}))$ spanned by the cocartesian squares with lower left corner (i.e. the entry at $(1,0)$) vanishing is on the one hand equivalent to the ∞-category of E_k-arrows in $\mathcal{C}$ by taking fibres, and on the other, to the ∞-category of E_k-Smith-ideals in $\mathcal{C}$ by taking cofibres; this is clearly true at the level of the underlying ∞-categories, and the claim follows from this since such cocartesian squares are closed under the Day convolution on all squares.

Convolving two Smith ideals provides the higher categorical way of taking the sum of ideals, and taking pointwise tensor products generalises taking the product of ideals. Since we used it in the proof above, we briefly explain why this pointwise operation is well-defined (we learned the following slick argument from Maxime Ramzi). Recall that E_k-monoids in the Day convolution structure are identified with lax E_k-monoidal functors. Consider then the functor $\mathrm{Fun}^{\text{lax-}\mathrm{E}_k}([1], -)\colon \mathrm{Alg}_{\mathrm{E}_k}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$. Since it preserves products, it lifts to a functor

$$\mathrm{Fun}^{\text{lax-}\mathrm{E}_k}([1], -)\colon \mathrm{Alg}_{\mathrm{E}_{k+l}}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Alg}_{\mathrm{E}_l}(\mathrm{Cat}_\infty)$$

or in other words, the ∞-category of lax E_k-monoidal functors to an E_{k+l}-monoidal ∞-category inherits an E_l-monoidal structure, which unwinds to be the pointwise tensor products.

To obtain the statements we used in the proof above, apply all of this in the ∞-category $\mathrm{E}_k\text{-}\mathrm{Mod}(A)$ of E_k-modules over A, as constructed in [Lur17, Sections 3.3 & 3.4], which is again E_k-monoidal under $\otimes_A$ (in contrast to Mod_A, which is only E_{k-1}-monoidal).

B. COMPARISONS TO PREVIOUS WORK

In this appendix, we compare our construction of Grothendieck-Witt spectra to two constructions in the literature: Schlichting's definition of Grothendieck-Witt spectra of rings with 2 invertible [Sch10a], and Spitzweck's definition of Grothendieck-Witt spaces for stable ∞-categories with duality [Spi16]. In our language, both cases pertain solely to symmetric Poincaré structures: In the case of Spitzweck's work, this largely consists of unfolding the definitions, whereas for exact categories, this is enforced by 2 being invertible, which makes the quadratic and symmetric Poincaré structures, and also their variants such as the genuine ones, agree. Spitzweck already gave a comparison between his definition and Schlichting's when applied to categories of chain complexes over a ring in which 2 is invertible, and our proof is a straightforward generalisation of his.

From Schlichting's work, we then also obtain that for a ring with involution R in which 2 is invertible and an invertible R-module M, the canonical map

$$\mathrm{Unimod}(R, M)^{\mathrm{grp}} \to \mathcal{GW}(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{s}}_M)$$

is an equivalence; here, $\mathrm{Unimod}(R, M)$ denotes the groupoid of unimodular, M-valued symmetric bilinear forms on finitely generated projective R-modules, symmetric monoidal under orthogonal sum. As explained in the introduction, this statement is no longer true if 2 is not invertible in R: The target has to be replaced by the Grothendieck-Witt space associated to the genuine Poincaré structure, and furthermore, one needs to distinguish between symmetric and quadratic forms. A proof of this more general statement, entirely independent from the discussion here, is given in [HS21] by adapting the parametrised surgery methods of Galatius and Randal-Williams from [GRW14] to the present setting.

Remark. We do not attempt here a full comparison of our work to the recent definitions in [HLAS16, HSV19, Sch21]. For the constructions in the first and third papers, a comparison requires a more detailed discussion of the genuine quadratic functors and the result is established in [HS21] (at least for the case of split-exact structures in the case of Schlichting's work). For the constructions of the second paper, we first note that Poincaré ∞-categories provide examples of Waldhausen ∞-categories with genuine duality by [HSV19, Section 6]. It is then easy to see that their space-valued functor KH from [HSV19, Definition 10.3] extends our $\mathcal{GW}$, and we record this in Remark B.1.2 below, but for the full KR-functor from [HSV19, Corollary 10.35] this is less obvious, not least because of the different models of genuine C_2-spectra employed in our construction and theirs.

B.1. **Spitzweck's Grothendieck-Witt space of a stable ∞-category with duality.**

We start by comparing our definition to Spitzweck's from [Spi16]. To this end, recall from Section [I].7.2 the forgetful functor

$$\mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathrm{Cat}^{\mathrm{ps}}_\infty,$$

where an object in the target consists of a stable ∞-category equipped with a perfect bilinear functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$. Informally, the functor is given by taking a Poincaré ∞-category $(\mathcal{C}, \Omega)$ to $(\mathcal{C}, \mathrm{B}_\Omega)$, and it has fully faithful left and a right adjoints given by taking $(\mathcal{C}, \mathrm{B})$ to $(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{B}})$ and $(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{B}})$, respectively; see Proposition [I].7.2.18. Extracting the duality from a perfect symmetric bilinear functor results in an equivalence

$$\mathrm{Cat}^{\mathrm{ps}}_\infty \longrightarrow (\mathrm{Cat}^{\mathrm{ex}}_\infty)^{\mathrm{hC}_2},$$

where C_2 acts on $\mathrm{Cat}^{\mathrm{ex}}_\infty$ by taking opposites, see Corollary [I].7.2.16. We use this equivalence and the right adjoint above to regard a stable ∞-category with duality as a (symmetric) Poincaré ∞-category throughout this section.

Let us denote by $\mathcal{GW}(\mathcal{C}, \mathrm{D})$ Spitzweck's Grothendieck-Witt space from [Spi16, Definition 3.4]; we recall the definition below. The purpose of this section is to show:

B.1.1. **Proposition.** *For any perfect symmetric bilinear functor* B *on a small stable ∞-category, there is a canonical equivalence*

$$\mathcal{GW}(\mathcal{C}, \mathrm{D}_\mathrm{B}) \simeq \mathcal{GW}(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{B}})$$

of E_∞*-groups, natural in the input.*

For the definition of $\mathcal{GW}(\mathcal{C}, \mathrm{D})$, Spitzweck employs the edgewise subdivision of Segal's S-construction: Recall the usual S-construction $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{sCat}^{\mathrm{ex}}_\infty$ given degreewise as the full subcategory of $\mathrm{Fun}(\mathrm{Ar}(\Delta^n), \mathcal{C})$ spanned by those diagrams φ with $\varphi(i \leq i) \simeq 0$ and having the squares

$$\begin{tikzcd}
\varphi(i \leq k) \arrow[r] \arrow[d] & \varphi(i \leq l) \arrow[d] \\
\varphi(j \leq k) \arrow[r] & \varphi(i \leq l)
\end{tikzcd}$$

bicartesian for every set of numbers $i \leq j \leq k \leq l$. The edgewise subdivision $\mathrm{S}^e(\mathcal{C})$ of $\mathrm{S}(\mathcal{C})$ is then given by precomposing this simplicial category with the functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, sending $[n]$ to $[n] * [n]^{\mathrm{op}}$. Now, Spitzweck equips the ∞-categories $\mathrm{Fun}(\mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}}), \mathcal{C})$ with the duality D_n induced by conjugation with respect to flipping the join factors in the source and the given duality D on $\mathcal{C}$; more formally, let us denote the internal mapping objects of the cartesian closed category $\mathrm{Cat}^{\mathrm{hC}_2}_\infty$ by $\mathrm{Fun}^{\mathrm{hC}_2}$. Then, the arrow categories inherit dualities via

$$\mathrm{Ar}(\mathcal{C}, \mathrm{D}) = \mathrm{Fun}^{\mathrm{hC}_2}((\Delta^1, \mathrm{fl}), (\mathcal{C}, \mathrm{D}))$$

and $\mathrm{S}^e_n(\mathcal{C}, \mathrm{D})$ is defined as the full subcategory of

$$\mathrm{Fun}^{\mathrm{hC}_2}\left((\mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}}), \mathrm{fl}), (\mathcal{C}, \mathrm{D})\right)$$

spanned by the diagrams in $\mathrm{S}^e_n(\mathcal{C}) = \mathrm{S}_{2n+1}(\mathcal{C})$, which is meaningful since the duality carries this subcategory into itself. Naturality in n then assembles $\mathrm{S}^e(\mathcal{C}, \mathrm{D})$ into a simplicial category with duality, i.e. a functor $\Delta^{\mathrm{op}} \to (\mathrm{Cat}^{\mathrm{ex}}_\infty)^{\mathrm{hC}_2}$. Spitzweck sets

$$\mathcal{GW}(\mathcal{C}, \mathrm{D}) = \mathrm{fib}(|\mathrm{CrS}^e(\mathcal{C})^{\mathrm{hC}_2}| \to |\mathrm{CrS}^e(\mathcal{C})|).$$

To start the comparison, we also recall that $\mathrm{S}^e(\mathcal{C})$ is canonically equivalent to $\mathrm{Q}(\mathcal{C})$: There is a canonical map $\mathrm{TwAr}(\Delta^n) \to \mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}})$ natural in n, taking $(i \leq j)$ to $(i_0 \leq j_1)$, where the subscript indicates the join factor. Pullback along this map is easily checked to give an equivalence

$$\mathrm{S}^e(\mathcal{C}) \longrightarrow \mathrm{Q}(\mathcal{C}).$$

In degree 1 for example it takes

$$\begin{tikzcd}
0 \arrow[r] & \phi(0_0 \leq 1_0) \arrow[r] \arrow[d] & \phi(0_0 \leq 1_1) \arrow[r] \arrow[d] & \phi(0_0 \leq 0_1) \arrow[d] \\
& 0 \arrow[r] & \phi(1_0 \leq 1_1) \arrow[r] \arrow[d] & \phi(1_0 \leq 0_1) \arrow[d] \\
& & 0 \arrow[r] & \phi(1_1 \leq 0_1) \arrow[d] \\
& & & 0
\end{tikzcd}$$

to

$$\phi(0_0 \leq 0_1) \longleftarrow \phi(0_0 \leq 1_1) \longrightarrow \phi(1_0 \leq 1_1).$$

Now, we claim that the duality described above corresponds exactly to that induced by $(\Qoppa^{\mathrm{s}}_{\mathrm{B}})_n\colon\ \mathrm{Q}_n(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}p$. To see this, recall that $\mathrm{Q}_n(\mathcal{C},\Qoppa)$ is a full subcategory of the cotensoring $(\mathcal{C},\Qoppa)^{\mathrm{TwAr}(\Delta^n)}$, and thus to refine the equivalence between the Q- and S-constructions to a hermitian functor, it suffices to give a functor

$$q_n\colon\ \mathrm{TwAr}(\Delta^n) \longrightarrow \mathrm{Fun}^{\mathrm{h}}\left((\mathrm{S}^e_n(\mathcal{C}),\Qoppa^{\mathrm{s}}_{\mathrm{D}_n}),(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}})\right)$$

refining the one on underlying ∞-categories described above. To this end, we note that assigning to $(i \leq j) \in \mathrm{TwAr}(\Delta^n)$ the arrow $(i_0 \leq j_1) \to (j_0 \leq i_1)$ in $\mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}})$ gives a functor

$$p_n\colon\ \mathrm{TwAr}(\Delta^n) \longrightarrow \mathrm{Ar}(\mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}},\mathrm{fl}))^{\mathrm{C}_2},$$

natural for $n \in \Delta$; here, the superscript indicates functors strictly commuting the with identifications of the input categories with their opposites. Pullback along this equivariant functor produces a map

$$\mathrm{TwAr}(\Delta^n) \longrightarrow \mathrm{Fun}^{\mathrm{hC}_2}\left((\mathrm{S}^e_n(\mathcal{C}),\mathrm{D}_n),\mathrm{Ar}(\mathcal{C},\mathrm{D})\right).$$

Now, by Remark [I].7.3.4, the Poincaré ∞-category $\mathrm{Ar}(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}})$ has underlying ∞-category with duality $\mathrm{Ar}(\mathcal{C},\mathrm{D})$, and since the functor $(\mathrm{Cat}^{\mathrm{ex}}_\infty)^{\mathrm{hC}_2} \simeq \mathrm{Cat}^{\mathrm{ps}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$, $(\mathcal{C},\mathrm{D}) \mapsto (\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}})$ is fully faithful by Proposition [I].7.2.18 and preserves products (as a right adjoint), we find that the underlying category with duality of $\mathrm{Fun}^{\mathrm{p}}((\mathrm{S}^e_n(\mathcal{C}),\Qoppa^{\mathrm{s}}_{\mathrm{D}_n}),\mathrm{Ar}(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}}))$ is the full subcategory of $\mathrm{Fun}^{\mathrm{hC}_2}((\mathrm{S}^e_n(\mathcal{C}),\mathrm{D}_n),\mathrm{Ar}(\mathcal{C},\mathrm{D}))$ spanned by the exact functors. But by construction of the Poincaré structure on $\mathrm{Ar}(\mathcal{C},\Qoppa)$, evaluation at the source defines a hermitian functor $\mathrm{Ar}(\mathcal{C},\Qoppa) \to (\mathcal{C},\Qoppa)$ (that is usually not Poincaré). In total, we obtain the desired functor q_n by composing the three steps just described. To see that its adjoint $(\mathrm{S}^e_n(\mathcal{C}),\Qoppa^{\mathrm{s}}_{\mathrm{D}_n}) \to \mathrm{Q}_n(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}})$ is Poincaré, and thus in fact an equivalence of Poincaré ∞-categories, it suffices to check this after postcomposition with the Segal maps $\mathrm{Q}_n(\mathcal{C},\Qoppa) \to \mathrm{Q}_1(\mathcal{C},\Qoppa)$ by Lemma 2.2.5, where it is a simple application of the formula for the duality in cotensor categories from Proposition [I].6.3.2.

The proof of Proposition B.1.1 is now simple:

Proof. The natural equivalence

$$(\mathrm{S}^e_n(\mathcal{C}),\Qoppa^{\mathrm{s}}_{\mathrm{D}_n}) \simeq \mathrm{Q}_n(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{D}})$$

constructed above implies that

$$(\mathrm{Cr}\mathrm{S}^e(\mathcal{C}))^{\mathrm{hC}_2} \simeq (\mathrm{Cr}\,\mathrm{Q}(\mathcal{C}))^{\mathrm{hC}_2} \simeq \mathrm{Pn}\,\mathrm{Q}(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{B}})$$

by Proposition [I].2.2.11, and therefore one obtains

$$\mathcal{GW}(\mathcal{C},\mathrm{D}_{\mathrm{B}}) = \mathrm{fib}(|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C},\Qoppa^{\mathrm{s}}_{\mathrm{B}})| \to |\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})|).$$

The proposition then follows from the metabolic fibre sequence of Corollary 4.1.6. □

B.1.2. **Remark.** Essentially the same argument identifies the functor $\mathrm{KH}\colon\ \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ arising from [HSV19, Definition 10.3] and the inclusion $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Wald}^{\mathrm{gd}}_\infty \subset \mathrm{DWald}^{\mathrm{gd}}_\infty$ constructed in [HSV19, Section 6] with our $\mathcal{GW}$, as we shall now indicate. First up, the inclusion $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Wald}^{\mathrm{gd}}$ is (in our notation) given by

$$(\mathcal{C},\Qoppa) \quad\longmapsto\quad (\mathcal{C},\mathrm{D}_\Qoppa,\mathrm{He}(\mathcal{C},\Qoppa)\xrightarrow{\mathrm{fgt}}\mathrm{He}(\mathcal{C},\Qoppa^{\mathrm{s}}),\mathrm{triv})$$

where triv denotes the trivial Waldhausen structure with all maps cofibrations and only the equivalences being weak equivalences. By [HSV19, Theorem 6.27], it identifies Poincaré ∞-categories with stable ∞-categories with genuine duality (and trivial Waldhausen structure, which we shall henceforth suppress from notation). Let us denote the Poincaré structure associated to a stable ∞-category with genuine duality $(\mathcal{C},\mathrm{D},\phi)$ by $\Qoppa^{\phi}_{\mathrm{D}}$. The construction of the equivalence above is such that, essentially by definition, we find

$$\mathrm{Pn}(\mathcal{C},\Qoppa^{\phi}_{\mathrm{D}}) \simeq \mathrm{Cr}(\mathcal{C},\mathrm{D},\phi)^{\mathrm{C}_2}$$

for every stable ∞-category with genuine duality.

Now, by [HSV19, Proposition 10.2], the space $\mathrm{KH}(\mathcal{C},\mathrm{D},\phi)$ is identified with the fibre of the forgetful map $|\mathrm{S}^e(\mathcal{C},\mathrm{D},\phi)^h| \to |\mathrm{Cr}\mathrm{S}^e(\mathcal{C})|$, so just as in the previous proof, we only need to identify $\mathrm{S}^e(\mathcal{C},\mathrm{D},\phi)^h$ with $\mathrm{Q}(\mathcal{C},\Qoppa^{\phi}_{\mathrm{D}})$ under the equivalence above.

The simplicial space $\mathrm{S}^e(\mathcal{C},\mathrm{D},\phi)^h$ is in turn defined by performing the cotensor construction of $(\mathcal{C},\mathrm{D},\phi)$ against the categories $\mathrm{Ar}(\Delta^n * (\Delta^n)^{\mathrm{op}})$ equipped with the flip involution, isolating the edge-wise subdivision of the S-construction in these ∞-categories with genuine duality, and then taking fixed points of the core, degreewise; see [HSV19, Section 8.2] and the discussion before [HSV19, Proposition 10.2]. The functor p_n from the proof above again identifies this with the full subcategory of the cotensor construction of $(\mathcal{C}, \mathrm{D}, \phi)$ against $\mathrm{TwAr}(\Delta^n)$ (with trivial duality) carved out by the conditions defining the Q-construction. But as the cotensor $(\mathcal{C}, \mathrm{D}, \phi)^K$ always has underlying ∞-category $\mathrm{Fun}(K, \mathcal{C})$, one finds that $\mathrm{Cat}^{\mathrm{p}}_\infty \subset \mathrm{Wald}^{\mathrm{gd}}_\infty$ is closed under such cotensors, which, comparing universal properties, therefore necessarily agree with those constructed in Section [I].6.3. We thus find

$$\mathrm{S}^e(\mathcal{C}, \mathrm{D}, \phi)^h \simeq \mathrm{Cr}\,\mathrm{Q}(\mathcal{C}, \mathrm{D}, \phi)^{\mathrm{C}_2} \simeq \mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, Ϙ^{\phi}_{\mathrm{D}})$$

and thus

$$\mathrm{KH}(\mathcal{C}, \mathrm{D}, \phi) \simeq \mathrm{fib}(|\mathrm{Pn}\,\mathrm{Q}(\mathcal{C}, Ϙ^{\phi}_{\mathrm{D}})| \xrightarrow{\mathrm{fgt}} |\mathrm{Cr}\,\mathrm{Q}(\mathcal{C})|) \simeq \mathcal{GW}(\mathcal{C}, Ϙ^{\phi}_{\mathrm{D}})$$

as desired.

As mentioned above, we expect the stronger result

$$\mathrm{KR}(\mathcal{C}, \mathrm{D}, \phi) \simeq \tau_{\geq 0}\,\mathrm{KR}(\mathcal{C}, Ϙ^{\phi}_{\mathrm{D}})$$

for the functor $\mathrm{KR} : \mathrm{Wald}^{\mathrm{gd}}_\infty \to \mathcal{S}p^{\mathrm{gC}_2}$ constructed in [HSV19, Corollary 10.35], but we will not attempt to prove this here.

B.2. **Schlichting's Grothendieck-Witt-spectrum of a ring with 2 invertible.** We now turn to the more delicate comparison to the classical set-up of exact categories with duality from [Sch10a] and [Sch10b]. It consists of an additive (ordinary) category $\mathcal{E}$, equipped with three special types of arrows, namely, inflations, deflations and weak equivalences, satisfying suitable properties, as well as a duality $\mathrm{D} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$, which switches between the inflations and deflations, and preserves weak equivalences. A symmetric object in $\mathcal{E}$ is then an object $X \in \mathcal{E}$ equipped with a self-dual map $\phi : X \to \mathrm{D}X$. Such a symmetric object is said to be Poincaré ϕ is a weak equivalence. Let us denote by $\mathrm{Poi}(\mathcal{E}, W, \mathrm{D})$ the category whose objects are the Poincaré objects (X, ϕ) in $\mathcal{E}$ and the morphisms are the weak equivalences $f : X \to X'$ such that $\mathrm{D}(f)\phi' f = \phi$. Similarly, let $\mathrm{Cor}(\mathcal{E}, W)$ denote the subcategory of $\mathcal{E}$ containing only the weak equivalences as morphisms. In both cases, we sometimes suppress W and D to declutter the notation.

To define a Grothendieck-Witt space in this context, one again uses the edgewise subdivision $\mathrm{S}^e_n\mathcal{E}$ of the S-construction (see the previous section for a recollection), which in the case at hand inherits an exact structure with pointwise weak equivalences, and the duality explained in the previous section; unwinding it is given on objects by sending a diagram X to

$$(\mathrm{D}X)(i_\epsilon \leq j_\delta) = \mathrm{D}(X(j_{1-\delta} \leq i_{1-\epsilon})).$$

One then defines the associated Grothendieck-Witt space $\mathcal{GW}(\mathcal{E}, W, \mathcal{D})$ as the fibre of the map

$$|\mathrm{Poi}(\mathrm{S}^e\mathcal{E})| \to |\mathrm{Cor}(\mathrm{S}^e\mathcal{E})|.$$

For a ring R and a (discrete) invertible R-module with involution M, one can consider the category $\mathrm{Proj}(R)$ of finitely generated projective R-modules as an exact category (with inflations the split injections and conflations the split surjections), with the duality $\mathrm{D}_M : \mathrm{Proj}(R) \to \mathrm{Proj}(R)^{\mathrm{op}}$ given by $X \mapsto \mathrm{Hom}_R(X, M)$ and weak equivalences the isomorphisms. Under the assumption that 2 is invertible in R Schlichting then proves that $\mathcal{GW}(\mathrm{Proj}(R), \mathrm{Iso}, \mathrm{D}_M)$ is naturally equivalent to the group completion of the symmetric monoidal E_∞-space $|\mathrm{Unimod}(R, M)| = |\mathrm{Poi}(\mathrm{Proj}(R), \mathrm{Iso}, \mathrm{D}_M)|$, see [Sch17, Appendix A].

On the other hand, one may also consider the exact category $\mathrm{Ch}^{\mathrm{b}}(R)$ of bounded chain complexes in $\mathrm{Proj}(R)$ with weak equivalences being the quasi-isomorphisms and with the exact structure and duality induced by those of $\mathrm{Proj}(R)$. Schlichting then shows that the natural map $\mathcal{GW}(\mathrm{Proj}(R), \mathrm{Iso}, \mathrm{D}_M) \to \mathcal{GW}(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}_M)$ is an equivalence, see [Sch10b, Proposition 6]; this does not require 2 being invertible in R.

The advantage of working with $\mathrm{Ch}^{\mathrm{b}}(R)$ instead of $\mathrm{Proj}(R)$ is that it enables one to refine the above definition into a Grothendieck-Witt spectrum. For this, one considers the shifted duality $\mathrm{D}^{[n]}_M : \mathrm{Ch}^{\mathrm{b}}(R) \to \mathrm{Ch}^{\mathrm{b}}(R)^{\mathrm{op}}$ obtained by post-composing D_M with the n-th suspension functor sending C to the shifted complex $C[n]$ defined by $C[n]_i = C_{i-n}$. Schlichting's Grothendieck-Witt (pre-)spectrum $\mathrm{GW}(R, M)$ is then defined as the sequence of spaces

$$\mathrm{GW}(R, M) = \big(|\mathrm{Poi}(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}_M)|, |\mathrm{Poi}\left(\mathrm{S}^e(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}^{[1]}_M)\right)|, |\mathrm{Poi}\left((\mathrm{S}^e)^{(2)}(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}^{[2]}_M)\right), \dots\big)$$

with bonding maps induced by the map into the 1-simplices

$$\mathrm{Poi}\left(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}_M^{[n]}\right) \longrightarrow \mathrm{Poi}\left(\mathrm{S}_1^e(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}_M^{[n+1]})\right)$$

given by the duality preserving functor $\mathrm{Ch}^{\mathrm{b}}(R) \to \mathrm{S}_1^e(\mathrm{Ch}^{\mathrm{b}}(R))$ that sends a chain complex C to the diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & 0 \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
& & 0 \arrow[r] & C[1] \arrow[d] \\
& & & 0
\end{tikzcd}$$

of $\mathrm{S}_1^e \mathrm{Ch}^{\mathrm{b}}(R) = \mathrm{S}_3 \mathrm{Ch}^{\mathrm{b}}(R)$.

B.2.1. **Remark.** The Grothendieck-Witt spectrum of [Sch17, Section 5] is defined more generally for dg-categories with duality, in which case the shifted duality requires a more careful construction. When applied to $\mathrm{Ch}^{\mathrm{b}}(R)$, this construction yields a different, but equivalent model for $(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{D}^{[n]})$, see the remarks immediately following [Sch17, (5.1)].

We refrain from carrying out the necessarily more elaborate comparison at this level of generality, as the present one suffices for our applications in Paper [III].

Now, recall that $\mathcal{D}^{\mathrm{p}}(R)$ is the ∞-categorical localisation of $\mathrm{Ch}^{\mathrm{b}}(R)$ with respect to quasi-isomorphisms. The duality $\mathrm{D}_M^{[n]}$ then induces the (opposite of the) duality on $\mathcal{D}^{\mathrm{p}}(R)$ associated to $(\Qoppa^{\mathrm{s}}_M)^{[n]}$, which we will also denote by $\mathrm{D}_M^{[n]}$. The localisation functor $\mathrm{Ch}^{\mathrm{b}}(R) \to \mathcal{D}^{\mathrm{p}}(R)$ then determines a compatible collection of duality preserving functors

$$(\mathrm{S}^e)^{(n)}(\mathrm{Ch}^{\mathrm{b}}(R)) \longrightarrow (\mathrm{S}^e)^{(n)}(\mathcal{D}^{\mathrm{p}}(R)) \longrightarrow \mathrm{Q}^{(n)}(\mathcal{D}^{\mathrm{p}}(R));$$

see the discussion after Proposition B.1.1. Using Proposition [I].2.2.11, it induces a compatible collection of maps

$$|\,\mathrm{Poi}((\mathrm{S}^e)^{(n)}(\mathrm{Ch}^{\mathrm{b}}(R), \mathrm{qIso}, \mathrm{D}_M^{[n]}))| \longrightarrow \mathrm{Pn}(\mathrm{Q}^{(n)}(\mathcal{D}^{\mathrm{p}}(R), (\Qoppa^s_M)^{[n]})) \ , \tag{73}$$

which fit together to give a natural map of spectra

$$\mathrm{GW}(R, M) \longrightarrow \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{\mathrm{s}}_M). \tag{74}$$

Our goal in this subsection is to prove:

B.2.2. **Proposition.** *Let M be a (discrete) invertible R-module with involution, such that 2 is invertible in R. Then the map (74) is an equivalence of spectra.*

We also show that Schlichting's definition of the Grothendieck-Witt space of an exact category in which 2 is invertible agrees with ours. To this end, let $\mathcal{E}$ be an exact category with duality D and weak equivalences W. We say that $\mathcal{E}$ is *homotopically sound* if the collection of deflations and weak equivalences on $\mathcal{E}$ exhibits it as a category of fibrant objects in the sense of [Cis19, Definition 7.5.7]. Since the duality D preserves weak equivalences and switches between inflations and deflations, this is also equivalent to saying that the collection of inflations and weak equivalences on $\mathcal{E}$ exhibits it as a category of cofibrant objects. In this case, we denote by $\mathcal{E}[W^{-1}]$ the ∞-categorical localisation of $\mathcal{E}$ with respect to the collection of weak equivalences.

B.2.3. **Proposition.** *Suppose that $\mathcal{E}$ is a homotopically sound, exact category with duality D and weak equivalences W, in which 2 is invertible. Suppose further that $\mathcal{E}[W^{-1}]$ is stable. Then the natural map*

$$|\,\mathrm{Poi}(\mathcal{E}, W, \mathrm{D})| \to \mathrm{Pn}(\mathcal{E}[W^{-1}], \Qoppa^{\mathrm{s}}_{\mathrm{D}})$$

is an equivalence.

Proof. Put $\mathcal{E}[W^{-1}] = \mathcal{E}_\infty$ for readability. Consider then the diagram

$$\begin{tikzcd}
\mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \arrow[r] \arrow[d] & \mathrm{Pn}(\mathcal{E}_\infty, \Qoppa^{s}_{\mathrm{D}}) \arrow[d] \\
W \arrow[r] & \mathrm{Cr}\,\mathcal{E}_\infty
\end{tikzcd} \tag{75}$$

where the vertical functors are forgetful. By [Cis19, Corollary 7.6.9], the bottom horizontal map becomes an equivalence upon realisation. By Quillen's Theorem B, it hence suffices to show that for every $X \in \mathcal{E}$, the map

$$\mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W W_{/X} \longrightarrow \mathrm{Pn}(\mathcal{E}_\infty, \Qoppa^{\mathrm{s}}_{\mathrm{D}}) \times_{\mathrm{Cr}\,\mathcal{E}_\infty} (\mathrm{Cr}\,\mathcal{E}_\infty)_{/X} \tag{76}$$

is an equivalence after realisation. Let $\mathcal{I}_X \subseteq W_{/X}$ be the full subcategory spanned by the deflations $Y \twoheadrightarrow X$ that are also weak equivalences. We claim that the map

$$i\colon\ \mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W \mathcal{I}_X \longrightarrow \mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W W_{/X} \tag{77}$$

induces an equivalence on realisations. To see this, let $X \to X^I \to X \times X$ be a path object for X, whose existence is guaranteed by our assumption that $\mathcal{E}$ is a category of fibrant objects with respect to deflations. Construct a functor

$$q\colon\ \mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W W_{/X} \to \mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W \mathcal{I}_X$$

by sending $(q\colon\ Y \to \mathrm{D}Y, Y \to X)$ to $(\mathrm{Dpr} \circ q \circ \mathrm{pr}, Y \times_X X^I \twoheadrightarrow X)$, where $\mathrm{pr}\colon\ Y \times_X X^I \to Y$ is the projection to the first component. The natural map $(q, Y \to X) \to (\mathrm{Dpr} \circ q \circ \mathrm{pr}, Y \times_X X^I)$ induced by the structure map $X \to X^I$ then determines natural transformations $\mathrm{id} \Rightarrow q \circ i$ and $\mathrm{id} \Rightarrow i \circ q$, showing that (77) is an equivalence after realisation. It will hence suffice to show that the map

$$|\,\mathrm{Poi}(\mathcal{E}, W, \mathrm{D}) \times_W \mathcal{I}_X| \xrightarrow{\ i\ } \mathrm{Pn}(\mathcal{E}_\infty, \Qoppa^{s}) \times_{\mathrm{Cr}\,\mathcal{E}_\infty} (\mathrm{Cr}\,\mathcal{E}_\infty)_{/X} \tag{78}$$

is an equivalence. We now observe that the left vertical map in (75) is a right fibration classified by the functor $X \mapsto \mathrm{Hom}_W(X, \mathrm{D}X)^{C_2} \simeq \mathrm{Hom}_W(X, \mathrm{D}X)^{\mathrm{hC}_2}$; recall that $\mathrm{Set} \subset \mathcal{S}$ is closed under limits. Similarly, the right vertical map is classified by $X \mapsto \mathrm{Map}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X, \mathrm{D}X)^{\mathrm{hC}_2}$; since $\mathrm{Cr}(\mathcal{E}_\infty)_{/X}$ is contractible, we do not need the full statement here, but rather only that the fibre of $\mathrm{Pn}(\mathcal{E}_\infty, \Qoppa^{s}_{\mathrm{D}}) \to \mathrm{Cr}(\mathcal{E}_\infty)$ over a point X is given by $\mathrm{Map}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X, \mathrm{D}X)^{\mathrm{hC}_2}$. This follows from the general fact that for a C_2-space $X \in \mathcal{S}^{\mathrm{hC}_2}$ the fibre of $X^{\mathrm{hC}_2} \to X$ over some $x \in X$ si either empty or may be computed as $\mathrm{Map}_x(\mathrm{S}^\sigma, X)^{\mathrm{hC}_2}$ from the fibre sequence

$$\mathrm{Map}_x(\mathrm{S}^\sigma, X) \longrightarrow \mathrm{Map}(*, X) \longrightarrow \mathrm{Map}(\mathrm{C}_2, X).$$

Since total spaces of right fibrations are given as the opposites of the colimits in Cat_∞ of their classified functors by [Lur09a, Corollary 3.3.4.6], and thus their realisation as the colimits in $\mathcal{S}$, we may identify (78) with the natural map

$$\operatorname*{colim}_{[Y \twoheadrightarrow X] \in \mathcal{I}_X^{\mathrm{op}}} \mathrm{Hom}_W(Y, \mathrm{D}Y)^{\mathrm{hC}_2} \longrightarrow \mathrm{Hom}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X, \mathrm{D}X)^{\mathrm{hC}_2} \tag{79}$$

in $\mathcal{S}$. Now, since 2 is assumed invertible in $\mathcal{E}$, multiplication by 2 acts invertibly on the E_∞-groups $\mathrm{Hom}_{\mathcal{E}}(Y, \mathrm{D}Y)$, which is of course an ordinary abelian group, and $\mathrm{Hom}_{\mathcal{E}_\infty}(Y, \mathrm{D}Y)$. It follows that the norm map identifies their homotopy fixed points with their homotopy orbits (in E_∞-groups). In particular, the homotopy fixed point functor commutes with colimits of $\mathcal{E}_\infty$-groups in which 2 is invertible. Now note that the category $\mathcal{I}_X$ admits products (given by fibre products in $\mathcal{E}$ over X), and so $\mathcal{I}_X^{\mathrm{op}}$ is sifted in the ∞-categorical sense. Since the forgetful functor from $\mathcal{E}_\infty$-groups to spaces preserves sifted colimits by [Lur17, Proposition 1.4.3.9], we conclude that

$$\operatorname*{colim}_{[Y \twoheadrightarrow X] \in \mathcal{I}_X^{\mathrm{op}}} \mathrm{Hom}_W(Y, \mathrm{D}Y)^{\mathrm{hC}_2} \simeq \left[\operatorname*{colim}_{[Y \twoheadrightarrow X] \in \mathcal{I}_X^{\mathrm{op}}} \mathrm{Hom}_W(Y, \mathrm{D}Y) \right]^{\mathrm{hC}_2},$$

so it suffices to establish that

$$\operatorname*{colim}_{[Y \to X] \in \mathcal{I}_X^{\mathrm{op}}} \mathrm{Hom}_W(Y, \mathrm{D}Y) \longrightarrow \mathrm{Hom}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X, \mathrm{D}X) \tag{80}$$

is an equivalence. Since $\mathcal{J}_X^{\mathrm{op}}$ is sifted, the map induced by the diagonal

$$\operatorname*{colim}_{[Y\twoheadrightarrow X]\in\mathcal{J}_X^{\mathrm{op}}}\mathrm{Hom}_{W}(Y,\mathrm{D}Y)\longrightarrow\operatorname*{colim}_{[Y\twoheadrightarrow X,Z\twoheadrightarrow X]\in\mathcal{J}_X^{\mathrm{op}}\times\mathcal{J}_X^{\mathrm{op}}}\mathrm{Hom}_{W}(Y,\mathrm{D}Z)$$

is an equivalence. We have thus reduced to showing that the natural map

$$\operatorname*{colim}_{[Y\twoheadrightarrow X,Z\twoheadrightarrow X]\in\mathcal{J}_X^{\mathrm{op}}\times\mathcal{J}_X^{\mathrm{op}}}\mathrm{Hom}_{W}(Y,\mathrm{D}Z)\longrightarrow\mathrm{Hom}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X,\mathrm{D}X)\tag{81}$$

is an equivalence. Since the duality switches inflations and deflations, we may rewrite this as

$$\operatorname*{colim}_{[Y\twoheadrightarrow X,\mathrm{D}X\hookrightarrow Z']\in\mathcal{J}_X^{\mathrm{op}}\times\mathcal{J}_{\mathrm{D}X}}\mathrm{Hom}_{W}(Y,Z')\longrightarrow\mathrm{Hom}_{\mathrm{Cr}\,\mathcal{E}_\infty}(X,\mathrm{D}X),\tag{82}$$

where $\mathcal{J}_{\mathrm{D}X}$ denotes the subcategory of $W_{\mathrm{D}X/}$ spanned by the inflations. Now, this last map is an equivalence on general grounds; it is one formula for derived mapping spaces in categories of fibrant/cofibrant objects [Cis10, Proposition 3.23]. ☐

B.2.4. **Lemma.** *Suppose that $\mathcal{E}$ is homotopically sound. Then for every $n\geq 0$, the exact category with weak equivalences $\mathrm{S}_n\mathcal{E}$ is homotopically sound and the natural functor $\mathrm{S}_n\mathcal{E}\to\mathrm{S}_n(\mathcal{E}[W^{-1}])$ exhibits the ∞-category $\mathrm{S}_n(\mathcal{E}[W^{-1}])$ as the localisation of $\mathrm{S}_n\mathcal{E}$ with respect to the pointwise weak equivalences.*

Proof. We note that $\mathrm{S}_n\mathcal{E}$ is equivalent to the category of sequences of inflations

$$X_1\hookrightarrow X_2\hookrightarrow\dots\hookrightarrow X_n$$

with the inflations in $\mathrm{S}_n\mathcal{E}$ being the Reedy inflations. It is then standard that if $\mathcal{E}$ is category of cofibrant objects, then the collection of Reedy inflations exhibit $\mathrm{S}_n\mathcal{E}$ as a category of cofibrant objects, see e.g. [Cis19, Theorem 7.4.20 & Example 7.5.8]. On the other hand, $\mathrm{S}_n(\mathcal{E}[W^{-1}])$ is equivalent to the ∞-category $\mathrm{Fun}(\Delta^n,\mathcal{E}[W^{-1}])$ of sequences of $n-1$ composable maps in $\mathcal{E}[W^{-1}]$. The fact that $\mathrm{Fun}(\Delta^n,\mathcal{E}[W^{-1}])$ is the ∞-categorical localisation of the category of Reedy sequences of inflations then follows from [Cis19, Theorems 7.5.18 & 7.6.17]. ☐

Proof of Proposition B.2.2. Let us denote the duality induced by M simply by D, and the induced duality on the r-fold S-construction by $\mathrm{D}_{(r)}$. Applying Proposition B.2.3 to the levels of the multisimplicial exact category with duality $(\mathrm{S}^e)^{(r)}(\mathrm{Ch}^{\mathrm{b}}(R),\mathrm{qIso},\mathrm{D}_M^{[r]})$, which is possible by Lemma B.2.4, we obtain an equivalence of Schlichting's $\mathrm{GW}(R,M)$ to the (pre-)spectrum formed by the sequence

$$\big(\mathrm{Pn}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa^{\mathrm{s}}_{\mathrm{D}}),|\mathrm{Pn}(\mathrm{S}^e(\mathcal{D}^{\mathrm{p}}(R)),\Qoppa^{\mathrm{s}}_{\mathrm{D}^{[1]}_{(1)}})|,|\mathrm{Pn}((\mathrm{S}^e)^{(2)}(\mathcal{D}^{\mathrm{p}}(R)),\Qoppa^{\mathrm{s}}_{\mathrm{D}^{[2]}_{(2)}})|,\dots\big)$$

But the latter agrees with

$$\big(\mathrm{Pn}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa^{\mathrm{s}}_{\mathrm{D}}),|\mathrm{Pn}\,\mathrm{Q}(\mathcal{D}^{\mathrm{p}}(R),(\Qoppa^{\mathrm{s}}_{\mathrm{D}})^{[1]})|,|\mathrm{Pn}\,\mathrm{Q}^{(2)}(\mathcal{D}^{\mathrm{p}}(R),(\Qoppa^{\mathrm{s}}_{\mathrm{D}})^{[2]})|,\dots\big)=\mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa^{\mathrm{s}}_{\mathrm{D}})$$

termwise by the discussion following Proposition B.1.1, and one readily checks that the bonding maps correspond as well. ☐

B.2.5. **Corollary.** *Let $\mathcal{E}$ be a homotopically sound, exact category with duality* D *and weak equivalences W, in which 2 is invertible, and such that $\mathcal{E}[W^{-1}]$ is stable. Then, there is a canonical equivalence*

$$\mathcal{GW}(\mathcal{E},W,\mathrm{D})\simeq\mathcal{GW}(\mathcal{E}[W^{-1}],\Qoppa^{\mathrm{s}}_{\mathrm{D}}).$$

Proof. From Proposition B.2.3 and Lemma B.2.4, we find that the defining map $|\,\mathrm{Poi}(\mathrm{S}^e\mathcal{E})|\to|\mathrm{Cor}(\mathrm{S}^e\mathcal{E})|$ is equivalently given by

$$|\mathrm{Pn}(\mathrm{S}^e(\mathcal{E}[W^{-1}]),\Qoppa^{\mathrm{s}}_{\mathrm{D}_{(1)}})|\to|\mathrm{Cr}(\mathrm{S}^e(\mathcal{E}[W^{-1}])|.$$

But the discussion following Proposition B.1.1 identifies this further with

$$|\mathrm{Pn}\,\mathrm{Q}(\mathcal{E}[W^{-1}],\Qoppa^{\mathrm{s}}_{\mathrm{D}})|\longrightarrow|\mathrm{Cr}\,\mathrm{Q}(\mathcal{E}[W^{-1}])|$$

and so Corollary 4.1.6 gives the claim. ☐

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