A MOTIVIC SPECTRUM REPRESENTING HERMITIAN K-THEORY

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ABSTRACT. In this paper we establish some fundamental motivic aspects of hermitian K-theory without assuming that 2 is invertible on the base scheme. In particular, we prove that both quadratic and symmetric Grothendieck-Witt theory satisfy Nisnevich descent, and that symmetric Grothendieck-Witt theory further satisfies dévissage and $\mathbb{A}^1$-invariance over a regular Noetherian base of finite Krull dimension, as well as a projective bundle formula. We use this to show that over a regular Noetherian base, symmetric Grothendieck-Witt theory is represented by a motivic E_∞-ring spectrum, which we then show is an absolutely pure spectrum, answering a question of Déglise. As with algebraic K-theory, we show that over a general base, one can also construct a hermitian K-theory motivic spectrum, representing this time a suitable homotopy invariant and Karoubi-localising version of Grothendieck-Witt theory.

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1. Introduction

Originally justified by Voevodsky's proof of Milnor's conjecture, motivic homotopy theory is now generally recognized as a deep and efficient framework to study cohomological invariants of rings and schemes. A key insight of this approach is that such invariants can be studied in a manner closely resembling that taken up by classical algebraic topology in the study of cohomological invariants of topological spaces. More precisely, the latter can be axiomatised via the notion of generalized cohomology theories, which in turn can be classified by the notion of a spectrum. Similarly, cohomological invariants of schemes can be classified by a suitable notion of a motivic spectrum. For example, motivic cohomology, algebraic K-theory, and algebraic cobordism, which are algebraic analogues of singular cohomology, complex K-theory, and cobordism, are each represented by a corresponding motivic spectrum.

On the side of topological spaces, another generalized cohomology theory which plays an important role in the study of smooth manifolds, is real K-theory. Its analogue on the side of schemes is hermitian K-theory, also called Grothendieck-Witt theory, which is the K-theory of unimodular quadratic forms. This somewhat less obvious analogy is due to the inclusions $\mathrm{O}_n(\mathbb{C}) \supseteq \mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ both being homotopy equivalences, so that the K-theory of real vector bundles coincides with that of complex vector bundles equipped with non-degenerate quadratic forms.

The behaviour of hermitian K-theory is notoriously sensitive to the prime 2. In particular, its behaviour simplifies considerably when considering schemes over which 2 is invertible. As a result, until recently a significant part of the relevant theory was developed under this assumption, and in particular, the realization of hermitian K-theory as a motivic spectrum, as was first done by Hornbostel [Hor05], was limited to $\mathbb{Z}[\frac{1}{2}]$-schemes.

In a recent series of papers [CDH+I, CDH+II, CDH+III, CDH+IV, CDH+V] by a group of authors including the authors of the present paper, the foundations of hermitian K-theory were revisited using the powerful recent technology of higher category theory, in a manner which enables one to dispense with the invertibility of 2 assumption. The goal of the present paper is to harness these results in the service of the motivic aspects of hermitian K-theory, allowing a base scheme S in which 2 is not assumed invertible. The first and primary outcome of our work is the following result, which is itself a prerequisite to any further investigations in the motivic direction:

Theorem 1.0.1. *Let S be a regular Noetherian scheme. Then symmetric Grothendieck-Witt theory is represented by a motivic spectrum over S.*

Let us take a moment to clarify what the term "symmetric" means in this context. First, note that when 2 is invertible, the notions of quadratic form and of symmetric bilinear form are essentially equivalent: every symmetric bilinear form admits a unique quadratic refinement. This is no longer true when 2 is not invertible, so in general, one needs to specify with respect to which flavour of forms Grothendieck-Witt theory is taken. For example, in the above theorem, we make reference to symmetric bilinear forms.

There are, however, more then one variant of those as well. Recall that by the Gillet-Waldhausen theorem, the algebraic K-theory of a ring can be calculated either by considering finitely generated projective modules and then taking group completion, or by considering all perfect complexes, in which case one needs to enforce relations for every exact sequence of such. Passing to the hermitian setting, projective modules can be endowed with symmetric bilinear forms in the usual manner, while perfect complexes can be endowed with a homotopical analogue of symmetric forms, where the symmetry property is interpreted as a homotopy fixed point structure. When 2 is not invertible, these two constructions lead to different Grothendieck-Witt groups (or spectra) in general.

In the paper series mentioned above, we use the term symmetric Grothendieck-Witt to refer to the latter construction using homotopy fixed points, and introduce the term *genuine symmetric* to refer to the former. While the genuine symmetric is the one more directly related to classical constructions in hermitian K-theory, the (non-genuine) symmetric admits better formal properties. Most notably from the present viewpoint, it is $\mathbb{A}^1$-invariant over regular Noetherian schemes, which is the main reason why it is this flavour of Grothendieck-Witt theory that features in Theorem 1.0.1. It also underlines a possible reason for why the motivic hermitian K-theory spectrum was previously only constructed when 2 is invertible: without this assumption, no flavour of Grothendieck-Witt groups defined using a classical notion of forms is $\mathbb{A}^1$-invariant (even over regular Noetherian bases); see Example 6.3.2.

While Theorem 1.0.1 is a necessary first step, it still leaves several obvious questions unanswered:

(1) What happens if S is not regular Noetherian?
(2) What about Witt theory, obtained from Grothendieck-Witt theory in principle by quotienting out the hyperbolic forms?
(3) What about the multiplicative structure of Grothendieck-Witt theory?
(4) What about all the natural maps relating Grothendieck-Witt theory, Witt theory, and algebraic K-theory?
(5) What happens to some of the expected properties of motivic Grothendieck-Witt theory, such as purity or the projective bundle formula, when 2 is not invertible?

In the present paper, we continue beyond Theorem 1.0.1 by giving complete answers to all the above questions. To describe how this is done, let us furthermore recall the theoretical framework of [CDH+I] and its sequels. The main point is that Grothendieck-Witt theory should be considered as an invariant defined on *Poincaré ∞-categories*. These are stable ∞-categories equipped with a spectrum valued functor $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ satisfying suitable axioms. These axioms guarantee in particular the existence of a uniquely determined duality $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that $\Omega(x \oplus y) = \Omega(x) \oplus \Omega(y) \oplus \hom(x, \mathrm{D}y)$. The functor Ω is the called a *Poincaré structure* on $\mathcal{C}$. For an object $x \in \mathcal{C}$, the underlying infinite loop space $\Omega^\infty \Omega(x)$ should be thought of as the space of forms on x. In particular, we call a point $\beta \in \Omega^\infty \Omega(x)$ a *form* on x. Such a form determines a map $\beta_\sharp\colon x \to \mathrm{D}x$, and we say that β is a *Poincaré form* if $\beta_\sharp$ is an equivalence. Poincaré forms are what corresponds in this setting to non-degenerate forms. A pair (x, β) of an object equipped with a Poincaré form $\beta \in \Omega^\infty \Omega(x)$ is called a Poincaré object.

The fact that Ω is an extra piece of structure (and not an inherent property of $\mathcal{C}$) is what allows one to specify the flavour of forms one is interested in. For example, if R is a commutative ring, then we may consider its perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(R)$, whose objects are the perfect R-complexes, and consider the functor

$$\Omega^{\mathrm{s}}_R(M) := \hom(M \otimes_R M, R)^{\mathrm{hC}_2},$$

where $(-)^{\mathrm{hC}_2}$ denotes homotopy fixed points with respect to the C_2-action induced by swapping the two M factors. Then Ω^{s} sends M to what can be considered as the homotopy theoretical analogue of the spectrum of symmetric bilinear forms on M. Alternatively, we may take the Poincaré structure $\Omega^{\mathrm{q}}_R(M) := \hom(M \otimes_R M, R)_{\mathrm{hC}_2}$ obtained by using homotopy orbits instead of fixed points. This is the homotopy theoretical analogue of quadratic forms. As mentioned above, these are not exactly the same as working with symmetric and quadratic forms on projective modules. The latter notions can also be encoded via suitable Poincaré structures Ω^{gs} and Ω^{gq}, called the genuine symmetric and genuine quadratic structures. All these Poincaré structures are generally different from each other, and are related by a sequence of natural transformations

$$\Omega^{\mathrm{q}}_R \Rightarrow \Omega^{\mathrm{gq}}_R \Rightarrow \Omega^{\mathrm{gs}}_R \Rightarrow \Omega^{\mathrm{s}}_R.$$

When 2 is invertible in R, these maps are all equivalences.

To a Poincaré ∞-category $(\mathcal{C}, Ϙ)$, one may associate an invariant $\mathrm{GW}(\mathcal{C}, Ϙ)$, called the Grothendieck-Witt spectrum, whose homotopy groups are the Grothendieck-Witt groups of $(\mathcal{C}, Ϙ)$. By [HS21], in the case of $(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gs}})$ and $(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{gq}})$, these reproduce the classical symmetric and quadratic Grothendieck-Witt groups of R, though, as mentioned earlier, the Grothendieck-Witt groups of the Poincaré ∞-category $(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{s}}_R)$ have better formal properties, and play a more important role in the current paper. More generally, one may replace the commutative ring R by a scheme X, and consider the perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(X)$ of perfect complexes on X. As above, one can define the symmetric Poincaré structure $Ϙ^{\mathrm{s}}_X$ using the same formulas. More generally, given a line bundle L on X, we may consider the Poincaré structure $Ϙ^{\mathrm{s}}_L$ defined by $Ϙ^{\mathrm{s}}_L(M) = \hom(M \otimes_{\mathcal{O}_X} M, L)^{\mathrm{hC}_2}$. It is also useful to consider the slightly more general case where, on the one hand, L is endowed with an involution, and on the other hand is allowed to be a shift of a line bundle (or more generally any tensor invertible perfect complex). The Grothendieck-Witt groups of $(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_L)$ are the main invariants of schemes we consider in this paper. To keep the introduction easily readable, we consider only line bundles with constant involution when describing our main results, though we provide references to the body of the paper where the reader can find the full version of each result.

The collection of Poincaré ∞-categories can be organized into a (large) category $\mathrm{Cat}^{\mathrm{p}}$, which is presentable and compactly generated (see [CDH+IV]). The functor $\mathrm{GW}(-)$ is Verdier localising, that is, it sends fibre-cofibre sequences of Poincaré ∞-categories to exact sequences of spectra, and is furthermore the universal such functor equipped with a natural transformation $\mathrm{Pn} \Rightarrow \Omega^\infty \mathrm{GW}$, where Pn is the functor which associates to a Poincaré ∞-category its space of Poincaré objects. It turns out that the property that is more relevant from the motivic point of view is not being Verdier localising, but being Karoubi-localising, which is equivalent to being Verdier localising and invariant under idempotent completion. While the functor GW is not Karoubi-localising, it can be turned into one in a universal manner, yielding a functor $\mathbb{GW}$ we call the *Karoubi-Grothendieck-Witt* functor. It comes equipped with a canonical natural transformation $\mathrm{GW} \Rightarrow \mathbb{GW}$, which is generally not equivalence. This transformation is closely related to the transformation $\mathrm{K} \Rightarrow \mathbb{K}$ between algebraic K-theory and Bass K-theory (also known as non-connective K-theory). In particular, if $\mathcal{C}$ is such that $\mathrm{K}(\mathcal{C}) \to \mathbb{K}(\mathcal{C})$ is an equivalence (e.g., $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(X)$ for X a regular Noetherian scheme) then $\mathrm{GW}(\mathcal{C}, Ϙ) \to \mathbb{GW}(\mathcal{C}, Ϙ)$ is an equivalence, see [CDH+V].

The importance of the Karoubi-localising property in the process of passing from invariants of Poincaré ∞-categories to invariants of schemes is expressed by the following key result of the present paper (see Corollary 4.4.2 below):

Proposition 1.0.2 (Nisnevich descent)**.** *Let S be a quasi-compact quasi-separated base scheme and $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$ a Karoubi-localising spectrum valued functor. Then, the functor*

$$\mathcal{F}^{\mathrm{s}}_S : \mathrm{Sch}^{\mathrm{qq}}_{/S} \to \mathcal{S}\mathrm{p} \qquad X \mapsto \mathcal{F}(X, Ϙ^{\mathrm{s}}_X)$$

is a Nisnevich sheaf, where $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ is the category of quasi-compact quasi-separated S-schemes.

In particular, the functor $\mathbb{GW}^{\mathrm{s}}_S(X) = \mathbb{GW}(X, Ϙ^{\mathrm{s}}_X)$ is a Nisnevich sheaf on $\mathrm{Sch}^{\mathrm{qq}}_{/S}$. We may also restrict attention to the full subcategory $\mathrm{Sm}_S \subseteq \mathrm{Sch}_{/S}$ spanned by the (quasi-compact quasi-separated) smooth S-schemes, and consider $\mathbb{GW}^{\mathrm{s}}_S$ as a Nisnevich sheaf on that site. If the base scheme S itself is regular, then every smooth S-scheme is regular as well, and, as mentioned above, for a regular Noetherian scheme X, the map $\mathrm{GW}(X, Ϙ^{\mathrm{s}}_X) \to \mathbb{GW}(X, Ϙ^{\mathrm{s}}_X)$ is an equivalence. This means that for a regular Noetherian S, the functor $\mathrm{GW}^{\mathrm{s}}_S(X) = \mathrm{GW}(X, Ϙ^{\mathrm{s}})$ is a Nisnevich sheaf on Sm_S. Alternatively, one may replace GW by the functor L which associates to a Poincaré ∞-category its L-spectrum (whose homotopy groups correspond to higher Witt groups in algebraic geometry), and obtain using similar arguments that the symmetric L-theory functor $\mathrm{L}^{\mathrm{s}}_S$ is a Nisnevich sheaf on Sm_S for any regular Noetherian S.

Now a motivic spectrum over S is not just a Nisnevich sheaf on Sm_S: it possess more structure (a tower of $\mathbb{P}^1$-deloopings) and satisfies more axioms ($\mathbb{A}^1$-invariance). In order to incorporate the former into the construction of the motivic Grothendieck-Witt spectrum, we need to understand the behaviour of GW under taking products with $\mathbb{P}^1$. More generally, we consider projective line bundles which are not necessarily constant (and eventually also higher rank projective bundles; see Theorem 6.1.6 below for the full version of this result, where also the assumptions on $\mathcal{F}$ and L are weakened):

Proposition 1.0.3 (Projective line formula)**.** *Let X be a qcqs scheme equipped with a line bundle L and let V be a vector bundle over X of rank 2. Let $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$ be a Karoubi-localising functor. Then we have* *a split fibre sequence*

$$\mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L}\big) \to \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L}\big) \to \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\big) ,$$

where the first arrow is induced by pullback along p.

The last two propositions combined can be leveraged to obtain the following:

Proposition 1.0.4 (Bott periodicity)**.** *Let S be a quasi-compact quasi-separated base scheme. Then, the association $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{s}}_S$ described in Proposition 1.0.2 refines to a functor*

$$\mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat}^{\mathrm{p}}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(\mathrm{Sm}_S, \mathcal{S}\mathrm{p})) \qquad \mathcal{F} \mapsto \mathcal{R}^{\mathrm{s}}_S(\mathcal{F}),$$

where the left hand side is the ∞-category of (spectrum valued) Karoubi-localising functors on $\mathrm{Cat}^{\mathrm{p}}$ *and the right hand side the ∞-category of $\mathbb{P}^1$-spectrum objects in Nisnevich (spectrum valued) sheaves on* Sm_S*. Furthermore, the $\mathbb{P}^1$-deloopings of $\mathcal{R}^{\mathrm{s}}_S(\mathcal{F})$ are governed by the rule*

$$\Sigma^n_{\mathbb{P}^1}\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}) \simeq \mathcal{R}^{\mathrm{s}}_S(\mathcal{F}^{[n]})$$

where $\mathcal{F}^{[n]}$ is the functor $\mathcal{F}^{[n]}(\mathcal{C}, \Omega) = \mathcal{F}(\mathcal{C}, \Omega^{[n]}) = \mathcal{F}(\mathcal{C}, \Sigma^n\Omega)$.

Applying the last result to $\mathbb{GW}$ yields a $\mathbb{P}^1$-spectrum object $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW})$ (infinitely) delooping the Nisnevich sheaf $\mathbb{GW}^{\mathrm{s}}_S$. In general $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW})$ can fail to be a motivic spectrum, as it might not be $\mathbb{A}^1$-invariant. However, if S is regular Noetherian, then the $\mathbb{A}^1$-invariance of $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW}) = \mathcal{R}^{\mathrm{s}}_S(\mathrm{GW})$ is assured, as we establish in the following key result (see Theorem 6.3.1):

Theorem 1.0.5. *Let S be a regular Noetherian scheme. Then, the Nisnevich sheaf* $\mathrm{GW}^{\mathrm{s}}_S$ *is $\mathbb{A}^1$-invariant. Furthermore, all the components of its $\mathbb{P}^1$-delooping* $\mathcal{R}^{\mathrm{s}}_S(\mathrm{GW})$ *are $\mathbb{A}^1$-invariant, so that* $\mathcal{R}^{\mathrm{s}}_S(\mathrm{GW})$ *is a motivic spectrum.*

This means that for a regular Noetherian S, symmetric Grothendieck-Witt theory is represented by a motivic spectrum, which we denote by $\mathrm{KQ}_S \in \mathrm{SH}(S)$. This establishes Theorem 1.0.1 above. The proof of Theorem 1.0.5 is primarily based on another key property of independent interest of symmetric Grothendieck-Witt theory - dévissage - whose proof occupies §5 in its entirety (see Theorem 5.2.1 for the full version of the result):

Theorem 1.0.6 (Dévissage)**.** *Let $i : Z \hookrightarrow X$ be a closed embedding of finite dimensional regular Noetherian schemes and L a line bundle on X. Then the map*

$$\mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\mathrm{s}}_{i^!L}\right) \to \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}_Z(X), \Omega^{\mathrm{s}}_L|_Z\right) .$$

induced by push-forward along i, is an equivalence.

If S is only assumed quasi-compact and quasi-separated, then the $\mathbb{P}^1$-spectrum $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW})$ is not necessarily a motivic spectrum. We can however turn it into a motivic spectrum in a universal manner by applying to it the $\mathbb{A}^1$-localisation functor

$$\mathrm{Loc}_{\mathbb{A}^1} : \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}(\mathrm{Sm}_{/S}, \mathcal{S}\mathrm{p})) \to \mathrm{SH}(S),$$

which can be written explicitly as a suitable geometric realization. We hence obtain a Grothendieck-Witt motivic spectrum $\mathrm{KQ}_S \in \mathrm{SH}(S)$ for an arbitrary quasi-compact quasi-separated S. These motivic spectra depend functorially on S, in the sense that if $f : T \to S$ is a map of quasi-compact quasi-separated schemes then we have an induced map

$$\eta_f : f^*\mathrm{KQ}_S \to \mathrm{KQ}_T .$$

We then verify the following property, which is of technical importance to future applications:

Proposition 1.0.7 (Base change invariance)**.** *The map η_f is an equivalence. In particular, the motivic spectra* KQ_S *are all pulled back from the absolute motivic spectrum* $\mathrm{KQ} := \mathrm{KQ}_{\mathrm{spec}(\mathbb{Z})} \in \mathrm{SH}(\mathrm{spec}(\mathbb{Z}))$.

In addition to Grothendieck-Witt theory, we also apply our construction to Karoubi L-theory $\mathbb{L}$ (the Karoubi-localising approximation of L-theory) and to Bass K-theory $\mathbb{K}$, thus yielding motivic spectra KW_S and KGL_S representing Karoubi Witt theory and Bass K-theory, respectively. Though the latter was previously constructed in the literature, reproducing the construction in the present setting allows one to use all the resulting functoriality. In particular, all the structural maps relating Grothendieck-Witt theory, K-theory and L-theory induce corresponding maps on the level of motivic spectra, and we obtain for example, the Tate fibre sequence

$$(\mathrm{KGL}_S)_{\mathrm{hC}_2} \to \mathrm{KQ}_S \to \mathrm{KW}_S$$

and the Wood sequence

$$\Omega_{\mathbb{P}^1}\,\mathrm{KQ}_S \to \mathrm{KGL}_S \to \mathrm{KW}_S\,.$$

Finally, we also consider the question of multiplicative structures on KQ, KW and the maps relating them to each other and to KGL. In fact, this accounts for most of the more technical parts of the paper, whose goal is to prove the following:

Proposition 1.0.8. *Let S be a quasi-compact quasi-separated base scheme. Then, the association $\mathcal{F} \mapsto \mathcal{R}^{\mathrm{s}}_S(\mathcal{F})$ described in Proposition 1.0.4 refines to a lax symmetric monoidal functor. In particular, if $\mathcal{F}$ is a Karoubi-localising lax symmetric monoidal functor then $\mathcal{R}^{\mathrm{s}}_S(\mathcal{F})$ is a commutative algebra object in the ∞-category $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(\mathrm{Sm}_S, \mathcal{S}\mathrm{p}))$ and $\mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_S(\mathcal{F})$ is a motivic commutative ring spectrum.*

This means in particular that KQ_S and KW_S are motivic commutative ring spectra, and that the canonical map $\mathrm{KQ}_S \to \mathrm{KW}_S$ is a commutative ring map. It also implies that the forgetful map $\mathrm{KQ}_S \to \mathrm{KGL}_S$ is one of motivic commutative ring spectra.

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2. Recollections on hermitian K-theory

We use the framework developed in [CDH+I, CDH+II]: hermitian K-theory is a functor sending a Poincaré ∞-category, i.e. a stable ∞-category endowed with a quadratic functor satisfying non-degeneracy conditions, to a spectrum. Let us walk the reader through some of the constructions and properties of this formalism.

2.1. Poincaré ∞-categories.

In the framework developed in [CDH+I] and [CDH+II], we view Grothendieck-Witt theory as an invariant of what we call a *Poincaré ∞-category*, a notion first introduced by Lurie [Lur11] as a novel framework for Ranicki's L-theory. By definition, a Poincaré ∞-category is a pair $(\mathcal{C}, \Omega)$ consisting of a (small) stable ∞-category $\mathcal{C}$ equipped with a *Poincaré structure* Ω, that is, a functor $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ which is reduced and 2-excisive in the sense of Goodwillie's functor calculus, and whose symmetric cross-effect $\mathrm{B}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is of the form $\mathrm{B}(X, Y) = \hom_{\mathcal{C}}(x, \mathrm{D}y)$ for some equivalence $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{C}$. In this case D is uniquely determined and endows $\mathcal{C}$ with the structure of a *perfect duality*, that is, a lift of $\mathcal{C}$ to a C_2-homotopy fixed point in $\mathrm{Cat}^{\mathrm{ex}}$ with respect to the op-action. The notion of a morphism of Poincaré ∞-categories $(\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ is that of a *Poincaré functor*, which consists of a pair (f, η) where $f\colon \mathcal{C} \to \mathcal{C}'$ is an exact functor and $\eta\colon \Omega \Rightarrow \Omega' \circ f^{\mathrm{op}}$ is a natural transformation for which the induced arrow $\mathrm{D}f^{\mathrm{op}} \Rightarrow f\mathrm{D}$ is an equivalence. In particular, Poincaré functors can be considered as a refinement of duality preserving functors. The collection of Poincaré ∞-categories and Poincaré functors between then assembles to form an ∞-category we denote by $\mathrm{Cat}^{\mathrm{p}}$.

We think of a Poincaré structure as providing an abstract notion of forms on the objects of $\mathcal{C}$. More precisely, we call points $q \in \Omega^\infty\Omega(x)$ hermitian forms on x, and we say that such a form is Poincaré if the map $q_\sharp\colon x \to \mathrm{D}x$ determined by q is an equivalence.

Example 2.1.1.

(1) Let $(\mathcal{C}, \mathrm{D})$ be a stable ∞-category with perfect duality. Then the functors

$$\Omega^{\mathrm{s}}(x) = \hom_{\mathcal{C}}(x, \mathrm{D}x)^{\mathrm{hC}_2} \qquad \Omega^{\mathrm{q}}(x) = \hom_{\mathcal{C}}(x, \mathrm{D}x)_{\mathrm{hC}_2}$$

are Poincaré structures with underlying duality D. We call them the symmetric and quadratic Poincaré structures associated to the duality $\mathcal{D}$, respectively. The corresponding notions of hermitian forms they encode can be considered as the homotopy coherent avatars of the classical notions of symmetric bilinear and quadratic forms, respectively.

(2) Let X be a scheme and L a line bundle. Then L determines a duality on the perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(X)$ of X, whose underlying duality $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(X)^{\mathrm{op}}$ sends M to the internal mapping object $\mathrm{D}_L(M) := \underline{\hom}_X(M, L)$. The symmetric and quadratic Poincaré structures associated to this duality are then denoted by Ω^{s}_L and Ω^{q}_L. More generally, we can introduce a C_2-action on L, which affects the structure of D_L as a duality (though not the underlying equivalence). The Poincaré ∞-categories of the form $(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L)$ constitute the main examples of interest for us in the present paper.

(3) For $\mathcal{C}$ a stable ∞-category consider $\overline{\mathrm{Hyp}}(\mathcal{C}) := \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ equipped with the Poincaré structure

$$\Omega_{\mathrm{hyp}}(X, Y) := \hom_{\mathcal{C}}(X, Y).$$

Then $\mathrm{Hyp}(\mathcal{C}) := (\overline{\mathrm{Hyp}}, \Omega_{\mathrm{hyp}})$ is a Poincaré ∞-category with underlying duality $\mathrm{D}_{\mathrm{hyp}}(X, Y) = (Y, X)$. We refer to it as the hyperbolic category of $\mathcal{C}$. We note that Ω_{hyp} is both the symmetric and the quadratic Poincaré structure associated to the duality $\mathrm{D}_{\mathrm{hyp}}$ in the sense of ((1)).

(4) If $(\mathcal{C}, \Omega)$ is a Poincaré ∞-category with underlying duality $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, then the pair $(\mathcal{C}, \Omega^{[n]}) := (\mathcal{C}, \Sigma^n \Omega)$ is a Poincaré ∞-category with underlying duality $\Sigma^n \mathrm{D}$.

(5) Let R be a discrete commutative ring. Then, there exists a unique Poincaré structure Ω^{gs} on the perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(R)$ of R whose value on any finitely generated projective module P is the (Eilenberg-Mac Lane spectrum of the) abelian group of symmetric bilinear forms on P. Similarly, there exists a unique Poincaré structure Ω^{gq} such that $\Omega^{\mathrm{gq}}(P)$ is the abelian group of quadratic forms on P for such P. We call these the *genuine symmetric* and *genuine quadratic* Poincaré structures, respectively. Their underlying duality is the usual duality $\mathrm{D}_R := \underline{\hom}(-, R)$, but Ω^{gs} and Ω^{gq} are generally *not* the associated symmetric and quadratic Poincaré structures in the sense of (1).

2.2. **Hermitian K-theory.** Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category.

A *Poincaré object* in $(\mathcal{C}, \Omega)$ is a pair (x, q) where x is an object of $\mathcal{C}$ and $q \in \Omega^\infty \Omega(x)$ is a Poincaré form on x, that is, a hermitian form such that the map $q_\sharp\colon X \to \mathrm{D}X$ determined by the image of q under $\Omega(x) \to \mathrm{B}(x, x) = \hom_{\mathcal{C}}(x, \mathrm{D}x)$, is an equivalence. We then write $\mathrm{Pn}(\mathcal{C}, \Omega) \subseteq \int_{\iota\mathcal{C}} \Omega^\infty$ for the full subgroupoid spanned by the Poincaré objects, where $\iota\mathcal{C}$ denotes the core ∞-groupoid of $\mathcal{C}$. The association $(\mathcal{C}, \Omega) \mapsto \mathrm{Pn}(\mathcal{C}, \Omega)$ assembles to a functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}$ by [CDH+I, Lemma 2.1.5], analogous to the groupoid core functor $\mathrm{Cr}\colon \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}$ yielding classes in K-theory, where $\mathrm{Cat}^{\mathrm{ex}}$ is the ∞-category of stable ∞-categories.

In order to describe the context in which hermitian K-theory naturally lives, we need to recall the notions of *additive*, *localising* and *Karoubi-localising* functors.

Both the ∞-categories $\mathrm{Cat}^{\mathrm{ex}}$ and $\mathrm{Cat}^{\mathrm{p}}$ are pointed (in fact, semi-additive), and we can consider fiber and cofibre sequences in them. A *Verdier* or *Poincaré-Verdier* sequence is a sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E} \qquad \text{or} \qquad (\mathcal{C}, \Omega) \to (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$$

that is both a fiber and a cofibre sequence in $\mathrm{Cat}^{\mathrm{ex}}$ or $\mathrm{Cat}^{\mathrm{p}}$ respectively. A Verdier sequence is furthermore *split* when both maps in the sequence have adjoints on both sides. Split Verdier sequences are equivalently described as stable recollements (see [CDH+II, Definition A.2.10 and Proposition A.2.11]). A Poincaré-Verdier sequence is *split* when its underlying Verdier sequence is (see [CDH+II, Definition 1.1.1]).

A functor $\mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$ is *additive* or *localising* if it sends split Poincaré-Verdier or Poincaré-Verdier respectively to fiber sequences of spectra. Analogous definitions apply to a functor $\mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}\mathrm{p}$, removing "Poincaré" everywhere (see [CDH+II, Definition 1.5.4 and Proposition 1.5.5]).

Hermitian K-theory, also known as the *Grothendieck-Witt spectrum* functor

$$\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$$

is the universal (initial) *additive* functor equipped with a transformation of functors $\mathrm{Pn} \to \Omega^\infty \mathrm{GW}$ (see [CDH+II, Definition 4.2.1 and Corollary 4.2.2]). Analogously, the K-theory spectrum functor is the universal additive functor $\mathrm{K}\colon \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}\mathrm{p}$ equipped with a map $\mathrm{Cr} \to \Omega^\infty \mathrm{K}$. A notable consequence of additivity is the Bott-Genauer sequence

$$\mathrm{GW}(\mathcal{C}, \Omega) \to \mathrm{K}(\mathcal{C}) \to \mathrm{GW}(\mathcal{C}, \Omega^{[1]})$$

which exhibits K-theory as an extension of GW-theory by GW-theory of a shifted Poincaré structure.

Another important invariant of a Poincaré ∞-category is its L-theory spectrum $\mathrm{L}(\mathcal{C}, \Omega)$, which, in the setting of Poincaré ∞-categories, was first defined in [Lur11], but was extensively studied in more classical contexts by Ranicki. For $n \in \mathbb{Z}$ the group $\mathrm{L}_n(\mathcal{C}, \Omega) := \pi_n(\mathcal{C}, \Omega)$ is the quotient of the monoid of $\pi_0\mathrm{Pn}(\mathcal{C}, \Omega^{[n]})$ of equivalences classes of n-shifted Poincaré objects by the submonoid consisting of the *metabolic Poincaré objects*, that is, those admitting a Lagrangian. As established in [CDH+II], The Grothendieck-Witt, L-theory and K-theory spectra are related via the *fundamental fibre sequence*

$$\mathrm{K}_{\mathrm{hC}_2} \to \mathrm{GW} \to \mathrm{L}\,.$$

One can show that both K and L are in fact Verdier-localising functors, and hence so is GW (see [CDH+II, Corollary 4.4.15]). In particular, GW is also universal as a Verdier-localising invariant.

2.3. **Karoubi-Grothendieck-Witt theory.** In this article, we actually mostly use the Karoubi-Grothendieck-Witt functor, i.e. a modified version of the above that is insensitive to idempotent completion of the underlying stable ∞-category. Let us now describe it.

A *Karoubi* or a *Poincaré-Karoubi* sequence is a sequence in $\mathrm{Cat}^{\mathrm{ex}}$ or $\mathrm{Cat}^{\mathrm{p}}$, respectively, that becomes a fiber-cofibre sequence in idempotent complete stable ∞-categories or idempotent complete Poincaré categories, respectively, after idempotent completion of its terms (see [CDH+II, Definition 1.3.6]). A functor $\mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}\mathrm{p}$ or $\mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$ is called *Karoubi-localising* if it sends Karoubi or Poincaré-Karoubi sequences, respectively, to fiber sequences (see [CDH+II, Definition 2.7.1 and Proposition 1.5.5]). In [CDH+V], we construct universal Karoubi-localising approximations

$$\mathrm{GW} \Rightarrow \mathbb{GW} \quad \text{and} \quad \mathrm{L} \Rightarrow \mathbb{L}$$

of Grothendieck-Witt- and L-theory, respectively. These are the analogues of the map $\mathrm{K} \Rightarrow \mathbb{K}$ from algebraic to Bass K-theory, which is the universal Karoubi-localising approximation of K-theory, see [BGT13]. These Karoubi-localising variants fit to form analogues of the Bott-Genauer sequence

$$\mathbb{GW} \Rightarrow \mathbb{K} \Rightarrow \mathbb{GW}((-)^{[1]}) \tag{1}$$

and fundamental fibre sequence

$$\mathbb{K}_{\mathrm{hC}_2} \Rightarrow \mathbb{GW} \Rightarrow \mathbb{L}. \tag{2}$$

In addition, we prove in [CDH+V] that the squares

$$\begin{array}{ccc} \mathrm{GW}(\mathcal{C}, \Omega) & \longrightarrow & \mathbb{GW}(\mathcal{C}, \Omega) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{C})^{\mathrm{hC}_2} & \longrightarrow & \mathbb{K}(\mathcal{C})^{\mathrm{hC}_2} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{L}(\mathcal{C}) & \longrightarrow & \mathbb{L}(\mathcal{C}, \Omega) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{C})^{\mathrm{tC}_2} & \longrightarrow & \mathbb{K}(\mathcal{C})^{\mathrm{tC}_2}\,. \end{array} \tag{3}$$

are exact, a result we call *Karoubi cofinality*. This means, in particular, that the map $\mathrm{GW}(\mathcal{C}, \Omega) \to \mathbb{GW}(\mathcal{C}, \Omega)$ is an equivalence on positive homotopy groups, and also on π_0 if $\mathcal{C}$ is idempotent complete. In addition, if $\mathcal{C}$ is such that the map $\mathrm{K}(\mathcal{C}) \to \mathbb{K}(\mathcal{C})$ is an equivalence then the maps $\mathrm{GW}(\mathcal{C}, \Omega) \to \mathbb{GW}(\mathcal{C}, \Omega)$ and $\mathrm{L}(\mathcal{C}, \Omega) \to \mathbb{L}(\mathcal{C}, \Omega)$ are equivalences as well.

For a scheme X with a line bundle L, the spectra

$$\mathbb{GW}^{\mathrm{s}}(X, L) := \mathbb{GW}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \quad \text{and} \quad \mathbb{L}^{\mathrm{s}}(X, L) := \mathbb{L}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L)$$

will be our principal object of interest in the present paper. When X is regular the map $\mathrm{K}(X) \to \mathbb{K}(X)$ is an equivalence and hence by the above these coincide with

$$\mathrm{GW}^{\mathrm{s}}(X, L) = \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \quad \text{and} \quad \mathrm{L}^{\mathrm{s}}(X, L) := \mathrm{L}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L),$$

respectively. The specific role played by Poincaré-Karoubi sequences is due to the fact that the localisation sequence of derived categories

$$\mathcal{D}^{\mathrm{p}}_Z(X) \to \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$$

associated to a closed subscheme $Z \subset X$ with open complement U is only a Karoubi sequence and not a Verdier sequence in general. As was emphasized in [TT90], although all categories are idempotent complete, the map $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$ is not essentially surjective in general: an object in the target is only a direct factor of an object in the image.

3. Poincaré structures on derived categories of schemes

For a scheme X, let $\mathcal{P}ic(X)$ denote the maximal subgroupoid of $\mathcal{D}^{\mathrm{p}}(X)$ whose objects are tensor invertible. To an invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$, one may associate a Poincaré structure $Ϙ^{\mathrm{s}}_L$ on the ∞-category $\mathcal{D}^{\mathrm{p}}(X)$ of perfect complexes as follows:

Definition 3.0.1. Let X be a scheme and $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$ an invertible object. We then define $Ϙ^{\mathrm{s}}_L : \mathcal{D}^{\mathrm{p}}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ to be the Poincaré structure given by

$$Ϙ^{\mathrm{s}}_L(M) = \hom_{\mathcal{D}^{\mathrm{p}}(X)}(M \otimes M, L)^{\mathrm{hC}_2}.$$

The fact that this is a Poincaré structure will be proven below, see Proposition 3.1.6. We refer to $Ϙ^{\mathrm{s}}_L$ as the associated *symmetric* Poincaré structure. We then write $\mathcal{E}_L$ for the quasi-coherent complex characterized by the existence of a natural equivalence

$$\hom_X(P \otimes P, L)^{\mathrm{tC}_2} \simeq \hom_X(P, \mathcal{E}_L)$$

for any perfect complex P on X. In other words, $\mathcal{E}_L$ is the object of $\mathcal{D}^{\mathrm{qc}}(X) = \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}(X))$ representing the linear part of $Ϙ^{\mathrm{s}}_L$. Similarly for every m we may consider the truncated Poincaré structure $Ϙ^{\geq m}_L$ on $\mathcal{D}^{\mathrm{p}}(X)$, given as the fibre product $Ϙ^{\geq m}_L(M) = Ϙ^{\mathrm{s}}_L(M) \times_{\hom_X(P, \mathcal{E}_L)} \hom_X(P, \tau_{\geq m}\mathcal{E}_L)$

Remark 3.0.2. The difference between the notion of an invertible perfect complex and that of a line bundle is not huge. By the main result of [Fau03], for any invertible perfect complex L on X, there exist a disjoint union decomposition $X = \coprod_{n \in \mathbb{Z}} X_n$ (where each X_n is potentially empty) such that $L|_{X_n}$ is a degree n shift of a line bundle. In particular, if X is connected then any invertible perfect complex L is globally a shift of a line bundle.

While the definition itself is simple, some effort is required in order to formally set up all the relevant functoriality and multiplicativity of this construction in X and L, summarized by the commutative diagram (12) below, and the present section is dedicated to these somewhat technical considerations. Specifically, in §3.1, we analyse in detail the formation of the symmetric Poincaré structure associated to a rigid stable symmetric monoidal ∞-category, in §3.2, we focus on the dependency on X and L, and in §3.3, we consider the genuine variants. A reader willing to accept all functorialities at faith is welcome to skip directly to Proposition 3.1.6.

3.1. **Poincaré ∞-categories associated to rigid symmetric monoidal structures.** Recall that a symmetric monoidal ∞-category is said to be *rigid* if every object $x \in \mathcal{C}$ is dualisable, that is, admits a dual $x^\vee \in \mathcal{C}$ equipped with a unit $1_{\mathcal{C}} \to x \otimes x^\vee$ and a counit map $x^\vee \otimes x \to 1_{\mathcal{C}}$ satisfying the triangle inequalities. If $\mathcal{C}$ is a rigid ∞-category then the association $x \mapsto x^\vee$ assembles to form a duality $(-)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ on $\mathcal{C}$ (that is, a C_2-fixed structure on $\mathcal{C}$ with respect to the op-action of C_2 on Cat) which can be encoded via the perfect symmetric bilinear functor $(x, y) \mapsto \hom(x \otimes y, 1_{\mathcal{C}})$. Furthermore, this duality is symmetric monoidal, that is, it is a C_2-fixed structure also with respect to the op-action of C_2 on CAlg(Cat), and is natural in $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat})$, so that symmetric monoidal functors are canonically duality preserving. A proof of these statements can be found in [HLAS16, Theorem 5.11]; more precisely, in loc. cit. the authors show that associating to each rigid ∞-category its duality determines a section of the forgetful functor $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})^{\mathrm{hC}_2} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})$, yielding a trivialization of the op-action on the full subcategory $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) \subseteq \mathrm{CAlg}(\mathrm{Cat})$ spanned by the rigid ∞-categories. In fact, within the proof of [HLAS16, Theorem 5.11], a slightly more general construction is considered: given a tensor invertible object $L \in \mathcal{C}$, one may consider the twisted duality $\mathrm{D}_L(x) = x^\vee \otimes L$, which is associated to the perfect symmetric bilinear functor $(x, y) \mapsto \hom(x \otimes y, L)$. This construction can be organized into a commutative square of product-preserving functors

$$\begin{array}{ccccc} & & (\mathcal{C}, L) & \longmapsto & (\mathcal{C}, \mathrm{D}_L) \\ & & \Cap & & \Cap \\ \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) \times_{\mathrm{Cat}} \mathrm{Cat}_* & \supseteq & \mathrm{Inv} & \longrightarrow & \mathrm{Cat}^{\mathrm{hC}_2} \\ & & \downarrow & & \downarrow \\ & & \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) & \longrightarrow & \mathrm{Cat}, \end{array} \tag{4}$$

where Inv is the full subcategory of $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) \times_{\mathrm{Cat}} \mathrm{Cat}_*$ spanned by those $(\mathcal{C}, L)$ such that L is tensor invertible, where Cat_* is the total category of the universal cocartesian fibration [Lur09, 3.3.2]. Passing to commutative monoid objects one obtains a functor

$$\mathrm{CAlg}(\mathrm{Inv}) \simeq \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}) \times_{\mathrm{CAlg}(\mathrm{Cat})} \mathrm{CAlg}(\mathrm{Cat})^{\mathrm{hC}_2} = \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})^{\mathrm{hC}_2},$$

yielding a section of the forgetful functor $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})^{\mathrm{hC}_2} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})$.

We would like to adapt the above constructions to the setting of stable ∞-categories and Poincaré structures. We first introduce a bit of notation. In what follows we say that a symmetric monoidal functor

$$p^{\otimes} : \mathcal{E}^{\otimes} \to \mathcal{B}^{\otimes}$$

is a Fin_*-*cartesian fibration* if the underlying functor $p: \mathcal{E} \to \mathcal{B}$ is a cartesian fibration of ∞-categories and for every $x \in \mathcal{E}$ the functor $x \otimes (-): \mathcal{E} \to \mathcal{E}$ preserves p-cartesian edges. Dually, we say that $p^{\otimes}$ is a Fin_*-*cocartesian fibration* if the opposite symmetric monoidal functor $(p^{\mathrm{op}})^{\otimes} : (\mathcal{E}^{\mathrm{op}})^{\otimes} \to (\mathcal{B}^{\mathrm{op}})^{\otimes}$ is a Fin_*-cartesian fibration.

Lemma 3.1.1. *Let $p^{\otimes} : \mathcal{E}^{\otimes} \to \mathcal{B}^{\otimes}$ be a symmetric monoidal functor. Then the following are equivalent:*

(1) $p^{\otimes}$ is a Fin_*-cartesian fibration in the above sense.
(2) $p: \mathcal{E} \to \mathcal{B}$ is a cartesian fibration and for every pair of p-cartesian edges $x \to y$ and $z \to w$ in $\mathcal{E}$, the edge $x \otimes z \to y \otimes w$ is again p-cartesian.
(3) For every $\langle n\rangle \in \mathrm{Fin}_*$ the functor $p^{\otimes}_{\langle n\rangle} : \mathcal{E}^{\otimes}_{\langle n\rangle} \to \mathcal{B}^{\otimes}_{\langle n\rangle}$ is a cartesian fibration, and for every $\alpha : \langle n\rangle \to \langle m\rangle$ in Fin_* the transition functor $\alpha_! : \mathcal{E}^{\otimes}_{\langle n\rangle} \to \mathcal{E}^{\otimes}_{\langle m\rangle}$ sends $p^{\otimes}_{\langle n\rangle}$-cartesian arrows to $p^{\otimes}_{\langle m\rangle}$-cartesian arrows.
(4) The opposite symmetric monoidal functor $(p^{\mathrm{op}})^{\otimes} \to (\mathcal{E}^{\mathrm{op}})^{\otimes} \to (\mathcal{B}^{\mathrm{op}})^{\otimes}$ is a cocartesian fibration of ∞-operads (in the sense of [Lur17a, Definition 2.1.2.13]*).*

We note that, in particular, Lemma 3.1.1 says that the notion of a Fin_*-cocartesian fibration identifies with that of a cocartesian fibration of ∞-operads (at least when the target is a symmetric monoidal ∞-category). The former notion is however sometimes easier to identify in practice.

Proof of Lemma 3.1.1. In (2), writing $x \otimes z \to y \otimes w$ as a composite $x \otimes z \to x \otimes w \to y \otimes w$ we see that one may as well assume that one of the arrows is an identity arrow. We conclude that (1) and (2) are equivalent.

Since $p^{\otimes}$ is a map of ∞-operads we have that for every $\langle n\rangle \in \mathrm{Fin}_*$ the map $\mathcal{E}^{\otimes}_{\langle n\rangle} \to \mathcal{B}^{\otimes}_{\langle n\rangle}$ breaks as the cartesian product $\prod_{i=1}^n p: \prod_{i=1}^n \mathcal{E} \to \prod_{i=1}^n \mathcal{B}$ of n copies of the underlying map $p: \mathcal{E} \to \mathcal{B}$, that is, of the induced map on fibres over $\langle 1\rangle \in \mathrm{Fin}_*$. We conclude that p is a cartesian fibration if and only if $p^{\otimes}_{\langle n\rangle}$ is a cartesian fibration for every $\langle n\rangle \in \mathrm{Fin}_*$, in which case the transition functors $\alpha_! : \mathcal{E}^{\otimes}_{\langle n\rangle} \to \mathcal{E}^{\otimes}_{\langle m\rangle}$ automatically preserve cartesian edges for every inert α. Now any active $\alpha : \langle n\rangle \to \langle m\rangle$ can be written as a composite of active maps $\alpha = \alpha_1 \circ \cdots \circ \alpha_k$ such that the fibres of each α_i are of size at most 2, and so we conclude that the transition functors $\alpha_! : \mathcal{E}^{\otimes}_{\langle n\rangle} \to \mathcal{E}^{\otimes}_{\langle m\rangle}$ all preserve cartesian edges if and only if this holds for the active maps $\langle 2\rangle \to \langle 1\rangle$ and $\langle 0\rangle \to \langle 1\rangle$. The latter case is automatic since $\mathcal{E}_{\langle 0\rangle} = \mathcal{B}_{\langle 0\rangle} = *$, and so we conclude that (2) and (3) are equivalent.

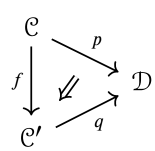
Finally, let us show that (3) and (4) are equivalent. First if $(p^{\mathrm{op}})^{\otimes} \to (\mathcal{E}^{\mathrm{op}})^{\otimes} \to (\mathcal{B}^{\mathrm{op}})^{\otimes}$ is a cocartesian fibration then each $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle} : (\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle n\rangle} \to (\mathcal{B}^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$ is a cocartesian fibration by base change, and on the other hand by [Lur09, Proposition 2.4.2.11] we have that if each $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$ is a cocartesian fibration then $(p^{\mathrm{op}})^{\otimes}$ is at least a locally cocartesian fibration, with the locally cocartesian edges of $(\mathcal{E}^{\mathrm{op}})^{\otimes}$ being exactly those of the form $e' \circ e''$ with e' a fibrewise cocartesian edge (lying over some $\langle n\rangle \in \mathrm{Fin}_*$) and e'' an edge which is cocartesian over Fin_*. At the same time, by [Lur09, Proposition 2.4.2.8] a locally cocartesian fibration is cocartesian if and only if the collection of locally cocartesian edges is closed under composition. To finish the proof it will hence suffice to show that when all the $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$ are cocartesian fibrations, the collection of locally cocartesian edges in $(p^{\mathrm{op}})^{\otimes}$ is closed under composition if and only if each transition functor $\alpha_! : (\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle n\rangle} \to (\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle m\rangle}$ preserves cocartesian edges. Since the collection of cocartesian edges with respect to any map is closed under composition the above explicit characterization of the collection of locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian edges of $(\mathcal{E}^{\mathrm{op}})^{\otimes}$ implies that this collection is closed under composition if and only if any composite of the form $e'' \circ e'$ is locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian whenever e' is a $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$-cocartesian edge of $(\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$ for some $\langle n\rangle \in \mathrm{Fin}_*$ and e'' is a cocartesian lift over Fin_* of some $\alpha\colon \langle n\rangle \to \langle m\rangle$. Now given such e' and e'' we may always lift the identity square on the arrow α in Fin_* to a commutative square in $\mathcal{E}$ of the form

$$\begin{array}{ccc} x & \xrightarrow{f''} & y \\ {\scriptstyle e'}\downarrow & \searrow & \downarrow{\scriptstyle f'} \\ z & \xrightarrow{e''} & w\,, \end{array}$$

where f'' is a cocartesian lift of α starting at x and f' lies in the fibre $(\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle m\rangle}$. In particular, such a square exhibits f' as the image under e' by the transition functor $\alpha_!\colon (\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle n\rangle} \to (\mathcal{E}^{\mathrm{op}})^{\otimes}_{\langle m\rangle}$, and $e''\circ e' = f'\circ f''$. To finish the proof it will hence suffice to show that in this scenario, $f'\circ f''$ is locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian if and only if f' is $(p^{\mathrm{op}})^{\otimes}_{\langle m\rangle}$-cocartesian. The "if" direction follows from the explicit characterization of locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian edges above. Now assume that $f'\circ f''$ is locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian. By [Lur09, Proposition 2.4.1.3] and the fact that $(p^{\mathrm{op}})^{\otimes}$ is symmetric monoidal we have that the edge f'' is $(p^{\mathrm{op}})^{\otimes}$-cocartesian, and hence by [Lur09, Lemma 2.4.2.7] we conclude that f' is locally $(p^{\mathrm{op}})^{\otimes}$-cocartesian. It is consequently also locally $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$-cocartesian. But $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$ is a cocartesian fibration, and hence f' is $(p^{\mathrm{op}})^{\otimes}_{\langle n\rangle}$-cocartesian by [Lur09, Proposition 2.4.2.8]. □

We now recall from [Lur17a, Remark 2.4.2.6] that for a symmetric monoidal ∞-category $\mathcal{B}^{\otimes}$, lax symmetric monoidal functors $\mathcal{B}^{\otimes} \to \mathrm{Cat}^{\times}$ correspond, via unstraightening, to symmetric monoidal cocartesian fibrations $\mathcal{E}^{\otimes} \to \mathcal{B}^{\otimes}$ of ∞-operads (where $\mathcal{E}^{\otimes}$ is necessarily a symmetric monoidal ∞-category, since $\mathcal{B}^{\otimes}$ is so). Passing to symmetric monoidal opposites, we get from Lemma 3.1.1 that lax symmetric monoidal functors $\chi\colon (\mathcal{B}^{\mathrm{op}})^{\otimes} \to \mathrm{Cat}^{\times}$ correspond via unstraightening to Fin_*-cartesian fibrations $\mathcal{E}^{\otimes} \to \mathcal{B}^{\otimes}$. The underlying cartesian fibration $\mathcal{E} \to \mathcal{B}$ is then the one classified by the underlying functor of χ.

A prototypical example of a Fin_*-cartesian fibration is the projection $\mathrm{Cat}^{\otimes}_{//\mathcal{D}} \to \mathrm{Cat}^{\times}$ constructed in [CDH+I, Lemma 5.2.3] for a given symmetric monoidal ∞-category $\mathcal{D}$. This is the Fin_*-cartesian fibration which classifies the lax symmetric monoidal functor $(\mathrm{Cat}^{\mathrm{op}})^{\times} \to \mathrm{Cat}$ sending $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In particular, for a functor $g\colon \mathcal{C} \to \mathcal{C}'$, the associated cartesian transition functor $(\mathrm{Cat}_{//\mathcal{D}})_{\mathcal{C}'} \to (\mathrm{Cat}_{//\mathcal{D}})_{\mathcal{C}}$ is given by the restriction functor $g^*\colon \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The underlying ∞-category $\mathrm{Cat}_{//\mathcal{D}}$ can be described as having objects $(\mathcal{C}, f)$ where $\mathcal{C}$ is an ∞-category and $f\colon \mathcal{C} \to \mathcal{D}$ is a functor, and morphisms are given by diagrams



filled by a non-invertible 2-cell $p \Rightarrow qf$. The tensor product in $\mathrm{Cat}^{\otimes}_{//\mathcal{D}}$ is given by $(\mathcal{C}, f) \otimes (\mathcal{C}', f') = (\mathcal{C} \times \mathcal{C}', f \otimes f')$, where $f \otimes f'$ is the composite

$$\mathcal{C} \times \mathcal{C}' \xrightarrow{f\times f'} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D},$$

obtained using the symmetric monoidal structure of $\mathcal{D}$. In particular, one readily checks that for any $(\mathcal{C}, f)$ the functor $(\mathcal{C}, f) \otimes (-)\colon \mathrm{Cat}_{//\mathcal{D}} \to \mathrm{Cat}_{//\mathcal{D}}$ preserves cartesian edges over Cat. A closely related variant is the Fin_*-cartesian fibration

$$\mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}} \to \mathrm{Cat}^{\times},$$

which is obtained from $\mathrm{Cat}^{\otimes}_{//\mathcal{D}} \to \mathrm{Cat}^{\times}$ by base changing along the op-equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat} \xrightarrow{\simeq} \mathrm{Cat}$. In particular, the underlying ∞-category $\mathrm{Cat}_{\mathrm{op}//\mathcal{D}}$ can be described as having objects $(\mathcal{C}, f)$ where $\mathcal{C}$ is an ∞-category and $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a functor.

We note that the construction of $\mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}}$ (or of $\mathrm{Cat}^{\otimes}_{//\mathcal{D}}$) works exactly the same if one takes $\mathcal{D}$ to be a presentable (and in particular large) ∞-category, while still considering only pairs $(\mathcal{C}, f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D})$ with $\mathcal{C}$

small. In this case, each of the cartesian transition functors $g^*\colon \mathrm{Fun}(\mathcal{C}'^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ admits a left adjoint $g_!\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}'^{\mathrm{op}}, \mathcal{D})$ in the form of left Kan extension, which means that the cartesian fibration $\mathrm{Cat}_{\mathrm{op}//\mathcal{D}} \to \mathrm{Cat}$ is also a cocartesian fibration. Furthermore, if the symmetric monoidal structure on $\mathcal{D}$ preserves colimits in each variable then the functor $(\mathcal{C}, f) \otimes (-)\colon \mathrm{Cat}_{\mathrm{op}//\mathcal{D}} \to \mathrm{Cat}_{\mathrm{op}//\mathcal{D}}$ preserves cocartesian edges for every $(\mathcal{C}, f) \in \mathrm{Cat}_{\mathrm{op}//\mathcal{D}}$, so that $\mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}} \to \mathrm{Cat}^{\times}$ is also a Fin_*-cocartesian fibration, and hence a cocartesian fibration of ∞-operads by Lemma 3.1.1. This presentable case is the one relevant to us, as we will later on set $\mathcal{D} = \mathcal{S}\mathrm{p}$, as is also eventually done in [CDH+I, §5.2].

Let us now fix a presentably symmetric monoidal ∞-category $\mathcal{D}$. Consider the ∞-category $\mathrm{CAlg}(\mathrm{Cat})$ of symmetric monoidal ∞-categories (and symmetric monoidal functors between them). We then set

$$\mathcal{Z}^{\otimes}_{\mathcal{D}} = \mathrm{CAlg}(\mathrm{Cat})^{\times} \times_{\mathrm{Cat}^{\times}} (\mathrm{Cat}_{\mathrm{op}//\mathcal{D}^{\mathrm{BC}_2}})^{\otimes},$$

so that $\mathcal{Z}^{\otimes}_{\mathcal{D}}$ is a symmetric monoidal ∞-category sitting in a Fin_*-cartesian fibration

$$\mathcal{Z}^{\otimes}_{\mathcal{D}} \to \mathrm{CAlg}(\mathrm{Cat})^{\times}$$

classified by the lax symmetric monoidal functor $\mathcal{C}^{\otimes} \mapsto \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{BC}_2})$. In particular, the underlying ∞-category of $\mathcal{Z}^{\otimes}_{\mathcal{D}}$ can be described as having objects pairs $(\mathcal{C}^{\otimes}, f)$ where $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category and $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{BC}_2}$ is a functor.

Construction 3.1.2. Given a functor $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{BC}_2}$, let us denote by $\overline{f}$ the composite

$$\overline{f}\colon (\mathcal{C}^{\mathrm{op}})^{\mathrm{BC}_2} \to \mathcal{D}^{\mathrm{BC}_2 \times \mathrm{BC}_2} \to \mathcal{D}^{\mathrm{BC}_2},$$

where the first functor is induced by f and the second is restriction along the diagonal $\mathrm{BC}_2 \to \mathrm{BC}_2 \times \mathrm{BC}_2$. Now, if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, we can associate to $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{BC}_2}$ the composite functor

$$T_f\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{x \mapsto x \otimes x} (\mathcal{C}^{\mathrm{op}})^{\mathrm{BC}_2} \xrightarrow{\overline{f}} \mathcal{D}^{\mathrm{BC}_2},$$

given by the formula $T_f(x) = \overline{f}(x \otimes x)$. Finally, if $\mathcal{D}$ admits BC_2-indexed limits, we can take homotopy fixed points of C_2-objects, in which case we denote by $\Qoppa^{\mathrm{s}}_f$ the functor $T_f^{\mathrm{hC}_2}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, given by

$$\Qoppa^{\mathrm{s}}_f(x) = T_f^{\mathrm{hC}_2}(x) = \overline{f}(x \otimes x)^{\mathrm{hC}_2}.$$

Proposition 3.1.3. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category which admits BC_2-indexed limits. Then the association $(\mathcal{C}, f) \mapsto (\mathcal{C}, \Qoppa^{\mathrm{s}}_f)$ of Construction 3.1.2 assembles to form a lax symmetric monoidal functor*

$$\mathcal{Z}^{\otimes}_{\mathcal{D}} \to \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}}$$

sitting in a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^{\otimes}_{\mathcal{D}} & \longrightarrow & \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{CAlg}(\mathrm{Cat})^{\times} & \xrightarrow{\mathcal{C}^{\otimes} \mapsto \mathcal{C}} & \mathrm{Cat}^{\times}, \end{array} \tag{5}$$

in which the vertical arrows are both Fin_*-cartesian fibrations and cocartesian fibrations of ∞-operads and the bottom horizontal arrow is a symmetric monoidal functor.

The proof of Proposition 3.1.3 will make use of a small variant of the Fin_*-cartesian fibration $(\mathrm{Cat}_{\mathrm{op}//\mathcal{D}})^{\otimes} \to \mathrm{Cat}^{\times}$, where ∞-categories are replaced with ∞-categories with C_2-action. For this, note that Fin_*-cartesian fibrations are closed under cotensoring by an ∞-category. More precisely, if $p^{\otimes}\colon \mathcal{E}^{\otimes} \to \mathcal{B}^{\otimes}$ is a Fin_*-cartesian fibration and $\mathcal{I}$ is an ∞-category, then

$$(p^{\mathcal{I}})^{\otimes}\colon (\mathcal{E}^{\mathcal{I}})^{\otimes} \to (\mathcal{B}^{\mathcal{I}})^{\otimes}$$

is again a Fin_*-cartesian fibration, where $(\mathcal{B}^{\mathcal{I}})^{\otimes} = (\mathcal{B}^{\otimes})^{\mathcal{I}} \times_{\mathrm{Fin}_*^{\mathcal{I}}} \mathrm{Fin}_*$ is the pointwise symmetric monoidal structure on $\mathcal{B}^{\mathcal{I}}$, and similarly for $(\mathcal{E}^{\mathcal{I}})^{\otimes}$. If p is classified by a lax symmetric monoidal functor $\chi\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ then $(p^{\mathcal{I}})^{\otimes}$ is classified by the functor $(\mathcal{B}^{\mathcal{I}})^{\mathrm{op}} \to \mathrm{Cat}$ sending $f\colon \mathcal{I} \to \mathcal{B}$ to $\lim_{i \in \mathcal{I}} \chi(f(i))$ (see [Lur09, Corollary 3.3.3.2]), with its induced lax symmetric monoidal structure. In particular, taking $\mathcal{I} = \mathrm{BC}_2$, we may consider, for a symmetric monoidal ∞-category $\mathcal{D}$, the Fin_*-cartesian fibration

$$\big((\mathrm{Cat}_{\mathrm{op}//\mathcal{D}})^{\mathrm{BC}_2}\big)^{\otimes} \to \big(\mathrm{Cat}^{\mathrm{BC}_2}\big)^{\times},$$

classified by the lax symmetric monoidal functor

$$\big(\mathrm{Cat}^{\mathrm{BC}_2}\big)^{\mathrm{op}} \to \mathrm{Cat} \qquad \mathcal{C} \mapsto \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{hC}_2} = \mathrm{Fun}_{\mathrm{C}_2}(\mathcal{C}, q^*\mathcal{D}), \tag{6}$$

where $q^*\colon \mathrm{Cat} \to \mathrm{Cat}^{\mathrm{BC}_2}$ is induced by restriction along the terminal map $q\colon \mathrm{BC}_2 \to *$ (otherwise put, $q^*\mathcal{D}$ is $\mathcal{D}$ endowed with the trivial C_2-action). Finally, since $\mathrm{Fun}_{\mathrm{C}_2}(q^*(-), q^*\mathcal{D}) = \mathrm{Fun}(-, \mathcal{D}^{\mathrm{BC}_2})$ by adjunction and restriction corresponds via unstraightening to base change we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}^{\mathrm{BC}_2}} & \longrightarrow & \big((\mathrm{Cat}_{\mathrm{op}//\mathcal{D}})^{\mathrm{BC}_2}\big)^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{Cat}^{\times} & \xrightarrow{(q^*)^{\times}} & \big(\mathrm{Cat}^{\mathrm{BC}_2}\big)^{\times} \end{array} \tag{7}$$

of symmetric monoidal ∞-categories and symmetric monoidal functors, where the vertical maps are Fin_*-cartesian fibrations.

Construction 3.1.4. Consider the composite functor $p\colon \mathrm{CAlg}(\mathrm{Cat}) \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} \mathrm{Cat} \xrightarrow{q^*} \mathrm{Cat}^{\mathrm{BC}_2}$, where the first functor just forgets the symmetric monoidal structure. We construct a symmetric monoidal natural transformation $\tau\colon p \Rightarrow p$ whose component at a given $\mathcal{C}^{\otimes}$ is the C_2-equivariant functor $q^*\mathcal{C} \xrightarrow{x\mapsto x\otimes x} q^*\mathcal{C}$ as follows. Let $\mathrm{Span}(\mathrm{Fin})$ be the span ∞-category of finite sets. Then $\mathrm{Span}(\mathrm{Fin})$ is semi-additive, that is, (finite) products and coproducts coincide, and are both given by disjoint union. Furthermore, for an ∞-category with finite products there is a natural equivalence

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{E}) = \mathrm{CMon}(\mathcal{E})$$

between product-preserving functors $\mathrm{Fin} \to \mathcal{E}$ and commutative monoid objects in $\mathcal{E}$ (see, e.g., [YH20, Proposition 5.14] for $m = 1$), where the underlying objects of a given monoid corresponds to the value of the associated functor $\mathrm{Span}(\mathrm{Fin}_*) \to \mathcal{E}$ at $\{1\} \in \mathrm{Span}(\mathrm{Fin}_*)$. In particular, we have

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathrm{Cat}) = \mathrm{CMon}(\mathrm{Cat}) = \mathrm{CAlg}(\mathrm{Cat})$$

and we may identify the functor $p\colon \mathrm{CAlg}(\mathrm{Cat}) \to \mathrm{Cat}^{\mathrm{BC}_2}$ above with the composite

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}_*), \mathrm{Cat}) \to \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}_*), \mathrm{Cat}) \xrightarrow{g^*} \mathrm{Fun}(\mathrm{BC}_2, \mathrm{Cat}),$$

where g^* denotes restriction along the functor $g\colon \mathrm{BC}_2 \to \Delta^0 \xrightarrow{\{1\}} \mathrm{Span}(\mathrm{Fin}_*)$.

Now the ∞-groupoid of maps $\{1\} \to \{1\}$ in $\mathrm{Span}(\mathrm{Fin})$ is given by the core ∞-groupoid $\iota\mathrm{Fin}$ of finite sets. In particular, this mapping space contains a copy of BC_2, sitting in the form of the span $\{1\} \leftarrow \{1,2\} \to \{1\}$ endowed with its order 2 group of automorphisms (or self homotopies). This copy of BC_2 in the mapping space $\{1\} \to \{1\}$ determines a functor $\Delta^1 \times \mathrm{BC}_2 \to \mathrm{Span}(\mathrm{Fin})$ whose restriction to $\partial\Delta^1 \times \mathrm{BC}_2$ is constant on $\{1\}$. Equivalently, this is the data of a natural transformation $g \Rightarrow g$, which then determines a natural transformation $g^* \Rightarrow g^*$ from the restriction functor $g^*\colon \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}_*), \mathrm{Cat}) \to \mathrm{Fun}(\mathrm{BC}_2, \mathrm{Cat})$ to itself. Horizontally pre-composing with the forgetful functor $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathrm{Cat}) \to \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathrm{Cat})$ we then obtain the desired natural transformation $\tau\colon p \Rightarrow p$. Since p is product-preserving we may consider it as a symmetric monoidal functor $p^{\times}\colon \mathrm{CAlg}(\mathrm{Cat})^{\times} \to (\mathrm{Cat}^{\mathrm{BC}_2})^{\times}$, in which case τ automatically refines to a symmetric monoidal natural transformation $\tau^{\times}\colon p^{\times} \Rightarrow p^{\times}$.

Proof of Proposition 3.1.3. Composing natural transformation $\tau^{\times}\colon p^{\times} \Rightarrow p^{\times}$ horizontally with the contravariant lax symmetric monoidal functor $\mathrm{Fun}_{\mathrm{C}_2}(-, q^*\mathcal{D})$ of (6) we obtain a natural transformation σ from the contravariant lax symmetric monoidal functor $\mathcal{C}^{\otimes} \mapsto \mathrm{Fun}_{\mathrm{C}_2}(q^*\mathcal{C}, q^*\mathcal{D})$ to itself. After unstraightening, this provides an extension of the pullback square (7) to a commutative rectangle

$$(8)\qquad \begin{array}{ccccc} \mathcal{Z}^{\otimes}_{\mathcal{D}} & \longrightarrow & \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}^{\mathrm{BC}_2}} & \longrightarrow & ((\mathrm{Cat}_{\mathrm{op}//\mathcal{D}})^{\mathrm{BC}_2})^{\otimes} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CAlg}(\mathrm{Cat})^{\times} & \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} & \mathrm{Cat}^{\times} & \xrightarrow{q^*} & (\mathrm{Cat}^{\mathrm{BC}_2})^{\times} \end{array}$$

of symmetric monoidal ∞-categories and symmetric monoidal functors, where the outer rectangle encodes the lax symmetric monoidal transformation σ, and the left square is formed from the outer rectangle by means of universal properties, since the right square is pullback.

If we now assume that the ∞-category $\mathcal{D}$ has BC_2-indexed limits then the functor $(-)^{\mathrm{hC}_2}\colon \mathcal{D}^{\mathrm{BC}_2} \to \mathcal{D}$ is canonically lax symmetric monoidal, and hence induces a lax symmetric monoidal functor

$$h\colon \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}^{\mathrm{BC}_2}} \to \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}} \qquad (\mathcal{C}, T\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{BC}_2}) \mapsto (\mathcal{C}, T^{\mathrm{hC}_2}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D})$$

covering the identity $\mathrm{Cat}^{\times} \to \mathrm{Cat}^{\times}$ (indeed, by its defining mapping property in [CDH+I, Lemma 5.2.3], the underlying ∞-operad of $\mathrm{Cat}^{\otimes}_{//\mathcal{D}}$ is functorial in the underlying ∞-operad of $\mathcal{D}^{\otimes}$). Composing this with the left square of (8) we then obtain a commutative diagram

$$\begin{array}{ccccc} (\mathcal{C}^{\otimes}, f) & \longmapsto & (\mathcal{C}, T_f) & \longmapsto & (\mathcal{C}, Ϙ^{\mathrm{s}}_f) \\ \Cap & & \Cap & & \Cap \\ \mathcal{Z}^{\otimes}_{\mathcal{D}} & \longrightarrow & \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}^{\mathrm{BC}_2}} & \xrightarrow{h} & \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CAlg}(\mathrm{Cat})^{\times} & \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} & \mathrm{Cat}^{\times} & = & \mathrm{Cat}^{\times} \end{array}$$

whose external rectangle is the desired square (5). (Here and later, we only made reference above the top row to the individual objects of the ∞-categories underlying the appearing symmetric monoidal ∞-categories, avoiding to mention arrows, tuples and multi-operations.) □

For our present setting of Poincaré ∞-categories, we now replace ∞-categories by stable ∞-categories and take $\mathcal{D}$ to be the ∞-category of spectra $\mathcal{S}\mathrm{p}$. Recall that the symmetric monoidal ∞-category $(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$ can be defined as the suboperad of $\mathrm{Cat}^{\times}$ spanned by the colors $\mathcal{C}$ which are stable ∞-categories, and by those multi-operations whose representing functors $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{C}$ are exact in each variable separately. As in [CDH+I, §5], we define $(\mathrm{Cat}^{\mathrm{h}})^{\otimes}$ as the suboperad of $\mathrm{Cat}^{\times}_{\mathrm{op}//\mathcal{S}\mathrm{p}}$ spanned by the colors $(\mathcal{C}, Ϙ)$ such that $\mathcal{C}$ is stable and $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is quadratic, and those multi-operations whose images in $\mathrm{Cat}^{\times}$ are contained in $(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$.

The symmetric monoidal structure $(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$ on $\mathrm{Cat}^{\mathrm{ex}}$ induces a symmetric monoidal structure $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$ on $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$ by means of internal mapping objects in ∞-operads, such that the forgetful functor

$$\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} \to (\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$$

is symmetric monoidal. The lax symmetric monoidal functor $(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} \to \mathrm{Cat}^{\times}$ then induces a lax symmetric monoidal functor $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} \to \mathrm{CAlg}(\mathrm{Cat})^{\times}$, and we define

$$\mathcal{Z}^{\otimes}_{\mathrm{ex}} \subseteq \mathcal{Z}^{\otimes}_{\mathcal{S}\mathrm{p}} \times_{\mathrm{CAlg}(\mathrm{Cat})^{\times}} \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$$

to be the full suboperad spanned by the colors $(\mathcal{C}^{\otimes}, L)$ such that $L\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}^{\mathrm{hC}_2}$ is exact. The square (5) then restricts to a commutative square of ∞-operads

$$(9)\qquad \begin{array}{ccc} (\mathcal{C}^{\otimes}, L) & \longmapsto & (\mathcal{C}, Ϙ^{\mathrm{s}}_L) \\ \Cap & & \Cap \\ \mathcal{Z}^{\otimes}_{\mathrm{ex}} & \longrightarrow & (\mathrm{Cat}^{\mathrm{h}})^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} & \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} & (\mathrm{Cat}^{\mathrm{ex}})^{\otimes}\ , \end{array}$$

where the bottom horizontal arrow is a symmetric monoidal functor between symmetric monoidal ∞-categories.

Lemma 3.1.5. *Both vertical maps in* (9) *are cocartesian fibrations of ∞-operads. In particular, $\mathcal{Z}_{\mathrm{ex}}^{\otimes}$ and $(\mathrm{Cat}^{\mathrm{h}})^{\otimes}$ are symmetric monoidal ∞-categories.*

To avoid confusion, let us point the top horizontal map in (9) does not preserve cocartesian edges. In fact, it doesn't even preserve cocartesian edges over Fin_*, namely, it is only a *lax* symmetric monoidal functor.

Proof of Lemma 3.1.5. For the right vertical map the claim is proven in [CDH+I, Theorem 5.2.7]. The argument for the left vertical map is similar (and easier), let us spell it out. First, note that the projection

$$\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} \times_{\mathrm{Cat}^{\times}} \mathrm{Cat}^{\otimes}_{\mathrm{op}//\mathcal{S}\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$$

is a cocartesian fibration of ∞-operads by base change, and that $\mathcal{Z}_{\mathrm{ex}}^{\otimes}$ is a full suboperad of the fibre product on the left, spanned by those colors $(\mathcal{C}, L)$ such that $L\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is exact. It will hence suffice to check that $\mathcal{Z}_{\mathrm{ex}}^{\otimes}$ contains a cocartesian arrow as soon as it contains its domain. Unwinding the definitions, this amounts to checking that if $\mathcal{C}$ and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are stable ∞-categories, $g\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{C}$ is a functor which is exact in each variable, and $L_i\colon \mathcal{C}_i^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ are exact functors for $i = 1, \dots, n$, then the left Kan extension of the top composite

$$\begin{array}{ccccc} \prod_i \mathcal{C}_i & \longrightarrow & \prod_i \mathcal{S}\mathrm{p} & \xrightarrow{\otimes} & \mathcal{S}\mathrm{p} \\ \big\downarrow{\scriptstyle g} & & & \nearrow & \\ \mathcal{C} & - & - & - & \end{array}$$

along g is again exact. Indeed, since g is exact in each variable its left Kan extension along the universal multi-exact functor $\prod_i \mathcal{C}_i \to \bigotimes_i \mathcal{C}$ yields an exact functor $g'\colon \bigotimes_i \mathcal{C}_i \to \mathcal{S}\mathrm{p}$, and the left Kan extension of g' along the exact functor $\bigotimes_i \mathcal{C}_i \to \mathcal{C}$ is again exact by [CDH+I, Lemma 1.4.1]. $\square$

Now as we have canonical equivalences

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{p}^{\mathrm{BC}_2}) = \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{p})^{\mathrm{BC}_2} = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{p})^{\mathrm{BC}_2} = \mathrm{Ind}(\mathcal{C})^{\mathrm{BC}_2}$$

we will generally identify the objects of the underlying ∞-category of $\mathcal{Z}_{\mathrm{ex}}^{\otimes}$ with pairs $(\mathcal{C}^{\otimes}, L)$ where $\mathcal{C}^{\otimes}$ is a stably symmetric monoidal ∞-category and L is an object of $\mathrm{Ind}(\mathcal{C})^{\mathrm{BC}_2}$. A morphism $(\mathcal{C}, L) \to (\mathcal{C}', L')$ in this description then corresponds to a pair (g, η) where $g\colon \mathcal{C} \to \mathcal{C}'$ is an exact functor and $\eta\colon g_! L \to L'$ is a morphism in $\mathrm{Ind}(\mathcal{C}')$, where $g_!\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ is the functor induced by g on Ind objects (given by left Kan extension when viewed as exact $\mathcal{S}\mathrm{p}$-valued presheaves). In this description, the functor $\Omega^{\mathrm{s}}_L\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ can be written as

$$\Omega^{\mathrm{s}}_L(x) = \hom_{\mathrm{Ind}(\mathcal{C})}(x \otimes x, L)^{\mathrm{hC}_2},$$

where we have implicitly identified $x \otimes x$ with its image in $\mathrm{Ind}(\mathcal{C})$ under the Yoneda embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

Proposition 3.1.6. *Suppose that $\mathcal{C}$ is a rigid stably symmetric monoidal ∞-category and $L \in \mathrm{Ind}(\mathcal{C})^{\mathrm{BC}_2}$ a C_2-equivariant Ind-object. Then the hermitian structure Ω^{s}_L above is non-degenerate if and only if L belongs to $\mathcal{C}^{\mathrm{BC}_2} \subseteq \mathrm{Ind}(\mathcal{C})^{\mathrm{BC}_2}$. It is furthermore Poincaré if and only if the underlying object of L is tensor invertible in $\mathcal{C}$. In addition, if $g\colon \mathcal{C} \to \mathcal{C}'$ is an exact functor, $L \in \mathcal{C}^{\mathrm{BC}_2}$ and $L' \in (\mathcal{C}')^{\mathrm{BC}_2}$ tensor invertible and $g(L) \to L'$ an equivalence, then the induced hermitian functor $(\mathcal{C}, \Omega^{\mathrm{s}}_L) \to (\mathcal{C}', \Omega^{\mathrm{s}}_{L'})$ is a Poincaré functor.*

Proof. Unwinding the definitions, the cross effect of Ω^{s}_L is given by

$$\mathrm{B}_L(x, y) = \hom_{\mathrm{Ind}(\mathcal{C})}(x \otimes y, L).$$

If B_L is represented by $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ then $\mathrm{D}(1_{\mathcal{C}}) = L$, so that L belongs to $\mathcal{C}$. On the other hand, if L belongs to $\mathcal{C}$ then

$$\hom_{\mathrm{Ind}(\mathcal{C})}(x \otimes y, L) = \hom_{\mathcal{C}}(x \otimes y, L) = \hom_{\mathcal{C}}(x, y^{\vee} \otimes L)$$

by the rigidity of $\mathcal{C}$, so that B_L is represented by $\mathrm{D}_L(y) = y^{\vee} \otimes L$. Moreover, the double duality map $1_{\mathcal{C}} \to \mathrm{D}_L \mathrm{D}_L$ is

$$y \to L \otimes L^{\vee} \otimes y$$

induced by the adjoint of the C_2-action $L \to L$, and so it is an equivalence if and only if L is invertible.

Finally, if L and L' are invertible objects of $\mathcal{C}$ and $\mathcal{C}'$ respectively, the natural transformation $g\mathrm{D}_L \to \mathrm{D}_{L'}g$ is given by

$$fL \otimes fx^\vee \to L' \otimes fx^\vee$$

and so it is an equivalence if and only if the map $fL \to L'$ is an equivalence. □

Let $\mathrm{Inv}^\otimes \subseteq \mathcal{Z}^\otimes_{\mathrm{ex}}$ denotes the suboperad spanned by those $(\mathcal{C}^\otimes, L)$ such that $\mathcal{C}^\otimes$ is rigid and L is (the Yoneda image of) a tensor invertible C_2-equivariant object of $\mathcal{C}$, and those multi-operations $((\mathcal{C}_1, L_1), \dots, (\mathcal{C}_n, L_n)) \to (\mathcal{C}, L)$ corresponding to a multi-exact functor $g\colon \mathcal{C}_1 \times \dots \times \mathcal{C}_n \to \mathcal{C}$ and an equivalence $g(L_1, \dots, L_n) \xrightarrow{\simeq} L$. Then the exact functor $\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_n \to \mathcal{C}$ induced by such a multi-operation sends the tensor invertible C_2-equivariant object $L_1 \otimes \dots \otimes L_n$ to L, and hence by Proposition 3.1.6 the associated hermitian functor

$$(\mathcal{C}_1, \Qoppa^{\mathrm{s}}_{L_1}) \otimes \dots \otimes (\mathcal{C}_n, \Qoppa^{\mathrm{s}}_{L_n}) \to (\mathcal{C}, \Qoppa^{\mathrm{s}}_L)$$

is a Poincaré functor between two Poincaré ∞-categories. As a consequence, the square (9) restricts to a square of ∞-operads

$$\begin{array}{ccc} (\mathcal{C}, L) & \longmapsto & (\mathcal{C}, \Qoppa^{\mathrm{s}}_L) \\ \Cap & & \Cap \\ \mathrm{Inv}^\otimes & \longrightarrow & (\mathrm{Cat}^{\mathrm{p}})^\otimes \\ \downarrow & & \downarrow \\ \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})^\otimes & \longrightarrow & (\mathrm{Cat}^{\mathrm{ex}})^\otimes . \end{array} \tag{10}$$

We also note that these are exactly the arrows in $\mathrm{Inv}^\otimes$ whose image in $\mathcal{Z}^\otimes_{\mathrm{ex}}$ is cocartesian over $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$. In particular,

$$\mathrm{Inv}^\otimes \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})^\otimes$$

is a left fibration of ∞-operads and $\mathrm{Inv}^\otimes$ is a symmetric monoidal ∞-category. By [CDH+I, Theorem 5.2.7(iii)] the ∞-operad $(\mathrm{Cat}^{\mathrm{p}})^\otimes$ is also a symmetric monoidal ∞-category and the right vertical arrow is a symmetric monoidal functor. We however point out that the top horizontal arrow in (10) is only an ∞-operad map, that is, a lax symmetric monoidal functor.

Let $\mathrm{Cat}^{\mathrm{ps}}$ denote the ∞-category of stable ∞-categories with duality. These can be defined in two equivalent manners, see [CDH+I, Corollary 7.2.16]. The first is as C_2-homotopy fixed points in $\mathrm{Cat}^{\mathrm{ex}}$ with respect to the C_2-action where the generator of C_2 acts by $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$. In particular, a stable ∞-category is equipped with an equivalence $\mathcal{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, called the duality, which satisfies all the coherence conditions of being a C_2-homotopy fixed point. The second is as stable ∞-category $\mathcal{C}$ equipped with a symmetric bilinear functor $\mathrm{B}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ such that there exists an equivalence $\mathcal{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ satisfying $\mathrm{B}(x, y) \simeq \hom(x, \mathrm{D}y)$ naturally in x, y. In this case $\mathcal{D}$ is uniquely determined, and the symmetric structure on B encodes the C_2-homotopy fixed structure of $\mathcal{D}$. There is a fully-faithful functor

$$\mathrm{Cat}^{\mathrm{ps}} \hookrightarrow \mathrm{Cat}^{\mathrm{p}} \qquad (\mathcal{C}, \mathrm{D}) \mapsto (\mathcal{C}, \Qoppa^{\mathrm{s}}_{\mathrm{D}}),$$

where $\Qoppa^{\mathrm{s}}_{\mathrm{D}}$ is the Poincaré structure $\Qoppa^{\mathrm{s}}_{\mathrm{D}}(x) = \hom(x, \mathrm{D}(x))^{\mathrm{hC}_2}$. This inclusion admits a left adjoint $\mathrm{Cat}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{ps}}$ sending $(\mathcal{C}, \Qoppa)$ to the underlying ∞-category with duality $(\mathcal{C}, \mathrm{D}_\Qoppa)$ (see [CDH+I, Proposition 7.2.18]). In particular, $\mathrm{Cat}^{\mathrm{ps}}$ is a left localisation of $\mathrm{Cat}^{\mathrm{p}}$. We refer the reader to [CDH+I, §7.2] for further details.

We refer to Poincaré ∞-categories in the image of this functor as *symmetric Poincaré ∞-categories*. In what follows, we sometimes abuse notation and do not distinguish between $\mathrm{Cat}^{\mathrm{ps}}$ and its image in $\mathrm{Cat}^{\mathrm{p}}$, that is, we identify an ∞-category with duality with its associated symmetric Poincaré ∞-category, and vice versa. We note that by construction, the lax symmetric monoidal functor $\mathrm{Inv}^\otimes \to (\mathrm{Cat}^{\mathrm{p}})^\otimes$ takes values in the full sub-operad $(\mathrm{Cat}^{\mathrm{ps}})^\otimes \subseteq (\mathrm{Cat}^{\mathrm{p}})^\otimes$ spanned by the symmetric Poincaré ∞-categories.

Proposition 3.1.7. *The localisation* $\mathrm{Cat}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{ps}}$ *is compatible with the symmetric monoidal structure. In particular, the ∞-operad* $(\mathrm{Cat}^{\mathrm{ps}})^\otimes$ *is a symmetric monoidal ∞-category and the left adjoint* $\mathrm{Cat}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{ps}}$ *canonically refines to a symmetric monoidal functor.*

Proof. Unwinding the definitions, what we need to prove is that if $(\mathcal{C}, \Phi)$ is a Poincaré ∞-category and $(\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ is a Poincaré functor such that the induced functor $(\mathcal{D}, \mathrm{D}_\Phi) \to (\mathcal{E}, \mathrm{D}_\Psi)$ is an equivalence of stable ∞-categories with duality then the Poincaré functor $(\mathcal{C}, \Qoppa) \otimes (\mathcal{D}, \Phi) \to (\mathcal{C}, \Qoppa) \otimes (\mathcal{D}, \Phi)$ is also an equivalence on underlying stable ∞-categories with duality. But this is clear, since an equivalence of stable ∞-categories with duality is detected on underlying stable ∞-categories and the forgetful functor $\mathrm{Cat}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{ex}}$ is symmetric monoidal, see [CDH+I, Theorem 5.2.7]. □

Since the forgetful functor $\mathrm{Cat}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{ex}}$ factors through $\mathrm{Cat}^{\mathrm{ps}}$ it follows from the universal property of symmetric monoidal localisations that the symmetric monoidal functor $(\mathrm{Cat}^{\mathrm{p}})^{\otimes} \to (\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$ factors through a symmetric monoidal functor $(\mathrm{Cat}^{\mathrm{ps}})^{\otimes} \to (\mathrm{Cat}^{\mathrm{ex}})^{\otimes}$, and the diagram (10) determines a diagram of ∞-operads

$$\begin{array}{ccc} (\mathcal{C}, L) & \longmapsto & (\mathcal{C}, \mathfrak{P}^{\mathrm{s}}_L) \\ \Cap & & \Cap \\ \mathrm{Inv}^{\otimes} & \longrightarrow & (\mathrm{Cat}^{\mathrm{ps}})^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes} & \longrightarrow & (\mathrm{Cat}^{\mathrm{ex}})^{\otimes} . \end{array} \tag{11}$$

Proposition 3.1.8. *In the square (11), all functors are symmetric monoidal functors between symmetric monoidal ∞-categories. In addition, the underlying ∞-categories all admit sifted colimits and these are preserved by all the functors in the square.*

The square (11) is the version of (4) relevant for our purposes.

Proof of Proposition 3.1.8. By construction all functors are symmetric monoidal functors between symmetric monoidal ∞-categories except possibly the top horizontal one, which is a-priori only a lax symmetric monoidal functor. However, since the right vertical functor is conservative by [CDH+I, Corollary 7.2.16] we have the top horizontal arrow must also be a symmetric monoidal functor.

Now $\mathrm{Cat}^{\mathrm{ex}}$ has all colimits [CDH+I, Proposition 6.1.1] and so $\mathrm{Cat}^{\mathrm{ps}} \simeq (\mathrm{Cat}^{\mathrm{ex}})^{\mathrm{hC}_2}$ has all colimits as well, and these are preserved by the forgetful functor $\mathrm{Cat}^{\mathrm{ps}} \to \mathrm{Cat}^{\mathrm{ex}}$. Now since the tensor product on $\mathrm{Cat}^{\mathrm{ex}}$ preserves colimits in each variable (indeed, the operation $\mathcal{C} \otimes (-)$ admits a right adjoint $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, -)$ by its defining mapping property) we have that $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$ has all colimits as well and that the forgetful functor $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathrm{Cat}^{\mathrm{ex}}$ preserves sifted colimits. We now claim that the full subcategory $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}) \subseteq \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$ is closed under sifted colimits. For this, note that since the forgetful functor $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathrm{Cat}^{\mathrm{ex}}$ is conservative, it also detects sifted colimits, that is, sifted colimits in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$ are calculated in $\mathrm{Cat}^{\mathrm{ex}}$. This means that if $\chi : \mathcal{I} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}})$ is a sifted diagram with colimit $\mathcal{C}^{\otimes} = \mathrm{colim}_{\mathcal{I}}\, \chi$, then $\mathcal{C}$ is the colimit of the underlying $\mathrm{Cat}^{\mathrm{ex}}$-valued diagram, and so the collection of essential images $\mathrm{im}[\chi(i) \to \mathcal{C}]$ generates $\mathcal{C}$ as a stable ∞-category. Now if we assume that each $\chi(i)$ is rigid then each of the objects in $\mathrm{im}[\chi(i) \to \mathcal{C}]$ is dualisable, and hence any object in $\mathcal{C}$ is dualisable since in any stable symmetric monoidal ∞-category the collection of dualisable objects forms a stable subcategory. We hence conclude that $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})$ admits sifted colimits and that these are preserved by the forgetful functor $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathrm{Cat}^{\mathrm{ex}}$.

Finally, the left vertical arrow $\mathrm{Inv} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})$ is the left fibration classified by the functor

$$\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathcal{S}$$

sending a rigid stably symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ to the ∞-groupoid of tensor-invertible objects in $\mathcal{C}$. Since any sifted category is weakly contractible and groupoids admit all colimits indexed by weakly contractible ∞-categories we conclude that ∞-groupoids admit sifted colimits, and hence the domain Inv of the left fibration $\mathrm{Inv} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})$ admit sifted colimits and these are preserved by the projection to $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})$. Since the forgetful functor $\mathrm{Cat}^{\mathrm{ps}} \to \mathrm{Cat}^{\mathrm{ex}}$ is conservative it now follows that the functor $\mathrm{Inv} \to \mathrm{Cat}^{\mathrm{ps}}$ preserves sifted colimits as well. □

3.2. **Derived Poincaré categories of schemes.**

Definition 3.2.1. Let $(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}})^{\otimes}$ be the symmetric monoidal ∞-category of stable ∞-categories equipped with a t-structure on its Ind-completion, that is, $(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}})^{\otimes}$ is the ∞-category sitting in the pullback square

$$\begin{array}{ccc} (\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}})^{\otimes} & \longrightarrow & (\mathrm{Pr}^{L}_{\mathrm{st,t}})^{\otimes} \\ \downarrow & & \downarrow \\ (\mathrm{Cat}^{\mathrm{ex}})^{\otimes} & \xrightarrow{\mathrm{Ind}} & (\mathrm{Pr}^{L}_{\mathrm{st}})^{\otimes} . \end{array}$$

By (52) of Appendix A, we have a functor

$$\mathrm{qSch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}) \qquad X \mapsto \mathcal{D}^{\mathrm{p}}(X) \quad (= (\mathcal{D}^{\mathrm{p}}(X), \mathcal{D}^{\mathrm{qc}}(X)_{\geq 0}, \mathcal{D}^{\mathrm{qc}}(X)_{\leq 0})),$$

sending a quasi-compact quasi-separated (qcqs for short) scheme to its perfect derived category, considered as a stably symmetric monoidal ∞-category endowed with t-structure on its Ind-completion. By [BZFN10, Proposition 4.6] this last functor almost preserves pushouts (which on the right hand side correspond to relative tensor products of stable ∞-categories): more precisely, this functor is pushout preserving up to idempotent completion, and hence becomes pushout preserving if instead of $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$ we consider its localisation $\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ by the collection of Karoubi equivalences (i.e. dense inclusions). This localisation is multiplicative, so that $\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ inherits from $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$ its symmetric monoidal structure. Concretely, if we identify the localisation $\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ with the full subcategory of $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$ spanned by those $(\mathcal{C}, t)$ such that $\mathcal{C}$ is idempotent complete, then the localised tensor product is computed by first taking the tensor product in $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$ and then taking idempotent completion (an operation which does not affect the Ind-completion, so that the t-structure remains unchanged). The localisation functor $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}} \to \mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ then canonically refines to a symmetric monoidal functor, and induces a functor

$$\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}) \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}).$$

By post-composition, we thus obtain a functor

$$\mathrm{qSch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}) \qquad X \mapsto \mathcal{D}^{\mathrm{p}}(X),$$

which by [BZFN10, Proposition 4.6] now preserves pushouts. Since each $\mathcal{D}^{\mathrm{p}}(X)$ is rigid as a symmetric monoidal ∞-category the functor above factors through the full subcategory $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}) \subseteq \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})$ spanned by those $(\mathcal{C}, t)$ for which the idempotent complete $\mathcal{C}$ is rigid. Since $\mathrm{spec}(\mathbb{Z})$ is initial in $\mathrm{qSch}^{\mathrm{op}}$, this functor canonically refines to a functor

$$\mathrm{qSch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})_{\mathcal{D}^{\mathrm{p}}(\mathbb{Z})/} = \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Mod}_{\mathcal{D}^{\mathrm{p}}(\mathbb{Z})}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}))$$

to rigid idempotent complete $\mathbb{Z}$-linearly symmetric monoidal ∞-categories, and this last functor now preserves both pushouts and initial objects by construction, so that it preserves all finite colimits. Similarly, for any base scheme $S \in \mathrm{qSch}^{\mathrm{op}}$ we obtain a finite colimit preserving functor

$$(\mathrm{qSch}_S)^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})_{\mathcal{D}^{\mathrm{p}}(S)/} = \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Mod}_{\mathcal{D}^{\mathrm{p}}(S)}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}))$$

from qcqs schemes over S to idempotent complete $\mathcal{D}^{\mathrm{p}}(S)$-linear symmetric monoidal ∞-categories. In particular, the last functor preserves coproducts, and hence can be considered as a symmetric monoidal functor between the cocartesian symmetric monoidal structures on both sides. On the side of $(\mathrm{qSch}_S)^{\mathrm{op}}$ this corresponds to taking fibre product over S, and on the side of $\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})_{\mathcal{D}^{\mathrm{p}}(S)/}$, this corresponds to tensoring over $\mathcal{D}^{\mathrm{p}}(S)$.

Construction 3.2.2. We define the following ∞-categories by means of pullback:

$$(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} := (\mathrm{Cat}^{\mathrm{ps}})^{\otimes} \times_{(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}} (\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes} \quad \text{and} \quad \mathrm{Inv}^{\otimes}_{\natural,\mathrm{t}} := \mathrm{Inv}^{\otimes} \times_{\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}})^{\otimes}} \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes}.$$

By Proposition 3.1.8 the square (11) then induces a commutative square

$$\begin{array}{ccc} (\mathcal{C}, L) & \longmapsto & (\mathcal{C}, Ϙ^{\mathrm{s}}_{L}) \\ \Cap & & \Cap \\ \mathrm{Inv}^{\otimes}_{\natural,\mathrm{t}} & \longrightarrow & (\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes} & \longrightarrow & (\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes} . \end{array}$$

of symmetric monoidal ∞-categories and symmetric monoidal functors between them. We note that the ∞-category $\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ admits all colimits and hence $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})$ admits all colimits and the forgetful functor $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}) \to \mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ preserves and detects sifted colimits. By 3.1.8 all the ∞-categories in the above square admit sifted colimits and all functors preserve and detect them. In addition, since the monoidal structure on $\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}$ preserves colimits in each variable separately it follows that the monoidal structures of all symmetric monoidal ∞-categories in the above square preserve sifted colimits in each variable. In particular, for any stable symmetric monoidal ∞-category $\mathcal{A}^{\otimes}$ with unit $\mathbb{1} \in \mathcal{A}$ the square (11) induces a square of symmetric monoidal ∞-categories and symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{Mod}_{(\mathcal{A}^{\otimes},\mathbb{1})}(\mathrm{Inv}_{\natural,\mathrm{t}})^{\otimes} & \longrightarrow & \mathrm{Mod}_{(\mathcal{A}^{\otimes},Ϙ^{\mathrm{s}}_{\mathbb{1}})}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes}_{\mathcal{A}^{\otimes}/} & \longrightarrow & \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes} . \end{array}$$

Pulling back the left vertical map of the square of Construction 3.2.2, we now obtain a commutative square of symmetric monoidal ∞-categories and symmetric monoidal functors

$$\begin{array}{ccc} (X, L) & \longmapsto & (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L}) \\ \Cap & & \Cap \\ (\mathrm{qSchPic}^{\mathrm{op}}_{S})^{\otimes} & \longrightarrow & \mathrm{Mod}_{(\mathcal{D}^{\mathrm{p}}(S),Ϙ^{\mathrm{s}}_{S})}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} \\ \downarrow & & \downarrow \\ (\mathrm{qSch}^{\mathrm{op}}_{S})^{\otimes} & \longrightarrow & \mathrm{Mod}_{\mathcal{D}^{\mathrm{p}}(S)}(\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}})^{\otimes} , \end{array} \tag{12}$$

where

$$\mathrm{qSchPic}_{S} = \mathrm{qSch}^{\otimes}_{S} \times_{(\mathrm{CAlg}_{\mathrm{rig}}(\mathrm{Cat})^{\mathrm{op}})^{\otimes}} (\mathrm{Inv}^{\mathrm{op}})^{\otimes}$$

is the symmetric monoidal ∞-category with objects (X, L) where X is a qcqs scheme over S and $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2} = \mathcal{P}ic(\mathcal{D}^{\mathrm{p}}(X))^{\mathrm{BC}_2}$ is a tensor invertible perfect complex on X with C_2-action. Here, the left vertical arrow is again a left fibration of ∞-operads by base change, the top horizontal arrow is a lax symmetric monoidal functor between symmetric monoidal ∞-categories and the lower horizontal arrow is symmetric monoidal. The square (12) then encodes the principal functoriality and multiplicativity enjoyed by the association $(X, L) \mapsto (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L})$.

Let us now spell out a variant of the above construction which will be useful as well. For this, consider the left fibration of ∞-operads $\pi\colon (\mathrm{qSchPic}^{\mathrm{op}}_{S})^{\otimes} \to (\mathrm{qSch}^{\mathrm{op}}_{S})^{\otimes}$. Let

$$\mathcal{N}^{\otimes} = \mathcal{N}\mathrm{m}_{\mathrm{qSch}^{\mathrm{op}}_{S}}(\mathrm{qSchPic}^{\mathrm{op}}_{S}) \to \mathrm{Com}^{\otimes}$$

be the norm of π along the cocartesian fibration of ∞-operads $(\mathrm{qSch}^{\mathrm{op}}_{S})^{\otimes} \to \mathrm{Com}^{\otimes}$, see [Lur17a, Theorem 2.2.6.2]. Then $\mathcal{N}^{\otimes}$ is an ∞-operad whose underlying ∞-category is the ∞-category of sections $\mathrm{Sch}^{\mathrm{op}}_{S} \to \mathrm{qSchPic}^{\mathrm{op}}_{S}$ of π. Now the unit $\mathrm{id}_{S} \in \mathrm{Sch}^{\mathrm{op}}_{S}$ can be viewed as a map of ∞-operads $\mathrm{Com}^{\otimes} \to (\mathrm{Sch}^{\mathrm{op}}_{S})^{\mathrm{op}}$, and as such it induces a map

$$\mathcal{N}^{\otimes} = \mathcal{N}\mathrm{m}_{\mathrm{Sch}^{\mathrm{op}}_{S}}(\mathrm{qSchPic}^{\mathrm{op}}_{S}) \to \mathcal{N}\mathrm{m}_{\mathrm{Com}^{\otimes}}((\mathrm{qSchPic}^{\mathrm{op}}_{S})^{\otimes} \times_{(\mathrm{Sch}^{\mathrm{op}}_{S})^{\otimes}} \mathrm{Com}^{\otimes}) = \left(\mathrm{qSchPic}^{\mathrm{op}}_{S} \times_{\mathrm{Sch}^{\mathrm{op}}_{S}} \{\mathrm{id}_{S}\}\right)^{\otimes} = \mathcal{P}ic(S)^{\mathrm{BC}_2},$$

and since id_{S} is initial in $\mathrm{Sch}^{\mathrm{op}}_{S}$ and π is a left fibration we see that this map is actually an equivalence of ∞-operads. By the universal property of the norm construction this means that for every ∞-operad $\mathcal{O}^{\otimes}$ we have a natural equivalence between $\mathcal{O}$-algebras in $\mathcal{P}ic(S)^{\mathrm{BC}_2}$ and lax $\mathcal{O}$-monoidal sections $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{qSchPic}_S^{\mathrm{op}}$ of π. Better yet, since π is a left fibration any lax $\mathcal{O}$-monoidal section is automatically $\mathcal{O}$-monoidal. In particular, taking the unit algebra object $\mathcal{O}_X \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$ yields a symmetric monoidal section $s_0 : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{qSchPic}_S^{\mathrm{op}}$, given by the formula $[p : X \to S] \mapsto (X, \mathcal{O}_X)$, and any object $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$ (which can be canonically considered as a module over the unit $\mathcal{O}_X$) determines a section $s_L : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{qSchPic}_S^{\mathrm{op}}$ of $\mathrm{Sch}_S^{\mathrm{op}}$-module ∞-categories, where the target is considered as an $\mathrm{Sch}_S^{\mathrm{op}}$-module via s_0. Composing with the formation of derived categories we obtain a composable pair of symmetric monoidal functors

$$\begin{array}{ccccc} X & \longmapsto & (X, \mathcal{O}_X) & \longmapsto & (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{\mathcal{O}_X}) \\ \Cap & & \Cap & & \Cap \\ (\mathrm{Sch}_S^{\mathrm{op}})^{\otimes} & \longrightarrow & (\mathrm{qSchPic}_S^{\mathrm{op}})^{\otimes} & \longrightarrow & \mathrm{Mod}_{(\mathcal{D}^{\mathrm{p}}(S), \Omega^{\mathrm{s}}_S)}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes}, \end{array} \tag{13}$$

and similarly, for every $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$, we obtain a composable pair of $\mathrm{Sch}_S^{\mathrm{op}}$-module functors

$$\begin{array}{ccccc} [p : X \to S] & \longmapsto & (X, p^*L) & \longmapsto & (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{p^*L}) \\ \Cap & & \Cap & & \Cap \\ \mathrm{Sch}_S^{\mathrm{op}} & \longrightarrow & \mathrm{qSchPic}_S^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_{(\mathcal{D}^{\mathrm{p}}(S), \Omega^{\mathrm{s}}_S)}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \end{array} \tag{14}$$

where the Sch_S-action on the last two terms is determined by the symmetric monoidal functors in (13). The diagrams (12), (13) and (14) summarize the key types of functoriality relating schemes and Poincaré ∞-categories, and will be exploited systematically throughout the present paper.

Notation 3.2.3. For simplicity, we now suppress the mention of the Poincaré structure on the base and set

$$\mathrm{Mod}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) = \mathrm{Mod}_{(\mathcal{D}^{\mathrm{p}}(S), \Omega^{\mathrm{s}}_S)}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}).$$

We also use similar notation when $\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}$ is replaced by other variants. Note that this always means that the Poincaré structure considered on the base is the *symmetric* one.

3.3. **Genuine structures.** In this section, we explain the general formalism for truncating the linear part of a Poincaré structure, in terms of an arbitrary t-structure, although we'll only use the standard one on derived categories of schemes in the rest of the paper.

Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category. Then the first excisive approximation $\mathrm{P}_1\Omega = \mathrm{colim}_n\, \Omega^n \Omega \Sigma^n$, is an exact functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ and so can be identified with an element of $\mathrm{Ind}\,\mathcal{C}$. Suppose now that $\mathrm{Ind}\,\mathcal{C}$ is equipped with a t-structure. Then we say that Ω is *m-connective* if $\mathrm{P}_1\Omega$ is m-connective when seen as an element of $\mathrm{Ind}\,\mathcal{C}$.

Lemma 3.3.1. *Let $\mathcal{C}$ be a stable ∞-category and t be a t-structure on* $\mathrm{Ind}\,\mathcal{C}$. *Then the inclusion of the subcategory* $\mathrm{Fun}^{\mathrm{q}}_{\geq m}(\mathcal{C}) \subseteq \mathrm{Fun}^{\mathrm{q}}(\mathcal{C})$ *of m-connective quadratic functors has a right adjoint sending Ω to $\Omega^{\geq m} = \Omega \times_{\mathrm{P}_1\Omega} (\tau_{\geq m}\mathrm{P}_1\Omega)$, where the connective cover of* $\mathrm{P}_1\mathcal{C}$ *is taken in* $\mathrm{Ind}\,\mathcal{C}$.

Proof. It suffices to show that the map $\Omega^{\geq m} \to \Omega$ induces an equivalence

$$\mathrm{Map}(\Omega', \Omega^{\geq m}) \to \mathrm{Map}(\Omega', \Omega)$$

for every m-connected quadratic functor Ω'. But

$$\begin{aligned} \mathrm{Map}(\Omega', \Omega^{\geq m}) &\simeq \mathrm{Map}(\Omega', \Omega) \times_{\mathrm{Map}(\Omega', \mathrm{P}_1\Omega)} \mathrm{Map}(\Omega', \tau_{\geq m}\mathrm{P}_1\Omega) \\ &\simeq \mathrm{Map}(\Omega', \Omega) \times_{\mathrm{Map}(\mathrm{P}_1\Omega', \mathrm{P}_1\Omega)} \mathrm{Map}(\mathrm{P}_1\Omega', \tau_{\geq m}\mathrm{P}_1\Omega) \end{aligned}$$

and the map

$$\mathrm{Map}(\mathrm{P}_1\Omega', \tau_{\geq m}\mathrm{P}_1\Omega) \to \mathrm{Map}(\mathrm{P}_1\Omega', \Omega)$$

is an equivalence since $\mathrm{P}_1\Omega'$ is m-connected. □

Recall from Definition 3.2.1 that we write $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}} = \mathrm{Cat}^{\mathrm{ex}} \times_{\mathrm{Pr}^L} \mathrm{Pr}^L_t$ for ∞-category of pairs $(\mathcal{C}, t)$ where $\mathcal{C}$ is a stable ∞-category and $\mathrm{t} = (\mathrm{Ind}(\mathcal{C})_{\geq 0}, \mathrm{Ind}(\mathcal{C})_{\leq 0})$ is a t-structure on $\mathrm{Ind}(\mathcal{C})$. Let $\mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}} = \mathrm{Cat}^{\mathrm{p}} \times_{\mathrm{Cat}^{\mathrm{ex}}} \mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$. We then write objects in $\mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}}$ as triples $(\mathcal{C}, \Omega, t)$ where $(\mathcal{C}, \Omega)$ is a Poincaré ∞-category t is a t-structure on $\mathrm{Ind}(\mathcal{C})$.

Proposition 3.3.2. *For every m the inclusion $\mathrm{Cat}^{\mathrm{p}}_{t\geq m} \subseteq \mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}}$ of the full subcategory of the Poincaré ∞-categories whose Poincaré structure is m-connective in the above sense has a right adjoint, sending $(\mathcal{C}, Ϙ, t)$ to the triple $(\mathcal{C}, Ϙ^{\geq m}, t)$ where $\mathcal{C}$ and t are the same and $Ϙ^{\geq m}$ is the functor of Lemma 3.3.1.*

Proof. Note that $Ϙ^{\geq m}$ has the same cross-effect as $Ϙ$, so it is a perfect quadratic functor on $\mathcal{C}$ with the same underlying duality. In particular $(\mathcal{C}, Ϙ^{\geq m})$ is a Poincaré ∞-category. Moreover the excisive approximation of $Ϙ^{\geq m}$ is m-connective by definition. So it suffices to show that if $(\mathcal{C}', Ϙ', t')$ is a Poincaré ∞-category such that $Ϙ'$ is m-connective, the map

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}}}((\mathcal{C}', Ϙ', t'), (\mathcal{C}, Ϙ^{\geq m}, t)) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}}}((\mathcal{C}', Ϙ', t'), (\mathcal{C}, Ϙ, t))$$

is an equivalence. Since both of these spaces lie above the space $\mathrm{Map}_{\mathrm{Cat}^{\mathrm{ex}}_{\natural,\mathrm{t}}}((\mathcal{C}', t'), (\mathcal{C}, t))$ of t-right exact functors, it suffices to show that the above map induces an equivalence on the fibre over every $F\colon \mathcal{C}' \to \mathcal{C}$. The map on the fibre is the restriction of the map

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{q}}(\mathcal{C}')}\left(Ϙ', Ϙ^{\geq m}\circ F\right) \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{q}}(\mathcal{C}')}\left(Ϙ', Ϙ\circ F\right)$$

to the subspaces of those maps that induce an equivalence $F \to \mathrm{D}\circ F\circ \mathrm{D}'$. Since the underlying duality of $Ϙ$ and of $Ϙ^{\geq m}$ is the same, it suffices to show that the above map is an equivalence. Finally we can rewrite it as the map

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{q}}(\mathcal{C})}\left(F_!Ϙ', Ϙ^{\geq m}\right) \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{q}}(\mathcal{C})}\left(F_!Ϙ', Ϙ\right)$$

so by Lemma 3.3.1 it suffices to show that $F_!Ϙ'$ is m-connective. But $\mathrm{P}_1(F_!Ϙ') = F_!(\mathrm{P}_1Ϙ')$ and $F_!$ is right t-exact by definition. So, since $\mathrm{P}_1Ϙ'$ is m-connective, so is $F_!Ϙ'$. □

Note that if the t-structure is left complete, $t_{\geq\infty}\mathrm{P}_1Ϙ = 0$ and so $Ϙ^{\geq\infty}$ coincides with the quadratic Poincaré structure associated with the duality of $Ϙ$.

Specializing to derived categories of schemes, this has the following consequences. Using the unit of the adjunction of Proposition 3.3.2 the square (12) induces for any line bundle with C_2-action L on S and any integer $m \in \mathbb{Z}$ a functor

$$\mathrm{qSch}^{\mathrm{op}}_S \to \mathrm{Cat}^{\mathrm{p}}_{\natural,\mathrm{t}} \subset \mathrm{Cat}^{\mathrm{p}}_{\mathrm{t}} \qquad [p\colon X \to S] \mapsto (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_{p^*L}),$$

fitting together in a sequence

$$(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq\infty}_{p^*L}) = (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{q}}_{p^*L}) \to \cdots \to (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_{p^*L}) \to \cdots \to (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq -\infty}_{p^*L}) = (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{p^*L}).$$

Definition 3.3.3. Given a scheme X and $L \in \mathcal{P}ic(X)^{\mathrm{hC}_2}$, we call the Poincaré structures $Ϙ^{\geq 0}_L$, $Ϙ^{\geq 1}_L$ and $Ϙ^{\geq 2}_L$ on $\mathcal{D}^{\mathrm{p}}(X)$ the *genuine symmetric*, *genuine even* and *genuine quadratic* Poincaré structures, respectively, and use the notation $Ϙ^{\mathrm{gs}}_L$, $Ϙ^{\mathrm{ge}}_L$ and $Ϙ^{\mathrm{gq}}_L$ to refer to them.

By [HS21, Theorem A], when $X = \mathrm{spec}R$ is affine the Grothendieck-Witt space of $(\mathcal{D}^{\mathrm{p}}(R), Ϙ^r_L)$ for $r = \mathrm{gs}, \mathrm{ge}, \mathrm{gq}$ coincides with the group completion of the symmetric monoidal groupoid of projective R-modules equipped with a perfect L-valued symmetric bilinear/even/quadratic form, respectively. In other words, the non-negative genuine symmetric/even/quadratic GW-groups of spec(R) coincide with the classical symmetric/even/quadratic Grothendieck-Witt groups of R, as defined by Karoubi-Villamayor [KV71]. For the genuine symmetric case we will extend this comparison to all divisorial schemes in §4.6.

Notation 3.3.4. For a functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$ and a scheme X and an invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{\mathrm{hC}_2}$ we write

$$\mathcal{F}^{\geq m}(X, L) := \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_X)$$

for every $m \in \mathbb{Z} \cup \{-\infty, \infty\}$. For shorthand we will also write

$$\mathcal{F}^{\geq m}(X) := \mathcal{F}^{\geq m}(X, \mathcal{O}_X)$$

and if $p\colon X \to S$ is a S-scheme and $L \in \mathcal{P}ic(X)^{\mathrm{hC}_2}$ is an invertible perfect complex with C_2-action on S then we write

$$\mathcal{F}^{\geq m}_L(X) := \mathcal{F}^{\geq m}(X, p^*L).$$

More generally, for $n, m \in \mathbb{Z}$ we define

$$\mathcal{F}^{\geq m,[n]}_L(X) = \mathcal{F}\left(\mathcal{D}^{\mathrm{p}}(X), (Ϙ^{\geq m}_L)^{[n]}\right).$$

When $n = 0$, we drop the shift in the notation, which becomes $\mathcal{F}^{\geq m}_L$. Similarly, for $m = \pm\infty$ we replace the superscripts $(-)^{\geq -\infty}$ and $(-)^{\geq \infty}$ with $(-)^{\mathrm{s}}$ and $(-)^{\mathrm{q}}$, respectively. and for $L = \mathcal{O}_S$ we use the subscript S instead of $\mathcal{O}_S$. Using the functoriality of (14) we may view $\mathcal{F}^{\geq m}_L$ as a functor on Sch_S, or on the full subcategory $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$ spanned by the smooth S-schemes.

4. Nisnevich descent

Our main goal in this section is to show that for a qcqs scheme S, a perfect invertible complex with C_2-action $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$, and a Karoubi-localising functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$, the functor

$$\mathcal{F}^{\geq m}_L\colon (\mathrm{Sch}^{\mathrm{qq}}_S)^{\mathrm{op}} \to \mathcal{E} \qquad (X, L) \mapsto \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \mathrm{Q}^{\geq m}_X)$$

of Construction 3.3.4 satisfies Nisnevich descent, that is, it is a sheaf with respect to the Nisnevich topology for every $m \in \mathbb{Z} \cup \{\pm\infty\}$. To arrive to this, we first dedicate §4.1 and 4.2 to introduce and study the notions of flat S-linear ∞-categories (with duality) and bounded Karoubi sequences between them. The main reason for this setting is that it will eventually allow us in §7 to bootstrap Nisnevich descent into the construction of motivic realization functors with suitable multiplicative properties. In §4.3 we then prove Zariski descent for $\mathcal{F}^{\geq m}_L$, followed by Nisnevich descent in §4.4. Finally, §4.5 and §4.6 contain applications of these descent results, the former to establishing a version of the coniveau filtration in the present setting, and the second for the study of genuine symmetric GW-theory of schemes.

4.1. **Flat linear categories.** As discussed in §3 and in Appendix §A, for every scheme X, the stable ∞-category $\mathcal{D}^{\mathrm{p}}(X)$ comes equipped with a t-structure on its Ind-completion $\mathcal{D}^{\mathrm{qc}}(X)$. To facilitate terminology, let us refer to the data of a t-structure on the Ind-completion of a given stable ∞-category $\mathcal{A}$ as an *orientation* on $\mathcal{A}$. If $\mathcal{A}$ carries a symmetric monoidal structure (e.g., if $\mathcal{A} = \mathcal{D}^{\mathrm{p}}(X)$) then $\mathrm{Ind}(\mathcal{A})$ acquires an induced symmetric monoidal structure, in which case we use the term *multiplicative orientation* to mean a t-structure $(\mathrm{Ind}(\mathcal{A})_{\geq 0}, \mathrm{Ind}(\mathcal{A})_{\leq 0})$ on $\mathrm{Ind}(\mathcal{A})$ which is compatible with the induced symmetric monoidal structure, that is, $\mathrm{Ind}(\mathcal{A})_{\geq 0}$ contains $\mathbb{1}_{\mathcal{A}}$ and is closed under tensor product.

Definition 4.1.1. Let $\mathcal{A}$ be a stably symmetric monoidal ∞-category equipped with a multiplicative orientation. Given $x \in \mathrm{Ind}(\mathcal{A})$, we say that x *has Tor amplitude* $\leq n$ for a given $n \in \mathbb{Z}$ if the operation $x \otimes (-)$ sends coconnective objects to n-coconnective objects. We say that x has *bounded Tor amplitude* if it has Tor amplitude $\leq n$ for some n.

Remark 4.1.2. If $x \in \mathrm{Ind}(\mathcal{A})$ has Tor amplitude $\leq n$ and $\mathrm{Ind}(y) \in \mathcal{A}$ has Tor amplitude $\leq m$ then $x \otimes y$ has Tor amplitude $\leq n + m$.

Lemma 4.1.3. *Let $\mathcal{A}$ be a rigid stably symmetric monoidal ∞-category equipped with a multiplicative orientation. Then for a given $n \in \mathbb{Z}$, an object $x \in \mathcal{A}$ has Tor amplitude $\leq n$ if and only if* $\mathrm{D}x$ *is $(-n)$-connective.*

Proof. The dual $\mathrm{D}x$ has in particular the property that the functor $\mathrm{D}x \otimes (-)$ is left adjoint to $x \otimes (-)$. In addition, as in any t-structure, we have for every i the equalities

$$\mathrm{Ind}(\mathcal{A})_{\leq i} = \mathrm{Ind}(\mathcal{A})^{\perp}_{\geq i+1} = \{z \in \mathcal{A} \mid \hom(y, z) = 0,\ \forall y \in \mathrm{Ind}(\mathcal{A})_{\geq i+1}\}$$

and

$$\mathrm{Ind}(\mathcal{A})_{\geq i+1} = {}^{\perp}\,\mathrm{Ind}(\mathcal{A})_{\leq i} = \{x \in \mathcal{A} \mid \hom(x, y) = 0,\ \forall y \in \mathrm{Ind}(\mathcal{A})_{\leq i}\}.$$

We conclude that $x \otimes (-)$ sends $\mathrm{Ind}(\mathcal{A})_{\leq 0}$ to $\mathrm{Ind}(\mathcal{A})_{\leq n}$ if and only if $\mathrm{D}x \otimes (-)$ sends $\mathrm{Ind}(\mathcal{A})_{\geq n+1}$ to $\mathrm{Ind}(\mathcal{A})_{\geq 1}$, hence if and only if $\mathrm{D}x \otimes (-)$ sends $\mathrm{Ind}(\mathcal{A})_{\geq 0}$ to $\mathrm{Ind}(\mathcal{A})_{\geq -n}$. Since the t-structure is multiplicative, this is equivalent to $\mathrm{D}x$ belonging to $\mathrm{Ind}(\mathcal{A})_{\geq -n}$. □

Let $\mathcal{A}$ be a stably symmetric monoidal ∞-category. By an $\mathcal{A}$-linear ∞-category $\mathcal{C}$, we mean an $\mathcal{A}$-module object in $\mathrm{Cat}^{\mathrm{ex}}$. Such an $\mathcal{A}$-module structure determines a refinement of the mapping spectra in $\mathcal{C}$ to $\mathrm{Ind}(\mathcal{A})$-valued mapping objects, which we denote by $\underline{\hom}_{\mathcal{C}}(x, y) \in \mathrm{Ind}(\mathcal{A})$. They are uniquely characterized by the property that

$$\hom_{\mathrm{Ind}(\mathcal{A})}(j(a), \underline{\hom}_{\mathcal{C}}(x, y)) = \hom_{\mathcal{C}}(a \otimes x, y)$$

for any $a \in \mathcal{A}$, where $j\colon A \to \mathrm{Ind}(\mathcal{A})$ is the Yoneda embedding. We refer to morphisms of $\mathcal{A}$-modules in $\mathrm{Cat}^{\mathrm{ex}}$ as $\mathcal{A}$-linear functors, and write $\mathrm{Fun}^{\mathcal{A}}(\mathcal{C}, \mathcal{D})$ for the ∞-category of $\mathcal{A}$-linear functors from $\mathcal{C}$ to $\mathcal{D}$. Note that our terminology is such that an $\mathcal{A}$-linear functor is by definition exact.

Definition 4.1.4. Let $\mathcal{A}$ be a stably symmetric monoidal ∞-category equipped with a multiplicative orientation. We say that an $\mathcal{A}$-linear ∞-category $\mathcal{E}$ is *flat* over $\mathcal{A}$ if the enriched mapping object $\underline{\hom}(x, y) \in \mathrm{Ind}(\mathcal{A})$ has bounded Tor amplitude for every $x, y \in \mathcal{E}$.

Example 4.1.5. Let $X \to S$ be a flat morphism of qcqs schemes. Then $\mathcal{D}^{\mathrm{p}}(X)$ is flat over $\mathcal{A} = \mathcal{D}^{\mathrm{p}}(S)$.

Proof. The assumption that f is flat implies that $f^*\mathcal{D}^{\mathrm{qc}}(S) \to \mathcal{D}^{\mathrm{qc}}(X)$ is exact and hence its right adjoint $f_*\colon \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(S)$ preserves objects with bounded Tor amplitude $\leq n$ by the projection formula. We may hence assume without loss of generality that f is the identity. In this case the result follows directly from the characterization of Lemma 4.1.3. □

Recall that the tensor product $(-) \otimes_{\mathcal{A}} (-)$ in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{ex}})$ is determined by the following universal property: there exists a natural map $i\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$ which is $\mathcal{A}$-linear in each variable separately and such that for every $\mathcal{A}$-linear ∞-category $\mathcal{E}$ restriction along i determines an equivalence between $\mathcal{A}$-linear functors $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D} \to \mathcal{E}$ and functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which are $\mathcal{A}$-linear in each variable separately. In particular, we have a canonical equivalence

$$\mathrm{Fun}^{\mathcal{A}}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}, \mathcal{E}) = \mathrm{Fun}^{\mathcal{A}}(\mathcal{C}, \mathrm{Fun}^{\mathcal{A}}(\mathcal{D}, \mathcal{E})),$$

so that the functor $(-) \otimes_{\mathcal{A}} \mathcal{D}\colon \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{ex}})$ admits a right adjoint given by $\mathcal{E} \mapsto \mathrm{Fun}^{\mathcal{A}}(\mathcal{D}, \mathcal{E})$. To obtain a more explicit description of the underlying stable ∞-category of $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$, we may ignore for the moment its $\mathcal{A}$-action and view the association $\mathcal{C} \mapsto \mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$ as a functor $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat}^{\mathrm{ex}}) \to \mathrm{Cat}^{\mathrm{ex}}$. As such, it also admits a right adjoint, given by the formula $\mathcal{E} \mapsto \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$, where $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$ is considered as an $\mathcal{A}$-module via the pre-composition action determined by its action on $\mathcal{D}$. We may hence describe $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$ by embedding it in its Ind-completion, which by the last adjunction can be described as

$$\begin{aligned}
\mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) &= \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D})^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \\
&= \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{A}^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \\
&= \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{S}\mathrm{p})) \\
&= \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))\,,
\end{aligned} \tag{15}$$

where the $\mathcal{A}^{\mathrm{op}}$-action on $\mathrm{Ind}(\mathcal{D}) = \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{p})$ is the pre-composition action induced by the $\mathcal{A}^{\mathrm{op}}$-action on $\mathcal{D}^{\mathrm{op}}$. To identify the full subcategory of $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))$ corresponding to $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$, we need to track the composite functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathcal{A}} \mathcal{D} \to \mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) = \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})).$$

For this, one needs to embed $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))$ as a reflective subcategory (and accessible localisation) of the ∞-category $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}-\mathrm{oplax}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))$ of oplax $\mathcal{A}^{\mathrm{op}}$-linear functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{D})$. Then, to every $(c, d) \in \mathcal{C} \times \mathcal{D}$ we may associate the oplax $\mathcal{A}^{\mathrm{op}}$-linear functor $f_{c,d}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{D})$ given by the formula

$$f_{c,d}(x) = \underline{\hom}(x, c) \otimes d.$$

The image of (c, d) in $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))$ is then given by applying to $f_{c,d}$ the reflection functor

$$T\colon \ \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}-\mathrm{oplax}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})) \to \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})),$$

and $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$ is the smallest stable subcategory of $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}))$ containing the objects $T(f_{c,d})$ for every $c, d \in \mathcal{C} \times \mathcal{D}$. In general, this reflection functor can be hard to describe. However, under the assumption that $\mathcal{A}$ is rigid, we have that $f_{c,d}$ is already itself $\mathcal{A}^{\mathrm{op}}$-linear (as opposed to just oplax) for every $c, d \in \mathcal{C} \times \mathcal{D}$. In fact:

Lemma 4.1.6. *Let $\mathcal{I}$ be a rigid symmetric monoidal ∞-category (not necessarily stable) and $\mathcal{C}, \mathcal{D}$ two $\mathcal{I}$-linear ∞-categories (that is, $\mathcal{I}$-module objects in* Cat*). Then any functor $f\colon \mathcal{C} \to \mathcal{D}$ which is either lax or oplax $\mathcal{I}$-linear is already $\mathcal{I}$-linear.*

Proof. Let $\mathrm{B}\mathcal{I}$ be the $(\infty, 2)$-category with one object whose endomorphism ∞-category is $\mathcal{I}$. Then the $\mathcal{I}$-module objects $\mathcal{C}$ and $\mathcal{D}$ correspond to $(\infty, 2)$-functors $F_{\mathcal{C}}, F_{\mathcal{D}}\colon \mathrm{B}\mathcal{I} \to \mathrm{Cat}$. In this setup, an (op)lax $\mathcal{I}$-linear functor corresponds to an (op)lax natural transformation $F_{\mathcal{C}} \Rightarrow F_{\mathcal{D}}$. By [Hau20, Corollary 4.8], any lax (resp. oplax) natural transformation restricts to an honest natural transformation on the subcategory spanned by the right (resp. left) adjoint 1-morphism, that is, on the subcategory $\mathrm{B}\mathcal{I}_0 \subseteq \mathrm{B}\mathcal{I}$ where $\mathcal{I}_0 \subseteq \mathcal{I}$ is the full subcategory spanned by the right (resp. left) dualisable objects. But if $\mathcal{I}$ is rigid, any object is both left and right dualisable, so the desired result follows. □

In particular, when $\mathcal{A}$ is rigid, we may then simply view $f_{c,d}$ as objects of $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}))$, and $\mathcal{C}\otimes_{\mathcal{A}}\mathcal{D}$ is simply the smallest stable subcategory of $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}))$ containing the "pure tensor" objects $f_{c,d}$ for every $c, d \in \mathcal{C} \times \mathcal{D}$. We then have that

$$\underline{\hom}(f_{a,b}, f_{c,d}) = \underline{\hom}_{\mathcal{D}}(b, \underline{\hom}_{\mathcal{C}}(a, c) \otimes d) = \underline{\hom}_{\mathcal{C}}(a, c) \otimes \underline{\hom}_{\mathcal{D}}(b, d), \tag{16}$$

where the second equivalence is due to the assumption that $\mathcal{A}$ is rigid. Our discussion thus leads to the following conclusion:

Corollary 4.1.7. *Let $\mathcal{A}$ be a stably symmetric monoidal ∞-category equipped with a multiplicative orientation. If $\mathcal{A}$ is rigid, the collection of flat $\mathcal{A}$-linear ∞-categories is closed under the operation of taking tensor products over $\mathcal{A}$.*

Proof. Since the collection of objects with bounded Tor amplitude in $\mathrm{Ind}(\mathcal{A})$ is closed under finite colimits, desuspensions, and tensor products (see Remark 4.1.2), it follows from the formula (16) that if $\mathcal{C}$ and $\mathcal{D}$ are flat over $\mathcal{A}$, the same holds for $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$. □

4.2. **Bounded Karoubi projections.** We now fix a stably symmetric monoidal ∞-category $\mathcal{A}$ equipped with a multiplicative orientation $(\mathrm{Ind}(\mathcal{A})_{\geq 0}, \mathrm{Ind}(\mathcal{A})_{\leq 0})$ satisfying the following assumptions:

Assumption 4.2.1.

(1) $\mathcal{A}$ is rigid.
(2) $\mathbb{1}_{\mathcal{A}}$ is m-coconnective for some m.

We write $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$ for the unit object.

Remark 4.2.2. Since $\mathbb{1}_{\mathcal{A}}$ is assumed to be m-coconnective we have that if $x \in \mathrm{Ind}(\mathcal{A})$ has Tor amplitude $\leq n$ then x is $(n+m)$-coconnective.

Example 4.2.3. If S is any qcqs scheme then $\mathcal{A} = \mathcal{D}^{\mathrm{p}}(S)$ with the standard t-structure on $\mathrm{Ind}(\mathcal{A}) = \mathcal{D}^{\mathrm{qc}}(S)$ satisfies the above assumptions. This is the main example we have in mind.

Example 4.2.4. If R is an E_∞-ring spectrum which is both connective and n-coconnective for some $n \geq 0$ then the ∞-category $\mathcal{A} = \mathrm{Mod}(R)^{\omega}$ of compact R-module spectra, together with the standard t-structure on $\mathrm{Ind}(\mathcal{A}) = \mathrm{Mod}(R)$, satisfies the above assumptions.

Recall from §3.1 that since $\mathcal{A}$ is rigid, it carries a canonical duality given by $\mathrm{D}_{\mathcal{A}}(x) = \underline{\hom}_{\mathcal{A}}(x, \mathbb{1}_{\mathcal{A}})$, which is furthermore a symmetric monoidal duality, that is, $(\mathcal{A}, \mathrm{D}_{\mathcal{A}})$ is a commutative algebra object in $\mathrm{Cat}^{\mathrm{ps}}$. The assumption that the orientation on A is multiplicative means that we can also consider it as an algebra object in $\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}$. We can consequently consider $(\mathcal{A}, \mathrm{D}_{\mathcal{A}})$-modules in either $\mathrm{Cat}^{\mathrm{ps}}$ or $\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}$. In concrete terms, a module in $\mathrm{Cat}^{\mathrm{ps}}$ is an $\mathcal{A}$-linear ∞-category equipped with a duality $\mathrm{D}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ which is also $\mathcal{A}$-linear. The morphisms of such a given by duality preserving $\mathcal{A}$-linear functors

$$(\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \to (\mathcal{D}, \mathrm{D}_{\mathcal{D}}).$$

Similarly, $(\mathcal{A}, \mathrm{D}_{A})$-modules in $\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}$ are such $\mathcal{A}$-linear ∞-categories with duality further equipped with an orientation which is compatible with the orientation of A. Morphisms of such are then further required to induce a right t-exact functor on Ind completions.

Definition 4.2.5. We say that a duality preserving $\mathcal{A}$-linear functor $(\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \to (\mathcal{D}, \mathrm{D}_{\mathcal{D}})$ is an $\mathcal{A}$-linear Karoubi inclusion/projection if it is so on the level of underlying stable ∞-categories. Similarly, we will say that a sequence of $\mathcal{A}$-linear ∞-categories with duality is a Karoubi sequence if it is such on the level of underlying stable ∞-categories.

Definition 4.2.6. Let $p \colon (\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \to (\mathcal{D}, \mathrm{D}_{\mathcal{D}})$ be an $\mathcal{A}$-linear Karoubi projection and $g \colon \mathcal{D} \to \mathrm{Pro}(\mathcal{C})$ the associated fully-faithful Pro-left adjoint (which is not necessarily duality preserving). We say that p is *bounded at* $x \in \mathcal{D}$ if there exists an $n \in \mathbb{Z}$ and a cofiltered diagram $\{y_i\}_{i\in\mathcal{I}}$ in $\mathcal{C}$ with limit $g(x) \in \mathrm{Pro}(\mathcal{C})$ such that $\underline{\hom}(y_i, \mathrm{D}_{\mathcal{C}} y_i) \in \mathrm{Ind}(\mathcal{A})$ has Tor amplitude $\leq n$ for every $i \in \mathcal{I}$. We say that a square

$$\begin{array}{ccc} (\mathcal{C}, \mathrm{D}_{\mathcal{C}}) & \longrightarrow & (\mathcal{D}, \mathrm{D}_{\mathcal{D}}) \\ \downarrow & & \downarrow \\ (\mathcal{C}', \mathrm{D}_{\mathcal{C}'}) & \longrightarrow & (\mathcal{D}', \mathrm{D}_{\mathcal{D}'}) \end{array}$$

in $\mathrm{Mod}_{(\mathcal{A},\mathrm{D}_{\mathcal{A}})}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ is a *bounded Karoubi square* if it is cartesian and the vertical arrows are bounded $\mathcal{A}$-linear Karoubi projections. Such a bounded Karoubi square whose lower left corner is zero is also called a *bounded Karoubi sequence*.

Example 4.2.7. If $(\mathcal{C}, \mathrm{D}_{\mathcal{C}})$ is a flat $\mathcal{A}$-linear ∞-category with duality, and $p\colon (\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \to (\mathcal{D}, \mathrm{D}_{\mathcal{D}})$ is a duality preserving $\mathcal{A}$-linear functor whose underlying functor is a split Verdier projection then p is a bounded Karoubi projection in the sense of Definition 4.2.6.

Our main interest in the notion of bounded Karoubi projection is due to the following property:

Proposition 4.2.8. *Let* $p\colon (\mathcal{C}, \mathrm{D}) \to (\mathcal{C}', \mathrm{D}')$ *be a bounded Karoubi projection and* $m \in \mathbb{Z} \cup \{-\infty, \infty\}$. *Then for* $r \in \{\mathrm{q}, \mathrm{s}\}$ *the associated Poincaré functor*

$$p^r\colon (\mathcal{C}, \Omega^r_{\mathrm{D}}) \to (\mathcal{C}', \Omega^r_{\mathrm{D}'})$$

is a Poincaré-Karoubi projection. Suppose in addition that $(\mathcal{C}, \mathrm{D})$ *and* $(\mathcal{C}', \mathrm{D}')$ *are equipped with orientations which are compatible with the* $\mathcal{A}$*-action such that the induced functor* $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ *is t-exact. Then for every* $m \in \mathbb{Z}$ *the Poincaré functor*

$$p^{\geq m}\colon (\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) \to (\mathcal{C}', \Omega^{\geq m}_{\mathrm{D}'})$$

is a Poincaré-Karoubi projection.

The proof of Proposition 4.2.8 will require a fairly standard lemma concerning stable ∞-categories, which we include here since we could not locate a reference.

Lemma 4.2.9. *Let* $f\colon \mathcal{C} \to \mathcal{D}$ *be an exact functor between stable* ∞*-categories. Suppose that both* $\mathcal{C}$ *and* $\mathcal{D}$ *admit t-structures such that the t-structure on* $\mathcal{D}$ *is right separated, and that there exists a* $d \geq 0$ *such that* $f(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\leq d}$ *(e.g.,* f *is left t-exact). Let* $m \in \mathbb{Z}$ *be an integer,* K *a finite type simplicial set (that is,* K *has finitely many non-degenerate simplices in each degree) and* $\varphi\colon K \to \mathcal{C}_{\leq m}$ *a diagram which extends to a limit diagram* $\overline{\varphi}\colon K^{\triangleleft} \to \mathcal{C}$ *in* $\mathcal{C}$*. Then* $f(\overline{\varphi})$ *is a limit diagram in* $\mathcal{D}$.

We will eventually need the following special case of the above standard lemma:

Corollary 4.2.10. *Let* $\mathcal{C}$ *be a stable* ∞*-category equipped with a right separated t-structure and suppose that* $\mathcal{C}$ *admits filtered colimits such that* $\mathcal{C}_{\leq 0}$ *is closed under filtered colimits. Then filtered colimits of uniformly bounded above diagrams commute with finite type limits.*

Proof of Lemma 4.2.9. Let K_n be the n-skeleton of K, so that K is a finite simplicial set. Since $\mathcal{C}$ is stable K_n-indexed limits exist in $\mathcal{C}$ and in $\mathcal{D}$. Furthermore, for any r and any diagram $\psi\colon K \to \mathcal{C}_{\leq r}$ which admits a limit in $\mathcal{C}$, the fibre of

$$\lim_{k\in K} \psi(k) \to \lim_{k\in K_n} \psi(k)$$

is in $\mathcal{C}_{\leq r-n-1}$ by the Bousfield-Kan formula, and similarly for any diagram $\psi\colon K \to \mathcal{D}_{\leq r}$ admitting a limit in $\mathcal{D}$. Now let $\varphi\colon K \to \mathcal{C}_{\leq m}$ be a diagram admitting a limit in $\mathcal{C}$. Consider the commutative square

$$\begin{array}{ccc} f(\lim_K \varphi) & \longrightarrow & f(\lim_{K_n} \varphi) \\ \downarrow & & \downarrow \simeq \\ \lim_K (f\circ\varphi) & \longrightarrow & \lim_{K_n}(f\circ\varphi) \end{array}$$

where the right vertical map is an equivalence since K_n is finite. Since φ takes values in $\mathcal{C}_{\leq m}$ and f raises truncatedness by at most d, the fibres of the horizontal maps lie in $\mathcal{D}_{\leq(m+d-n-1)}$. Since the right vertical map is an equivalence the fibre of the left vertical map is in $\mathcal{D}_{\leq(m+d-n-1)}$. Since the t-structure is right separated we conclude that the left vertical map is an equivalence. □

Proof of Proposition 4.2.8. We prove all statements simultaneously by taking $m \in \mathbb{Z} \cup \{-\infty, \infty\}$, so that $Ϙ^{\geq \infty} = Ϙ^{q}$ and $Ϙ^{\geq -\infty} = Ϙ^{s}$. The Poincaré functor $p^{\geq m}$ determines a natural transformation

$$\mathrm{Lan}_p(Ϙ^{\geq m}_{\mathrm{D}}) \Rightarrow Ϙ^{\geq m}_{\mathrm{D}'}$$

from the left Kan extension of $Ϙ^{\geq m}_{\mathrm{D}}$ along $p^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}')^{\mathrm{op}}$ to $Ϙ^{\geq m}_{\mathrm{D}'}$, and since we already know that p is a Karoubi projection on the level of underlying stable ∞-categories, we only need to show that this transformation is an equivalence. Since $\mathcal{D}$ is generated as a stable ∞-category by the objects $x \in \mathcal{C}'$ at which p is bounded, it will suffice to show that the components of this natural transformation at such x is an equivalence. Choose a cofiltered diagram $\{y_i\}_{i \in \mathcal{J}}$ in $\mathcal{C}$ with limit $g(x)$, where $g\colon \mathcal{C}' \to \mathrm{Pro}(\mathcal{C})$ is the fully-faithful Pro-left adjoint of p. Then the left Kan extension can be computed via this presentation of $g(x)$, and we are reduced to showing that the induced map

$$\operatorname*{colim}_{i \in \mathcal{J}^{\mathrm{op}}} Ϙ^{\geq m}_{\mathrm{D}}(y_i) \to Ϙ^{\geq m}_{\mathrm{D}'}(x)$$

is an equivalence. If $m = \infty$ then

$$Ϙ^{\geq m}_{\mathrm{D}}(y_i) = Ϙ^{q}_{\mathrm{D}}(y_i) = \hom_{\mathcal{C}}(y_i, \mathrm{D}y_i)_{\mathrm{hC}_2},$$

and since $(-)_{\mathrm{hC}_2}$ commutes with colimits the statement follows from the fact that the induced map

$$\operatorname*{colim}_{i \in \mathcal{J}^{\mathrm{op}}} \hom_{\mathcal{C}}(y_i, \mathrm{D}y_i) = \operatorname*{colim}_{i,j \in \mathcal{J}^{\mathrm{op}}} \hom_{\mathcal{C}}(y_i, \mathrm{D}y_j) \to \hom_{\mathcal{C}'}(x, \mathrm{D}x)$$

is an equivalence by virtue of $p\colon \mathcal{C} \to \mathcal{C}'$ being a Karoubi projection. On the other extremity, if $m = -\infty$ then

$$Ϙ^{\geq m}_{\mathrm{D}}(y_i) = Ϙ^{s}_{\mathrm{D}}(y_i) = \hom_{\mathcal{C}}(y_i, \mathrm{D}y_i)^{\mathrm{hC}_2},$$

and we can rewrite the above map is the limit-colimit interchange map

$$\operatorname*{colim}_{i \in \mathcal{J}}[\hom_{\mathcal{C}}(y_i, \mathrm{D}y_i)^{\mathrm{hC}_2}] \to \left[\operatorname*{colim}_{i \in \mathcal{J}} \hom_{\mathcal{C}}(y_i, \mathrm{D}y_i)\right]^{\mathrm{hC}_2}.$$

Now, by assumption, there exists an n such that $\underline{\hom}(y_i, \mathrm{D}y_i) \in \mathrm{Ind}(\mathcal{A})$ has Tor amplitude $\leq n$ for every i, and is hence $(n+k)$-coconnective for every i, where k is such that $\mathbb{1}_{\mathcal{A}}$ is k-coconnective, see Remark 4.2.2. Since $\mathbb{1}_{\mathcal{A}}$ is also connective (by the assumption that the t-structure is multiplicative) we conclude that the family of spectra $\{\hom_{\mathcal{C}}(y_i, \mathrm{D}y_i)\}$ is uniformly bounded above, and so the desired result follows from the fact that limits of finite type diagrams of spectra commute with uniformly bounded above colimits (see Corollary 4.2.10). Finally, given that the case $m = \pm\infty$ is established, to show the case $m \neq \pm\infty$ it will suffice to prove that the induced map

$$\mathrm{Lan}_p\, \Lambda^{\geq m}_{\mathrm{D}} \to \Lambda^{\geq m}_{\mathrm{D}'}$$

is an equivalence, where $\Lambda^{\geq m}_{\mathrm{D}}$ and $\Lambda^{\geq m}_{\mathrm{D}}$ are the linear parts of $Ϙ^{\geq m}_{\mathrm{D}}$ and $Ϙ^{\geq m}_{\mathrm{D}'}$, respectively. Indeed, by definition these are the m-connective covers of the linear parts of $Ϙ^{s}_{\mathrm{D}}$ and $Ϙ^{s}_{\mathrm{D}'}$, and $\mathrm{Lan}_p = \mathrm{Ind}(p)\colon\ \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ is assumed t-exact when $m \neq \pm\infty$. □

Our principal examples of bounded Karoubi projections are the following:

Proposition 4.2.11. *Let S be a qcqs scheme and $j\colon U \to X$ an open embedding of qcqs smooth S-schemes. Let $L \in \mathcal{D}^{\mathrm{p}}(X)^{\mathrm{BC}_2}$ be an invertible perfect complex with C_2-action. Then, the duality preserving functor*

$$j^*\colon (\mathcal{D}^{\mathrm{p}}(X), \mathrm{D}_L) \to (\mathcal{D}^{\mathrm{p}}(U), \mathrm{D}_{j^*L})$$

is a bounded Karoubi projection (over $\mathcal{A} = \mathcal{D}^{\mathrm{p}}(S)$).

Corollary 4.2.12. *Let $j\colon U \to X$ an open embedding of qcqs schemes and let $L \in \mathcal{D}^{\mathrm{p}}(X)^{\mathrm{BC}_2}$ be an invertible perfect complex with C_2-action. Then for every $m \in \mathbb{Z} \cup \{-\infty, \infty\}$ the Poincaré functor*

$$(j^*, \eta)\colon (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_{L}) \to (\mathcal{D}^{\mathrm{p}}(U), Ϙ^{\geq m}_{j^*L})$$

is a Poincaré-Karoubi projection. In particular, the sequence

$$(\mathcal{D}^{\mathrm{p}}_Z(X), Ϙ^{\geq m}_{L}) \to (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_{L}) \to (\mathcal{D}^{\mathrm{p}}(U), Ϙ^{\geq m}_{L|_U}).$$

is a Poincaré-Karoubi sequence.

Corollary 4.2.13 (Localisation sequence)**.** *Let S be a qcqs scheme and $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$ be a Karoubi-localising functor valued in some stable presentable ∞-category $\mathcal{E}$ (e.g., $\mathcal{F} = \mathbb{GW}$ or $\mathcal{F} = \mathbb{L}$). Let X be a qcqs scheme equipped with a invertible perfect complex with* C_2*-action $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$. Then for every qc open subset U with $Z = X \smallsetminus U$ and every $m \in \mathbb{Z} \cup \{-\infty, \infty\}$ we have a fiber sequence*

$$\mathcal{F}(\mathcal{D}^{\mathrm{p}}_Z(X), \mathfrak{P}^{\geq m}_L) \to \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{P}^{\geq m}_L) \to \mathcal{F}(\mathcal{D}^{\mathrm{p}}(U), \mathfrak{P}^{\geq m}_{L|_U}) .$$

The proof of Proposition 4.2.11 uses the following lemma:

Lemma 4.2.14. *Let $\mathcal{A}$ be a stably symmetric monoidal ∞-category equipped with a multiplicative orientation* $(\mathrm{Ind}(\mathcal{A})_{\geq 0}, \mathrm{Ind}(\mathcal{A})_{\leq 0})$*. Suppose that every object in* $\mathrm{Ind}(\mathcal{A})_{\geq 0}$ *is a filtered colimit of objects in* $\mathcal{A} \cap \mathrm{Ind}(\mathcal{A})_{\geq 0}$*. Then every object $x \in \mathrm{Ind}(\mathcal{A})$ which has Tor amplitude $\leq n$ is a filtered colimits of objects with Tor amplitude $\leq n$ which furthermore belong to $\mathcal{A} \subseteq \mathrm{Ind}(\mathcal{A})$.*

Proof. Let us fix $x \in \mathrm{Ind}(\mathcal{A})$ with Tor amplitude $\leq n$. Then x is the colimit of the canonical filtered diagram $\mathcal{A}_{/x} \to \mathrm{Ind}(\mathcal{A})$. Let $\mathcal{I} \subseteq \mathcal{A}_{/x}$ be the full subcategory spanned by those $z \in x$ where z has Tor amplitude $\leq n$. We claim that the inclusion $\mathcal{I} \subseteq \mathcal{A}_{/x}$ is cofinal. By Quillen's theorem A what we need to verify is that if $f\colon z \to x$ is a map with $z \in \mathcal{A}$, then the ∞-category $\mathcal{A}_{z//x} \times_{\mathcal{A}} \mathcal{I}$ of factorizations of $z \to w \to x$ such that w is in $\mathcal{A}$ and has Tor amplitude $\leq n$, is weakly contractible.

First let us prove that $\mathcal{A}_{z//x} \times_{\mathcal{A}} \mathcal{I}$ is non empty. Since x has Tor amplitude $\leq n$ the object $x \otimes \tau_{\leq -n-1}(\mathrm{D}z)$ is (-1)-coconnective, and since the unit $\mathbb{1}_{\mathcal{A}}$ is connective (by the assumption that the orientation is multiplicative) we get that the dual map

$$\tilde{f}\colon \mathbb{1}_{\mathcal{A}} \to x \otimes \mathrm{D}z$$

lifts to $x \otimes t_{\geq -n}(\mathrm{D}z)$. Now, by our assumption, we have that $t_{\geq -n}(\mathrm{D}z)$ is a filtered colimit of $(-n)$-connective objects which belong to $\mathcal{A}$, and so we can find a $(-n)$-connective object $y \in \mathcal{A}$ such that $\tilde{f}$ factors as

$$\mathbb{1}_{\mathcal{A}} \to x \otimes y \to x \otimes \mathrm{D}z.$$

It then follows that $f\colon z \to x$ factors as

$$z \to \mathrm{D}y \to z,$$

where $\mathrm{D}y$ has Tor amplitude $\leq n$ by Lemma 4.1.3, and so $\mathcal{A}_{z//x} \times_{\mathcal{A}} \mathcal{I}$ is non-empty. □

Proof of Proposition 4.2.11. We first note that the underlying exact functor $j^*\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$ is indeed a Karoubi projection by A.5.8 (2). Let $g\colon \mathcal{D}^{\mathrm{p}}(U) \to \mathrm{Pro}\,\mathcal{D}^{\mathrm{p}}(X)$ be its Pro-left adjoint. We need to show that for every $R \in \mathcal{D}^{\mathrm{p}}(U)$ the object $g(R) \in \mathrm{Pro}\,\mathcal{D}^{\mathrm{p}}(X)$ can be written as the limit of a cofiltered diagram $\{P_\alpha\}_{\alpha \in \mathcal{I}}$ such that

$$\underline{\hom}(P_\alpha, \mathrm{D}_L P_\alpha) = \underline{\hom}(P_\alpha, \mathrm{D}_X P_\alpha \otimes L) = p_*(\mathrm{D}_X P_\alpha \otimes \mathrm{D}_X P_\alpha \otimes L) \in \mathcal{D}^{\mathrm{p}}(S)$$

has bounded Tor amplitude uniformly in α, where $p\colon X \to S$ is the structure map of X. Now the duality $\mathrm{D}_X\colon \mathcal{D}^{\mathrm{p}}(X) \xrightarrow{\simeq} \mathcal{D}^{\mathrm{p}}(X)^{\mathrm{op}}$ induces an equivalence

$$\widetilde{\mathrm{D}}_X\colon \mathcal{D}^{\mathrm{qc}}(X) = \mathrm{Ind}\,\mathcal{D}^{\mathrm{p}}(X) \xrightarrow{\simeq} \mathrm{Ind}\,\mathcal{D}^{\mathrm{p}}(X)^{\mathrm{op}} = \mathrm{Pro}(\mathcal{D}^{\mathrm{p}}(X))^{\mathrm{op}}$$

which intertwines the Pro-left and Ind-right adjoints of j^*. More precisely, we have

$$g(R) = \widetilde{\mathrm{D}}_X j_*(\mathrm{D}_U(R)),$$

where $j_*\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{qc}}(U)$ is the pushforward functor, which is the Ind-right adjoint of j^*. Replacing R with $\mathrm{D}_U R$, it will hence suffice to show that for every $R \in \mathcal{D}^{\mathrm{p}}(U)$, the quasi-coherent complex $j_*U \in \mathcal{D}^{\mathrm{qc}}$ can be written as a filtered colimit of perfect complexes $\{Q_\beta\}$ such that $p_*(Q_\beta \otimes Q_\beta \otimes L)$ has bounded Tor amplitude uniformly in β.

Since $p\colon X \to S$ is smooth it is in particular flat, and hence the pullback functor $p^*\colon \mathcal{D}^{\mathrm{qc}}(S) \to \mathcal{D}^{\mathrm{qc}}(X)$ preserves coconnective objects. By the projection formula, it follows that for any given n the pushforward functor $p_*\colon \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(S)$ preserves the property of having bounded Tor amplitude $\leq n$. It will hence suffice to find $\{Q_\beta\}$ such that $\{Q_\beta \otimes Q_\beta \otimes L\}$ has uniformly bounded Tor amplitude in $\mathcal{D}^{\mathrm{qc}}(X)$. Now since L is a perfect complex it has Tor amplitude $\leq n'$ for some n', and so it will suffice to show that Q_β can be chosen to have a uniformally bounded Tor amplitude. Indeed, since $j\colon U \to X$ is flat we have that j^* preserves coconnective objects and hence by the projection formula j_* preserves Tor amplitudes. In particular, since R is perfect it has Tor amplitude $\leq n$ for some n by Lemma 4.1.3, which means that j_*R

has Tor amplitude $\leq n$, and so by Lemma 4.2.14 it can be written as a filtered colimit of objects with Tor amplitude $\leq n$, as desired. ☐

Finally, let us prove that bounded Karoubi sequences are stable under tensoring with flat ∞-categories with duality. This property is what makes the notion of bounded Karoubi sequence useful to set up a multiplicative framework for motivic realization in §7.2.

Proposition 4.2.15. *Let $\mathcal{A}$ be stably symmetric monoidal ∞-category equipped with a multiplicative orientation and satisfying Assumption 4.2.1. Let $(\mathcal{C}, \mathrm{D})$ be an $\mathcal{A}$-linear ∞-category with duality. Then the operation of tensoring with $(\mathcal{C}, \mathrm{D})$*

$$(\mathcal{C}, \mathrm{D}) \otimes_{A} (-): \ \mathrm{Mod}_{(\mathcal{A},\mathrm{D}_{\mathcal{A}})}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathrm{Mod}_{(\mathcal{A},\mathrm{D}_{\mathcal{A}})}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$$

preserves Karoubi sequences of $\mathcal{A}$-linear ∞-categories with duality. Furthermore, if $(\mathcal{C}, \mathrm{D})$ is flat then $(\mathcal{C}, \mathrm{D}) \otimes_{A} (-)$ preserves bounded Karoubi sequences.

Proof. For the first claim, note that since the operation $\mathcal{C} \otimes_{\mathcal{A}} (-)$ preserves colimits, it suffices to show that it preserves Karoubi inclusions, that is, fully-faithful embeddings. For this, it will suffice to show that if $\mathcal{D} \to \mathcal{D}'$ is fully-faithful then the induced functor

$$\mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) \to \mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}')$$

is fully-faithful. The formula (15) gives us canonical equivalences

$$\begin{aligned}\mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) &= \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C} \otimes_{A} \mathcal{D})^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \\ &= \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))\ .\end{aligned}$$

To avoid confusion, let us point out that the functoriality of $\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}))$ in $\mathcal{D}$ is slightly delicate: given an $\mathcal{A}$-linear functor $f : \mathcal{D} \to \mathcal{D}'$, one has an associated $\mathcal{A}^{\mathrm{op}}$-linear functor $f^* : \ \mathrm{Ind}(\mathcal{D}') \to \mathrm{Ind}(\mathcal{D})$ induced by restriction along f, and hence an induced $\mathcal{A}^{\mathrm{op}}$-linear functor

$$\mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}')) \to \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})),$$

whose left adjoint is the $\mathcal{A}$-linear functor encoding the covariant dependence in $\mathcal{D}$. This left adjoint is generally *not* induced by post-composition with $f_! : \ \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{D}')$, since the latter is generally not $\mathcal{A}^{\mathrm{op}}$-linear (only oplax $\mathcal{A}^{\mathrm{op}}$-linear). However, if the oplax $\mathcal{A}^{\mathrm{op}}$-linear structure on $f_!$ happens to be strict, then post composition with $f_!$ is well-defined and hence yields a left adjoint to post-composition with $f^* : \ \mathrm{Ind}(\mathcal{D}') \to \mathrm{Ind}(\mathcal{D})$. This phenomenon happens for example when $\mathcal{A}$ is assumed to be rigid, since then by adjunction the $\mathcal{A}^{\mathrm{op}}$-action on $\mathrm{Ind}(\mathcal{D})$ is given by pre-composing the $\mathcal{A}$-action on it with the duality $\mathrm{D}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$, and hence any $\mathcal{A}$-linear functor $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{D}')$ is also $\mathcal{A}^{\mathrm{op}}$-linear. In particular, if $f : \mathcal{D} \to \mathcal{D}'$ is fully-faithful then $f_! : \ \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{D}')$ is fully-faithful and so the post-composition functor

$$f_! \circ : \ \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})) \to \mathrm{Fun}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D}'))$$

is fully-faithful, so that

$$\mathrm{Ind}(\mathcal{C} \otimes_{A} \mathcal{D}) \to \mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}')$$

is fully-faithful, as desired.

To prove the second claim, let us assume that $(\mathcal{C}, \mathrm{D})$ is flat and let $p : \ (\mathcal{D}, \mathrm{D}_{\mathcal{D}}) \to (\mathcal{E}, \mathrm{D}_{\mathcal{E}})$ be a bounded Karoubi projection of $\mathcal{A}$-linear ∞-categories with duality. By the first part of the proposition we have that the induced functor

$$\mathcal{C} \otimes_{\mathcal{A}} p : \ (\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \otimes_{\mathcal{A}} (\mathcal{D}, \mathrm{D}_{\mathcal{D}}) \to (\mathcal{C}, \mathrm{D}_{\mathcal{C}}) \otimes_{\mathcal{A}} (\mathcal{E}, \mathrm{D}_{\mathcal{E}})$$

is also a Karoubi projection. We now show that $\mathcal{C} \otimes_{\mathcal{A}} p$ is bounded. Since $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{E}$ is generated as a stable ∞-category by the pure tensors $z \otimes x \in \mathcal{C} \otimes_{\mathcal{A}} \mathcal{E}$, and by assumption $\mathcal{E}$ is generated as a stable ∞-category by the collection of $x \in \mathcal{E}$ at which p is bounded, it will suffice to show that if p is bounded at x then $\mathcal{C} \otimes_{\mathcal{A}} p$ is bounded at each $z \otimes x$ for every $z \in \mathcal{E}$. Let $\{y_i\}_{i \in \mathcal{I}}$ be a cofiltered in $\mathcal{D}$ family with limit $g(x)$, where $g : \ \mathcal{E} \to \mathrm{Pro}(\mathcal{D})$ is a Pro-left adjoint of p. Then $\{z \otimes y_i\}_{i \in \mathcal{I}}$ is a cofiltered family in $\mathcal{C} \otimes \mathcal{D}$ with limit $z \otimes g(x)$ and by (16) we get that

$$\{\underline{\mathrm{hom}}_{\mathcal{C}}(z \otimes y_i, \mathrm{D}_{\mathcal{C}}(z) \otimes \mathrm{D}_{\mathcal{C}}(y_i))\}_{i \in \mathcal{I}} = \{\underline{\mathrm{hom}}_{\mathcal{C}}(z, \mathrm{D}_{\mathcal{C}}(z)) \otimes \underline{\mathrm{hom}}_{\mathcal{C}}(y_i, \mathrm{D}(y_i))\}_{i \in \mathcal{I}},$$

and so $\mathcal{C} \otimes_{\mathcal{A}} p$ is bounded at $z \otimes x$ by Remark 4.1.2. ☐

4.3. **Zariski descent.** In this section, we assemble the results of the previous two sections into various forms of Zariski descent.

Proposition 4.3.1. *Let X be a qcqs scheme and $j_0 : U_0 \hookrightarrow X \hookleftarrow U_1 : j_1$ a pair of qcqs open subschemes and write $j_{01} : U_0 \cap U_1 \hookrightarrow X$ for the inclusion of the intersection. Then for any invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$ on X and for every $-\infty \le m \le \infty$ the corresponding square of Poincaré ∞-categories*

$$\begin{array}{ccc} (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\geq m}_{L}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(U_1), \Omega^{\geq m}_{j_1^* L}) \\ \downarrow & & \downarrow \\ (\mathcal{D}^{\mathrm{p}}(U_0), \Omega^{\geq m}_{j_0^* L}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(U_0 \cap U_1), \Omega^{\geq m}_{j_{01}^* L}) \end{array} \tag{17}$$

is a Poincaré-Karoubi square.

Corollary 4.3.2 (Excision)**.** *Let X be a qcqs scheme, $U_0 \subseteq X$ a qcqs open subscheme with closed complement $Z = X \setminus U_0$ and U_1 an open subscheme which contains Z. Then, for every invertible perfect complex with C_2-action L and every $m \in \{-\infty, \infty\} \cup \mathbb{Z}$ the functor*

$$(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\geq m}_{L}) \to (\mathcal{D}^{\mathrm{p}}_{Z}(U_1), \Omega^{\geq m}_{L|_{U_1}})$$

is an equivalence of Poincaré ∞-categories.

Proof. This is the induced Poincaré functor on vertical fibres in the square (17). □

Proof of Proposition 4.3.1. By Corollary 4.2.12 all the Poincaré functors in the square (17) are Poincaré-Karoubi projections. To finish the proof we now show that the square is cartesian in $\mathrm{Cat}^{\mathrm{p}}$. For this, note that the underlying square of stable ∞-categories is cartesian by Remark A.5.7, and hence to prove that the square is cartesian in $\mathrm{Cat}^{\mathrm{p}}$ it will suffice to show that the square of Poincaré structures

$$\begin{array}{ccc} \Omega^{\geq m}_{L}(-) & \longrightarrow & \Omega^{\geq m}_{j_1^* L}(j_1^*(-)) \\ \downarrow & & \downarrow \\ \Omega^{\geq m}_{j_0^* L}(j_0^*(-)) & \longrightarrow & \Omega^{\geq m}_{j_{01}^* L}(j_{01}^*(-)) \end{array}$$

is exact. Now since the square (17) is cartesian on the level of underlying stable ∞-categories it is also cartesian on the level of underlying stable ∞-categories with duality, and hence the above square of Poincaré structures is cartesian for $m = -\infty, \infty$. For a general m it will hence suffice to show that this square becomes cartesian after passing to linear parts. Now, by definition, the linear part of Ω^{s}_{L} is represented by the object $\tau_{\geq m}\mathcal{E}_L \in \mathcal{D}^{\mathrm{qc}}(X)$. Since the Poincaré functors in the square (17) are Karoubi projections we have that for $\epsilon \in \{0, 1, 01\}$ the natural transformation

$$\Omega^{\geq m}_{L}(-) \Rightarrow \Omega^{\geq m}_{j_\epsilon^* L}(j_\epsilon^*(-))$$

exhibits $\Omega^{\geq m}_{j_\epsilon^* L}(-)$ as the left Kan extension of $\Omega^{\geq m}_{L}$ along j_ϵ^*. To finish the proof it will hence suffice to show that the square

$$\begin{array}{ccc} \tau_{\geq m}\mathcal{E}_L & \longrightarrow & (j_1)_*(j_1)^*(\tau_{\geq m}\mathcal{E}_L) \\ \downarrow & & \downarrow \\ (j_0)_*(j_0)^*(\tau_{\geq m}\mathcal{E}_L) & \longrightarrow & (j_{01})_*(j_{01})^*(\tau_{\geq m}\mathcal{E}_L) \end{array}$$

is exact in $\mathcal{D}^{\mathrm{qc}}(X)$. Indeed, this follows from the fact that the square

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{qc}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{qc}}(U_0) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}(U_0 \cap U_1) \end{array}$$

is cartesian, see Corollary A.3.5. □

Corollary 4.3.3. *Let X be a scheme equipped with a finite open covering $X = \cup_{i=1}^n U_i$ by opens. For every non-empty subsets $S \subseteq \{1, \dots, n\}$ let us denote by $U_S = \cap_{i\in S} U_i$ and $j_S : U_S \hookrightarrow X$ the associated inclusion. Then for any invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{BC_2}$ on X and for every $-\infty \leq m \leq \infty$ the Poincaré functor*

$$(\mathcal{D}^p(X), Ϙ^{\geq m}_L) \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} (\mathcal{D}^p(U_S), Ϙ^{\geq m}_{j_S^* L}) \tag{18}$$

is an equivalence of Poincaré ∞-categories.

Corollary 4.3.4. *Let S be a qcqs scheme and $L \in \mathcal{P}ic(S)^{BC_2}$ a invertible perfect complex with C_2-action. Then for every $m \in \mathbb{Z} \cup \{-\infty, +\infty\}$, the functor*

$$\mathrm{Sch}^{qq}_{/S} \to \mathrm{Cat}^p \qquad [p\colon X \to S] \mapsto (\mathcal{D}^p(X), Ϙ^{\geq m}_{p^* L})$$

is a Cat^p-valued sheaf with respect to the Zariski topology.

Corollary 4.3.5 (Zariski descent)**.** *Let S be a qcqs scheme equipped with a invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{BC_2}$. Let $\mathcal{F}\colon \mathrm{Cat}^p \to \mathcal{A}$ be a Karoubi-localising functor valued in some presentable ∞-category $\mathcal{A}$ (e.g., $\mathcal{A} = \mathcal{S}\mathrm{p}$ and $\mathcal{F}$ is $\mathbb{GW}$ or $\mathbb{L}$, or, say, $\mathcal{A} = \mathcal{S}$ and $\mathcal{F} = \mathrm{GW}(-^\natural)$). Then for every $m \in \mathbb{Z} \cup \{-\infty, +\infty\}$, the functor*

$$\mathcal{F}^{\geq m}_L \colon \mathrm{Sch}^{qq}_{/S} \to \mathcal{A} \qquad \mathcal{F}^{\geq m}_L(p\colon X \to S) = \mathcal{F}(\mathcal{D}^p(X), Ϙ^{\geq m}_{p^* L})$$

is a Zariski sheaf.

Proof. Combine Proposition 4.3.1 and Corollary B.2.7. □

Corollary 4.3.6. *Let X be qcqs scheme and $Z, W \subseteq X$ two closed subschemes with quasi-compact complements V, U, respectively. Let $L \in \mathcal{P}ic(X)^{BC_2}$ be an invertible perfect complex with C_2-action. Then for every $m \in \mathbb{Z} \cup \{-\infty, +\infty\}$, the Poincaré functor*

$$(\mathcal{D}^p_Z(X), Ϙ^{\geq m}_L) \to (\mathcal{D}^p_{U\cap Z}(U), Ϙ^{\geq m}_{L|_U})$$

is a Poincaré-Karoubi projection. In particular, the sequence

$$(\mathcal{D}^p_{Z\cap W}(X), Ϙ^{\geq m}_L) \to (\mathcal{D}^p_Z(X), Ϙ^{\geq m}_L) \to (\mathcal{D}^p_{U\cap Z}(U), Ϙ^{\geq m}_{L|_U}).$$

is a Poincaré-Karoubi sequence.

Proof. By excision (Corollary 4.3.2) the Poincaré functor in question fits in a cartesian square of Poincaré ∞-categories

$$\begin{array}{ccc} (\mathcal{D}^p_Z(X), Ϙ^{\geq m}_L) & \to & (\mathcal{D}^p_{U\cap Z}(U \cup V), Ϙ^{\geq m}_{L|_{U\cup V}}) \simeq (\mathcal{D}^p_{U\cap Z}(U), Ϙ^{\geq m}_{L|_U}) \\ \downarrow & & \downarrow \\ (\mathcal{D}^p(X), Ϙ^{\geq m}_L) & \longrightarrow & (\mathcal{D}^p(U \cup V), Ϙ^{\geq m}_{L|_{U\cup V}}) \end{array}$$

By [CDH+II, Proposition 1.1.9] Poincaré-Verdier projections are preserved under base change. Since the bottom horizontal arrow is a Poincaré-Verdier projection by Corollary 4.3.6 we conclude that the top horizontal arrow is a Poincaré-Verdier projection as well. □

4.4. **Nisnevich descent.** In this section, we upgrade the Zariski descent result of the previous section to Nisnevich descent.

Proposition 4.4.1. *Let $f : A \to B$ be a flat homomorphism of commutative rings and $S \subseteq A$ a set of elements such that for every $s \in S$, the map $A/sA \to B/sB$ is an isomorphism of A-modules. Let M be an invertible A-module with A-linear involution and $N = B \otimes_A M$ the induced invertible B-module (with B-linear involution). Then for every $m \in \mathbb{Z} \cup \{-\infty, +\infty\}$, the square*

$$\begin{array}{ccc} (\mathcal{D}^p(A), Ϙ^{\geq m}_M) & \longrightarrow & (\mathcal{D}^p(B), Ϙ^{\geq m}_M) \\ \downarrow & & \downarrow \\ (\mathcal{D}^p(A[S^{-1}]), Ϙ^{\geq m}_{M[S^{-1}]}) & \longrightarrow & (\mathcal{D}^p(B[f(S)^{-1}]), Ϙ^{\geq m}_{N[f(S)^{-1}]}) \end{array}$$

is a Poincaré-Karoubi square.

Proof. We apply [CDH+II, Proposition 4.4.20], and thus need to verify that its assumptions (i)-(v) hold in the present case. For (i)-(iv) this is clear: assumption (i) is by construction, (ii) holds in the commutative case whenever the involution on M is A-linear (see [CDH+II, Example 1.4.4]), (iii) is assumed directly and (iv) (the Ore condition) is automatic in the commutative case. To prove that (v) holds, we need to show that the boundary maps $\hat{\mathrm{H}}^n(\mathrm{C}_2, N[f(S)^{-1}]) \to \hat{\mathrm{H}}^n(\mathrm{C}_2, M)$ associated to the short exact sequence of abelian groups

$$0 \to M \to N \oplus M[S^{-1}] \to N[f(S)^{-1}] \to 0$$

vanish. Equivalently, we need to show that the map

$$0 \to \hat{\mathrm{H}}^n(\mathrm{C}_2, M) \to \hat{\mathrm{H}}^n(\mathrm{C}_2, N) \oplus \hat{\mathrm{H}}^n(\mathrm{C}_2, M[S^{-1}])$$

is injective for every n. Now, for odd n, the Tate cohomology group $\hat{\mathrm{H}}^n(\mathrm{C}_2, M)$ is given by the kernel of the norm map $M_{\mathrm{C}_2} \to M^{\mathrm{C}_2}$, while for even n by the cokernel of this map, or, equivalently, by the kernel of the norm map $M(-1)_{\mathrm{C}_2} \to M(-1)^{\mathrm{C}_2}$, where $M(-1)$ denotes M with its C_2-action sign twisted. It will hence suffice to show that the maps

$$M_{\mathrm{C}_2} \to N_{\mathrm{C}_2} \oplus M[S^{-1}]_{\mathrm{C}_2}$$

and

$$M(-1)_{\mathrm{C}_2} \to N(-1)_{\mathrm{C}_2} \oplus M(-1)[S^{-1}]_{\mathrm{C}_2}$$

are injective. Indeed, since the C_2-action on M is A-linear we can identify these maps with the result of applying the functors $M_{\mathrm{C}_2} \otimes_A (-)$ and $M(-1)_{\mathrm{C}_2} \otimes_A (-)$ to the injective A-module map

$$A \to B \oplus A[S^{-1}],$$

and so the desired result follows from the fact that the cokernel of the last map is $B[p(S)^{-1}]$, which is flat as an A-module by assumption. □

Corollary 4.4.2 (Nisnevich descent)**.** *Let S be a qcqs scheme equipped with a invertible perfect complex with C_2-action $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$. Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{A}$ be a Karoubi-localising functor valued in a presentable ∞-category $\mathcal{A}$ (e.g., $\mathcal{A} = \mathcal{S}\mathrm{p}$ and $\mathcal{F}$ is $\mathbb{GW}$ or $\mathbb{L}$, or, say, $\mathcal{A} = \mathcal{S}$ and $\mathcal{F} = \mathrm{GW}(-^{\natural})$) Then for every $m \in \{-\infty, +\infty\} \cup \mathbb{Z}$, the functor*

$$\mathcal{F}^{\geq m}_L\colon \mathrm{Sch}^{\mathrm{qq}}_{/S} \to \mathcal{A} \qquad \mathcal{F}^{\geq m}_L(p\colon X \to S) = \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\geq m}_{p^*L})$$

is a Nisnevich sheaf.

Proof. We first reduce to the case where L is a shift of a line bundle. For this, recall from Remark 3.0.2 that there exists a disjoint union decomposition $S = \coprod_n S_n$, and for each n a line bundle L_n on S_n, such that $L|_{S_n} = L_n[n]$. Now, since S is quasi-compact, the S_n's must be empty for all but finitely many n. In particular, we may write $S = S_{n_1} \coprod \cdots \coprod S_{n_r}$ for some $r \geq 1$. We now note that the category of qcqs schemes is extensive, so that, in particular, we have an equivalence of categories

$$\prod_{i=1}^{r} \mathrm{Sch}^{\mathrm{qq}}_{/S_{n_i}} \xrightarrow{\simeq} \mathrm{Sch}^{\mathrm{qq}}_{/S} \qquad \{[X_i \to S_{n_i}]\}_{i=1}^{r} \mapsto \Big[\coprod_{i=1}^{r} X_i \to S\Big]. \tag{19}$$

Let $\mathcal{F}^{\geq m}_i$ be the restriction of $\mathcal{F}^{\geq m}_L$ along the functor $(\mathrm{Sch}^{\mathrm{qq}}_{/S_{n_i}})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}}$ induced by post-composition with $S_{n_i} \hookrightarrow S$. Since $\mathcal{F}^{\geq m}_L$ is a Zariski sheaf by Corollary 4.3.5 it sends finite disjoint unions of schemes to products in $\mathcal{E}$. We may hence identify $\mathcal{F}^{\geq m}_L$ with the composite

$$\prod_{i=1}^{r} \mathrm{Sch}^{\mathrm{qq}}_{/S_{n_i}} \xrightarrow{\prod \mathcal{F}^{\geq m}_i} \prod_{i=1}^{r} \mathcal{E} \xrightarrow{\times} \mathcal{E}.$$

In addition, by Lemma B.2.4, the equivalence of categories (19) identifies Nisnevich epis on the right hand side with tuples of Nisnevich epis on the right hand side. It will hence suffice to show that each of the presheafs $\mathcal{F}^{\geq m}_i\colon (\mathrm{Sch}^{\mathrm{qq}}_{/S_{n_i}})^{\mathrm{op}} \to \mathcal{E}$ is a Nisnevich sheaf. Replacing S with S_{n_i} we may consequently assume without loss of generality that L is a shift of a line bundle.

Now since we already know that $\mathcal{F}_L^{\geq m}$ is a Zariski sheaf it will suffice by Proposition B.3.1 to show the restriction $\mathcal{F}_{\mathrm{aff}}^{\geq m} := (\mathcal{F}_L^{\geq m})|_{\mathrm{Aff}_{/S}}$ is a Nisnevich sheaf. We note that since all coverings in the Nisnevich topology consist of étale maps we have that $\mathcal{F}_{\mathrm{aff}}^{\geq m}$ is a Nisnevich sheaf if and only if for very affine scheme $\mathrm{spec}(R) \to S$ the restriction of $\mathcal{F}_{\mathrm{aff}}^{\geq m}$ to the category of affine étale R-schemes is a Nisnevich sheaf. By [Lur17b, Theorem B.5.0.3] the latter property is equivalent to $\mathcal{F}_{\mathrm{aff}}^{\geq m}$ vanishing on the empty scheme and satisfying Nisnevich excision for affine schemes in the sense of [Lur17b, Definition B.5.0.1], that is, for every map $f\colon A \to B$ of étale R-algebras, and every $s \in A$ such that the map $A/sA \to B/f(s)B$ is an isomorphism of A-modules, the square

$$\begin{array}{ccc} \mathcal{F}_{\mathrm{aff}}^{\geq m}(\mathrm{spec}(A)) & \longrightarrow & \mathcal{F}_{\mathrm{aff}}^{\geq m}(\mathrm{spec}(B)) \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathrm{aff}}^{\geq m}(\mathrm{spec}(A[s^{-1}]) & \longrightarrow & \mathcal{F}_{\mathrm{aff}}^{\geq m}(\mathrm{spec}(B[f(s)^{-1}]) \end{array}$$

is cartesian. To finish the proof, we now point out that $\mathcal{F}_{\mathrm{aff}}^{\geq m}$ indeed vanishes on the empty scheme since it is a Zariski sheaf, and satisfies Nisnevich excision for affine schemes since $\mathcal{F}$ is Karoubi-localising and for every invertible A-module with A-linear involution M and any $n \in \mathbb{Z}$ and the square

$$\begin{array}{ccc} \big(\mathcal{D}^{\mathrm{p}}(A), (\Omega_M^{\geq m})^{[n]}\big) & \longrightarrow & \big(\mathcal{D}^{\mathrm{p}}(B), (\Omega_M^{\geq m})^{[n]}\big) \\ \downarrow & & \downarrow \\ \big(\mathcal{D}^{\mathrm{p}}(A[s^{-1}]), (\Omega_{M[s^{-1}]}^{\geq m})^{[n]}\big) & \longrightarrow & \big(\mathcal{D}^{\mathrm{p}}(B[f(s)^{-1}]), (\Omega_{N[f(s)^{-1}]}^{\geq m})^{[n]}\big) \end{array}$$

is a Poincaré-Karoubi square by Proposition 4.4.1, where $N = M \otimes_A B$. ☐

For later use in §8, let us also record here the following mild variant of Corollary 4.4.2. Recall from (13) and Notation 3.2.3 that the association $(X \to S) \mapsto (\mathcal{D}^{\mathrm{p}}(X), \Omega_X^{\mathrm{s}})$ can be assembled to a functor of the form

$$\left(\mathrm{Sch}_{/S}^{\mathrm{qq}}\right)^{\mathrm{op}} \to \mathrm{Mod}_S(\mathrm{Cat}_{\natural,\mathrm{t}}^{\mathrm{ps}})$$

By Example 4.1.5, if $X \to S$ is flat, then its image under this functor lies in the full subcategory $\mathrm{Mod}_S^{\flat}(\mathrm{Cat}_{\natural,\mathrm{t}}^{\mathrm{ps}}) \subseteq \mathrm{Mod}_S(\mathrm{Cat}_{\natural,\mathrm{t}}^{\mathrm{ps}})$ spanned by those $\mathcal{D}^{\mathrm{p}}(S)$-linear ∞-categories with duality which are flat over $\mathcal{D}^{\mathrm{p}}(S)$ in the sense of Definition 4.1.4.

Corollary 4.4.3. *Let S be a qcqs scheme equipped with a invertible perfect complex with* C_2*-action $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$. Let $\mathcal{F}\colon \mathrm{Mod}_S^{\flat}(\mathrm{Cat}_{\natural,\mathrm{t}}^{\mathrm{ps}}) \to \mathcal{A}$ be a functor which sends bounded Karoubi squares (see Definition 4.2.6) to fibre squares. Then the functor*

$$\mathcal{F}_L^{\mathrm{s}}\colon \left(\mathrm{Sch}_{/S}^{\mathrm{qq}}\right)^{\mathrm{op}} \to \mathcal{A}, \qquad \mathcal{F}_L^{\mathrm{s}}(p\colon X \to S) = \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \Omega_L^{\mathrm{s}})$$

is a Nisnevich sheaf.

Proof. The proof is the same as the proof of 4.4.2: one shows that $\mathcal{F}_L^{\mathrm{s}}$ is a Zariski sheaf whose restriction to $\mathrm{Aff}_{/S}$ satisfies Nisnevich excision. In light of Proposition 4.2.11, the Zariski statement follows from the $m = -\infty$ case of Proposition 4.3.1 and the Nisnevich excision statement from the $m = -\infty$ case of Proposition 4.4.1. ☐

4.5. **The coniveau filtration.** Our goal in this subsection is to prove the following Poincaré analogue of the coniveau filtration along the lines of [Bal07]. Let $Z \subseteq X$ a closed subscheme with quasi-compact complement. For $0 \leq c \leq \infty$ we denote by $Z^{(c)}$ the set of points of Z of codimension c, that is, the set of points for which $\mathcal{O}_{Z,x}$ is $\geq c$. We write $\mathcal{D}_Z^{\mathrm{p}}(X)^{\geq c} \subseteq \mathcal{D}_Z^{\mathrm{p}}(X)$ for the full subcategory of $\mathcal{D}_Z^{\mathrm{p}}(X)$ spanned by those perfect complexes such that any point in the support of P belongs to $Z^{(c)}$. In particular $\mathcal{D}_Z^{\mathrm{p}}(X)^{\geq 0} = \mathcal{D}_Z^{\mathrm{p}}(X)$, and if Z has Krull dimension e then $\mathcal{D}_Z^{\mathrm{p}}(X)^{\geq e+1} = 0$. Since the duality preserves the support, the subcategory $\mathcal{D}_Z^{\mathrm{p}}(X)^{\geq c}$ inherits from $\mathcal{D}_Z^{\mathrm{p}}(X)$ its Poincaré structure.

Proposition 4.5.1. *Let X be a Noetherian scheme and $Z \subseteq X$ a closed subset with quasi-compact complement. Then the restriction functors $(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \mathfrak{Q}^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,x}), \mathfrak{Q}^{\geq m}_L)$ for $x \in Z^{(c)}$ fit in a Poincaré-Verdier sequence*

$$(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c+1}, \mathfrak{Q}^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \mathfrak{Q}^{\geq m}_L) \to \bigoplus_{x \in Z^{(c)}} (\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,x}), \mathfrak{Q}^{\geq m}_L)\,.$$

The main ingredient in constructing the Poincaré-Verdier sequence of Proposition 4.5.1 is the Poincaré-Verdier sequence of Corollary 4.3.6. The proof will require however a couple of additional lemmas.

Lemma 4.5.2. *Let X be a quasi-compact and quasi-separated scheme and let $Z_1, Z_2 \subseteq X$ two disjoint closed subsets such that $X \smallsetminus Z_i$ is quasi-compact. Then the canonical map of Poincaré ∞-categories*

$$(\mathcal{D}^{\mathrm{p}}_{Z_1}(X), \mathfrak{Q}^{\geq m}_L) \oplus (\mathcal{D}^{\mathrm{p}}_{Z_2}(X), \mathfrak{Q}^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_{Z_1 \cup Z_2}(X), \mathfrak{Q}^{\geq m}_L) \qquad (P_1, P_2) \mapsto P_1 \oplus P_2$$

is an equivalence.

Proof. For every $P_i \in \mathcal{D}^{\mathrm{p}}_{Z_i}(X)$ both $P_1 \otimes P_2$ and $P_1 \otimes P_2^\vee$ are supported on $Z_1 \cap Z_2 = \emptyset$ and so are 0. In particular, the map in the statement is a fully faithful Poincaré inclusion. It suffices to show it is essentially surjective, that is that every $P \in \mathcal{D}^{\mathrm{p}}_{Z_1 \cup Z_2}(X)$ can be written as $P \simeq P_1 \oplus P_2$ with P_i supported on Z_i.

Write $U_i = X \smallsetminus Z_i$ for the open complements and consider the $\mathcal{O}_X$-algebras $\mathcal{O}_{U_i}$. Since $U_1 \cap U_2 = X \smallsetminus (Z_1 \cup Z_2)$ we have that $P \otimes_{\mathcal{O}_X} \mathcal{O}_{U_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_2} = 0$. We can write $P_1 = P \otimes_{\mathcal{O}_X} \mathcal{O}_{U_2}$ and $P_2 = P \otimes_{\mathcal{O}_X} \mathcal{O}_{U_1}$, so that P_i is supported on Z_i. Consider now the map

$$P \to P_1 \oplus P_2\,,$$

obtained by tensoring $\mathcal{O}_X \to \mathcal{O}_{U_1} \oplus \mathcal{O}_{U_2}$ with P. This is an equivalence after restricting to U_i, and so it is an equivalence on $U_1 \cup U_2 = X$. Finally P_1 and P_2 are perfect because summand of a perfect complex. □

Lemma 4.5.3. *Let X be a quasi-compact quasi-separated scheme and $Z \subseteq X$ a closed subset with quasi-compact complement. Let $x \in Z$ be a point. Then there is a natural equivalence of Poincaré ∞-categories*

$$\operatorname*{colim}_{V \ni x} (\mathcal{D}^{\mathrm{p}}_{Z \cap V}(V), \mathfrak{Q}^{\geq m}_L) \simeq (\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,x}), \mathfrak{Q}^{\geq m}_L)\,,$$

where the colimit runs over all open neighborhoods of x which contain $X \setminus Z$.

Proof. Write $V_x = \mathrm{spec}(\mathcal{O}_{X,x})$. Given an open neighborhood V of x, write $U = V \setminus (V \cap Z)$ and consider the commutative diagram

$$\begin{array}{ccccc}
(\mathcal{D}^{\mathrm{p}}_{V \cap Z}(V), \mathfrak{Q}^{\geq m}_{L|_V}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(V), \mathfrak{Q}^{\geq m}_{L|_V}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(U), \mathfrak{Q}^{\geq m}_{L|_U}) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{D}^{\mathrm{p}}_x(V_x), \mathfrak{Q}^{\geq m}_{L|_{V_x}}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(V_x), \mathfrak{Q}^{\geq m}_{L|_{V_x}}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(V_x \setminus x), \mathfrak{Q}^{\geq m}_{L|_{V_x \setminus x}})
\end{array}$$

whose rows are Poincaré-Verdier sequences by Corollary 4.2.12. Since x is in Z we may identify $V_x \setminus x$ with $V_x \cap U$. We claim that the middle and right vertical arrows become equivalences when one takes the colimit over all open neighborhoods V of x. Equivalently, since the family of neighborhoods is a filtered poset, this is the same as saying that the middle and right vertical arrows exhibit their targets as the colimit of their domains when V ranges over all neighborhoods of x. Indeed, this holds on the level of underlying stable ∞-categories by Lemma A.5.14. At the same time, for every V the middle and right most vertical maps are Poincaré-Karoubi projections by Corollary A.5.13, and so their colimit is again a Poincaré-Karoubi projection. But any Poincaré-Karoubi projection whose underlying exact functor is an equivalence is itself an equivalence of Poincaré ∞-categories.

Now since $\mathrm{Cat}^{\mathrm{p}}$ is compactly generated filtered colimits commute with finite limits, and hence we conclude that the family of left most vertical Poincaré functors also exhibit their target as the colimit of their domains when V ranges over all neighborhoods of x. It is left to explain why one may as well consider only neighborhoods which contain $X \setminus Z$. For this, note that if $V \subseteq V'$ is a pair of neighborhoods of x such that $V \cap Z = V' \cap Z$ then the induced functor

$$(\mathcal{D}^{\mathrm{p}}_{V \cap Z}(V), \mathfrak{Q}^{\geq m}_{L|_V}) \to (\mathcal{D}^{\mathrm{p}}_{V' \cap Z}(V'), \mathfrak{Q}^{\geq m}_{L|_{V'}})$$

is an equivalence of Poincaré ∞-categories by excision (Corollary 4.3.2). It follows that the diagram $V \mapsto (\mathcal{D}^{\mathrm{p}}_{V\cap Z}(V), \Omega^{\geq m}_{L|_V})$ factors through the poset map $V \mapsto V \cup (X \setminus Z)$, considered as a map from the poset of open neighborhoods of x (with reverse inclusion) to the poset of open neighborhoods of x which contain $X \setminus Z$. This last map is cofinal and hence the desired result follows. □

Proof of Proposition 4.5.1. For ease of readability we will suppress the Poincaré structures from the proof.

For every $c \geq 0$, write $\mathcal{A}_Z(c)$ for the poset consisting of those closed subsets $W \subseteq Z$ such that all points of W have codimension $\geq c$ in Z, where the partial order is given by inclusion. Let $\mathcal{B}_Z(c) \subseteq \mathcal{A}_Z(c+1) \times \mathcal{A}_Z(c)$ be the full subposet spanned by those pairs (W', W) such that $W' \subseteq W$. Since the collection of subsets $\mathcal{A}_Z(c)$ is non-empty (it always contains the empty set) and closed under disjoint union we see that $\mathcal{A}_Z(c)$ is filtered. Similarly, $\mathcal{B}_Z(c)$ is filtered. For every $(W', W) \subseteq \mathcal{B}_Z(c)$ we have by Corollary 4.3.6 a Poincaré-Karoubi sequence

$$(\mathcal{D}^{\mathrm{p}}_{W'}(X), \Omega^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_{W}(X), \Omega^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L)\,.$$

Now since $\mathrm{Cat}^{\mathrm{p}}$ is compactly generated filtered colimits there compute with finite limits, and hence the collection of Poincaré-Verdier sequences is closed under filtered colimits. Taking the colimit of the above Poincaré-Verdier sequences for $(W', W) \in \mathcal{B}_Z(c)$ and using that both projections $\mathcal{A}_Z(c+1) \leftarrow \mathcal{B}_Z(c) \to \mathcal{A}_Z(c)$ are cofinal (their corresponding comma categories are non-empty and hence automatically filtered) we thus obtain a Poincaré-Verdier sequence

$$\operatorname*{colim}_{W'\in\mathcal{A}_Z(c+1)}(\mathcal{D}^{\mathrm{p}}_{W'}(X), \Omega^{\geq m}_L) \to \operatorname*{colim}_{W\in\mathcal{A}_Z(c)}(\mathcal{D}^{\mathrm{p}}_{W}(X), \Omega^{\geq m}_L) \to \operatorname*{colim}_{(W',W)\in\mathcal{B}_Z(c)}(\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L)\,.$$

Now for $W \in \mathcal{A}_Z(c)$ we have that $(\mathcal{D}^{\mathrm{p}}_W(X), \Omega^{\geq m}_L)$ is by construction a full Poincaré subcategory of $\mathcal{D}^{\mathrm{p}}(X)$, and so their colimit in $\mathrm{Cat}^{\mathrm{p}}$ coincides with their union, considered as a full subcategory of $\mathcal{D}^{\mathrm{p}}(X)$. This union is nothing but $\mathcal{D}^{\mathrm{p}}(X)^{\geq c}$: indeed, by definition a perfect complex $P \in \mathcal{D}^{\mathrm{p}}_Z(X)$ belongs to $\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq 0}$ if and only if its support, which is a closed subset of Z, contains only points of codimension $\geq c$, that is, if and only if $\mathrm{supp}(P) \in \mathcal{A}_Z(c)$. We hence conclude that the Poincaré-Verdier inclusion $(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c+1}, \Omega^{\geq m}_L) \subseteq (\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \Omega^{\geq m}_L)$ fits in a Poincaré-Verdier sequence of the form

$$(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c+1}, \Omega^{\geq m}_L) \to (\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \Omega^{\geq m}_L) \to \operatorname*{colim}_{(W',W)\in\mathcal{B}_Z(c)}(\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L)\,.$$

To conclude the proof it will now suffice to show that the last term in the above sequence is naturally equivalent to $\bigoplus_{x\in Z^{(c)}}(\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,x}), \Omega^{\geq m}_L)$. Indeed, the projection $\mathcal{B}_Z(c) \to \mathcal{A}_Z(c)$ is a cocartesian fibration and hence we may calculate the colimit in question as

$$\operatorname*{colim}_{(W',W)\in\mathcal{B}_Z(c)}(\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L) = \operatorname*{colim}_{W\in\mathcal{A}_Z(c)} \operatorname*{colim}_{\substack{W'\in\mathcal{A}_Z(c+1)\\ W'\subseteq W}}(\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L).$$

Let us fix a $W \in \mathcal{A}_Z(c)$ and assume first that it is irreducible of codimension c in Z. In particular, it contains a single point w of $Z^{(c)}$. But then $\mathcal{A}_W(c+1)$ is exactly the poset of proper closed subsets of W. That is, it is the opposite of the poset of open neighborhoods of the generic point w of W containing $X \smallsetminus W$. Therefore, by Lemma 4.5.3 we have that

$$\operatorname*{colim}_{\substack{W'\in\mathcal{A}_Z(c+1)\\ W'\subseteq W}}(\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W'), \Omega^{\geq m}_L) \simeq (\mathcal{D}^{\mathrm{p}}_w(\mathcal{O}_{X,w}), \Omega^{\geq m}_L)$$

Now for a general $W \in \mathcal{A}_W(c)$, as X is noetherian, we can write $W = W_1 \cup \cdots \cup W_n \cup \tilde{W}$, where the W_i are the irreducible components of codimension c and $\tilde{W} \in \mathcal{A}_W(c+1)$ is the union of all the irreducible components of codimension $\geq c+1$. Set

$$W_0 = \tilde{W} \cup \bigcup_{i\neq j} W_i \cap W_j \in \mathcal{A}_W(c+1)\,,$$

so that for every W' containing W_0 we have $W \smallsetminus W' = (W_1 \smallsetminus W') \amalg \cdots \amalg (W_n \smallsetminus W')$, where the unions are disjoint. Therefore by Lemma 4.5.2 we have

$$\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X \smallsetminus W') \simeq \bigoplus_{i=1}^{n} \mathcal{D}^{\mathrm{p}}_{W_i\smallsetminus W'}(X \smallsetminus W')\,.$$

Taking the colimit in W' we obtain, by the previous case

$$\operatorname*{colim}_{\substack{W'\in\mathcal{A}_Z(c+1)\\ W'\subseteq W}} (\mathcal{D}^{\mathrm{p}}_{W\smallsetminus W'}(X\smallsetminus W'), \Omega^{\geq m}_L) \simeq \bigoplus_{x\in Z^{(c)}\cap W} (\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,x}), \Omega^{\geq m}_L)\,.$$

Finally taking the colimit for $W \in \mathcal{A}_Z(c)$ we get the desired result. □

4.6. **Genuine symmetric GW-theory of schemes.** For a scheme X equipped with a line bundle L, its classical GW-groups were defined and studied in Hornbostel's thesis [Hor01] under the assumption that 2 is invertible, and later in the works of Schlichting [Sch10b, Sch10a] without this assumption, by considering the collection of vector bundles on X as an exact category with duality $\mathrm{D}_L = \underline{\mathrm{Hom}}_X(-, L)$, to which one can associate a symmetric GW-space using the hermitian Q-construction. More general flavours of forms were later considered in [Sch21] (see in particular [Sch21, Theorem 1.1] for a proof that this approach indeed generalizes classical Grothendieck-Witt groups of rings).

Using vector bundles gives a well-behaved theory when the scheme X is divisorial, a condition which insures that X has a sufficiently supply of such bundles. Our goal in this subsection is to show that for a divisorial scheme X, its classical symmetric GW-space in the above sense coincides with the GW-space of the Poincaré ∞-category $(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{gs}}_L)$, where $\Omega^{\mathrm{gs}}_L = \Omega^{\geq 0}_L$ is the genuine symmetric Poincaré structure, see 3.3.3. The idea is to use the Zariski descent result of Corollary 4.3.4 in order to reduce the comparison to the case where X is affine, which in turn follows from the main result of [HS21].

Despite its relation with classical symmetric GW-theory, our main focus in the present paper is not genuine symmetric GW-theory, but rather the homotopy symmetric one, that is, the one associated to the Poincaré structure Ω^{s}_L. These are not completely unrelated to each other: for regular Noetherian X of finite Krull dimension d, the genuine symmetric and homotopy symmetric GW-groups coincide in degrees $\geq d-1$, see Proposition 4.6.2 below.

Let us now fix a qcqs scheme X and a line bundle L on X. Recall that by the Bott-Genauer sequence (see [CDH+II, § 4.3]), the genuine symmetric GW-space $\mathcal{GW}^{\mathrm{gs}}(X, L) := \mathcal{GW}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{gs}}_L)$ fits into a fibre sequence of E_∞-groups

$$\mathcal{GW}^{\mathrm{gs}}(X, L) \to |\mathrm{CrQ}_\bullet(\mathcal{D}^{\mathrm{p}}(X))| \to |\mathrm{PnQ}_\bullet(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{gs}}_L)|.$$

To compare this with the classical symmetric GW-space of X defined using vector bundles note that by definition, the cofibre of the natural map $\Omega^{\mathrm{gs}}_L \Rightarrow \Omega^{\mathrm{s}}_L$ is an exact functor on $\mathcal{D}^{\mathrm{p}}(X)$ which is represented by the (-1)-truncated quasi-coherent complex $\tau_{\leq -1}\mathcal{E}_L \in \mathcal{D}^{\mathrm{qc}}(X) = \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}(X))$. For any vector bundle V on X the cofibre of $\Omega^{\mathrm{gs}}_L(V) \to \Omega^{\mathrm{s}}(V)$ is hence (-1)-truncated, and so the map

$$\Omega^\infty \Omega^{\mathrm{gs}}_L(V) \to \Omega^\infty \Omega^{\mathrm{s}}_L(V) = \mathrm{Map}(V \otimes V, L)^{\mathrm{hC}_2}$$

is an equivalence of spaces. In particular, $\Omega^\infty \Omega^{\mathrm{gs}}_L(V)$ is a discrete space whose π_0 is the abelian group $\mathrm{Hom}(V \otimes V, L)^{\mathrm{C}_2} = \mathrm{Hom}_X(V, \mathrm{D}_L(V))^{\mathrm{C}_2}$. We may consequently consider the commutative square

$$\begin{array}{ccc} |\mathrm{CrQ}_\bullet(\mathrm{Vect}(X))| & \longrightarrow & |\mathrm{PnQ}_\bullet(\mathrm{Vect}(X), \mathrm{D}_L)| \\ \downarrow & & \downarrow \\ |\mathrm{CrQ}_\bullet(\mathcal{D}^{\mathrm{p}}(X))| & \longrightarrow & |\mathrm{PnQ}_\bullet(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{gs}}_L)|\,, \end{array}$$

where on the top left corner we have the Q-construction of the exact category $\mathrm{Vect}(X)$ of vector bundles on X, and the top right corner the hermitian Q-construction of the exact category Vect equipped with the duality $\mathrm{D}_L = \underline{\mathrm{Hom}}_X(-, L)$, see [Sch10b]. Taking horizontal fibres yields a map of E_∞-groups

$$\varphi_{X,L}\colon\ \mathcal{GW}^{\mathrm{cl}}(X, L) \to \mathcal{GW}^{\mathrm{gs}}(X, L),$$

from the classical symmetric GW-space $\mathcal{GW}^{\mathrm{cl,s}}(X, L) := \mathcal{GW}^{\mathrm{s}}(\mathrm{Vect}(X), \mathrm{D}_L)$ of X to the associated genuine symmetric GW-space.

Proposition 4.6.1. *If X is divisorial then the map $\varphi_{X,L}$ is an equivalence.*

Proof. Let us consider the associations $U \mapsto \mathcal{GW}^{\mathrm{cl}}(U, L|_U)$ and $U \mapsto \mathcal{GW}^{\mathrm{gs}}(X, L|_U)$ as presheaves of spaces on the site of qcqs open subschemes of X. By [Sch10a, Theorem 16] the former is a sheaf for the Zariski topology, and by Corollary 4.3.5 the latter is one as well. The collection of maps $\varphi_{U,L|_U}$ determines a map between these two sheaves, and the claim we wish to prove is that this map is an equivalence on global sections. But by the main result of [HS21] the map $\varphi_{U,L|_U}$ is an equivalence whenever U is affine. It is hence an equivalence on stalks, and so an equivalence of sheaves, and in particular an equivalence on global sections. □

Proposition 4.6.1 tells us that, when X is a divisorial qcqs scheme, its classical symmetric Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{cl,s}}_n(X, L)$ with coefficient in a line bundle L are recovered in the present setting by the genuine symmetric Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{gs}}_n(X, L)$ for $n \geq 0$. If X is not divisorial we don't expect this to be always true. At the same time, for non-divisorial schemes we don't expect the GW-theory of vector bundles to be a well-behaved invariant, and in particular do not expect it to satisfy Zariski descent. We may hence consider genuine symmetric GW-theory as the natural extension of classical symmetric GW-theory from rings to schemes, in the sense that it agrees with the latter on affine schemes, while also satisfying Zariski, and even Nisnevich (Corollary 4.4.2), descent. In fact, by [Lur17b, Proposition 3.7.4.5] it is the unique such extension.

In the present paper we are mostly interested in homotopy symmetric GW-theory, which, as we prove in §6.3 has the additional advantage of being $\mathbb{A}^1$-invariant on regular Noetherian schemes of finite Krull dimension, and that constitutes our principal input in order to construct the hermitian K-theory spectrum in §8. In this context let us mention that the genuine symmetric GW-spectrum does not satisfy such $\mathbb{A}^1$-invariance. It is hence natural to try and compare genuine and homotopy symmetric GW-theories. The proposition below uses the Zariski descent and the results of [CDH⁺III] in order to obtain such a comparison in sufficiently high degrees for regular Noetherian schemes of finite Krull dimension.

In what follows, recall that a space $\mathcal{Z}$ is said to be (-1)-truncated if it is either empty or contractable, and n-truncated for $n \geq 0$ if the homotopy groups of each of its connected components have vanishing homotopy groups vanish in degrees $> n$. For $n < -1$ we take being n-truncated as meaning contractible.

Proposition 4.6.2. *Let X be a regular Noetherian scheme of finite Krull dimension d and L a line bundle with C_2-action on X. Then the homotopy fibres of the map*

$$\mathcal{GW}^{\mathrm{gs}}(X, L) \to \mathcal{GW}^{\mathrm{s}}(X, L)$$

are all $(d-3)$-truncated. In particular, if $d = 0, 1$ then this map is an equivalence.

Proof. As in the proof of Proposition 4.6.1, let us consider the associations $U \mapsto \mathcal{GW}^{\mathrm{gs}}(U, L|_U)$ and $U \mapsto \mathcal{GW}^{\mathrm{s}}(U, L|_U)$ as presheaves of spaces on the site of quasi-compact open subsets on X, which by Corollary 4.3.5 are actually Zariski sheaves. A point $s \in \mathcal{GW}^{\mathrm{s}}(X, L)$ is hence by definition a global section of the sheaf $\mathcal{GW}^{\mathrm{s}}(-, L|_{-})$, which we can equally consider as a map $s: \ast \to \mathcal{GW}^{\mathrm{s}}(-, L|_{-})$ from the terminal sheaf $\ast$. Let

$$\mathcal{K}_s := \mathcal{GW}^{\mathrm{gs}}(-, L|_{-}) \times_{\mathcal{GW}^{\mathrm{s}}(-, L|_{-})} \ast$$

be the associated fibre product sheaf, which we can equally view as the fibre of the map $\mathcal{GW}^{\mathrm{gs}}(-, L|_{-}) \to \mathcal{GW}^{\mathrm{s}}(-, L|_{-})$ over s. The desired claim can be reformulated as saying that for every s the spectrum of global sections of $\mathcal{K}_s$ is $(d-3)$-truncated. Since the collection of $(d-3)$-truncated spectra is closed under limits, it will suffice to show that the stalks of $\mathcal{K}_s$ are $(d-3)$-truncated. This now follows from the fact that $\mathcal{K}_s(U)$ is $(d-3)$-truncated for any affine $U \subseteq X$ by [CDH⁺III, Corollary 1.3.10]. □

5. Dévissage

Our goal in this section is to establish a dévissage result for symmetric GW- and L-theory of schemes, see Theorem 5.2.1 below. Technical details aside, this theorem says that if X is a regular Noetherian scheme of finite Krull dimension, $L \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$, and $i: Z \subseteq X$ is a regular embedding, then the Z-supported symmetric GW-theory of X with coefficients in L identifies with symmetric GW-theory of Z with coefficients in a suitable invertible complex $i^! L$ (and similarly for symmetric L-theory). To properly formulate this result we first investigate in §5.1 the conditions under which the push-forward functor refines to a Poincaré functor. It turns out that this holds in particular for regular embeddings, which allows us to formulate Theorem 5.2.1 below. After reducing the claim to the level of L-theory, our strategy consists of first addressing the case where X is the spectrum of a regular local ring in §5.3, and then extending to the general case in §5.4 using the coniveau filtration of §4.5.

5.1. **Push-forward as a Poincaré functor.** Recall that a map $f : X \to Y$ of qcqs schemes is said to be *quasi-perfect* if $f_* : \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(Y)$ preserves perfect complexes (equivalently, if its right adjoint $f^!$ preserves colimits, see Definition A.6.3 in Appendix A and the surrounding discussion). In particular, when f is quasi-perfect we may view f_* as an exact functor $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(Y)$.

Definition 5.1.1. Let $f : X \to Y$ be a map of qcqs schemes. It is called *quasi-Gorenstein* if $f^!\mathcal{O}_Y$ is an invertible perfect complex on X. We say that f is *qGqp* if it is quasi-Gorenstein and quasi-perfect. Equivalently (by Lemma A.6.2), f is qGqp if it is quasi-perfect and $f^!$ preserves tensor invertible objects.

Remark 5.1.2. By its second characterization we see that the property of being a qGqp map is closed under composition.

Construction 5.1.3. Let X, Y be qcqs schemes, let $N, M \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$ be invertible complexes with C_2-action on X and let $N \in \mathcal{P}ic(Y)^{\mathrm{BC}_2}$ be an invertible complex with C_2-action on Y. Let $f : X \to Y$ be a qGqp map and $\tau : N \otimes M^{\otimes 2} \xrightarrow{\simeq} f^!L$ a natural equivalence. We construct a natural transformation $\eta_{f,\tau} : Ϙ^{\mathrm{s}}_N \Rightarrow Ϙ^{\mathrm{s}}_L(f_*(- \otimes M))$ as the composite

$$\begin{aligned}
\eta_{f,\tau} : Ϙ^{\mathrm{s}}_N(P) &= \hom_X(P \otimes P, N)^{\mathrm{hC}_2} \\
&\xrightarrow{\simeq} \hom_X(M^{\otimes 2} \otimes P \otimes P, M^{\otimes 2} \otimes N)^{\mathrm{hC}_2} && (20) \\
&\xrightarrow{\simeq} \hom_X((M \otimes P) \otimes (M \otimes P), f^!L)^{\mathrm{hC}_2} && (21) \\
&= \hom_Y(f_*(P \otimes M \otimes P \otimes N), L)^{\mathrm{hC}_2} \\
&\to \hom_Y(f_*(P \otimes M) \otimes f_*(P \otimes M), L)^{\mathrm{hC}_2} && (22) \\
&= Ϙ^{\mathrm{s}}_L(f_*(P \otimes M))) .
\end{aligned}$$

where (20) is an equivalence since M is invertible, the arrow (21) is induced by post-composing with τ (and rearranging the terms in the domain) and is hence an equivalence, and (22) is induced from the lax symmetric monoidal structure of f_* (which in turn is induced by the symmetric monoidal structure on its left adjoint f^*, see [Lur17a, Corollary 7.3.2.7]). We then consider the pair $(f_*(- \otimes M), \eta_{f,\tau})$ as a hermitian functor

$$(f_*(- \otimes M), \eta_{f,\tau}) : (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_N) \to (\mathcal{D}^{\mathrm{p}}(Y), Ϙ^{\mathrm{s}}_L) .$$

When f and τ are implied, we denote $\eta_{f,\tau}$ simply by η.

Our next goal is to show that the hermitian functor of Construction 5.1.3 is always Poincaré. For this, we first show that the construction $(f_*(- \otimes M), \eta_{f,\tau})$ is compatible with composition in the following sense. Suppose given a composable pair

$$X \xrightarrow{f} X \xrightarrow{f'} Z$$

and invertible perfect complexes with C_2-action $N, M \in \mathcal{P}ic(X)^{\mathrm{BC}_2}$, $L, M' \in \mathcal{P}ic(X')^{\mathrm{BC}_2}$ and $P \in \mathcal{P}ic(Z)$ together with natural equivalences

$$\tau : N \otimes M^{\otimes 2} \xrightarrow{\simeq} f^!L \quad \text{and} \quad \tau' : L \otimes (M')^{\otimes 2} \xrightarrow{\simeq} (f')^!P.$$

Applying Construction 5.1.3 to (f, τ) and (f', τ'), we then obtain a composable pair of hermitian functors

$$(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_N) \xrightarrow{(f_*(-\otimes M), \eta_{f,\tau})} (\mathcal{D}^{\mathrm{p}}(Y), Ϙ^{\mathrm{s}}_L) \xrightarrow{(f'_*(-\otimes M'), \eta_{f',\tau'})} (\mathcal{D}^{\mathrm{p}}(Z), Ϙ^{\mathrm{s}}_P) .$$

On the other hand, we can also apply Construction 5.1.3 to the map $f'' := f' \circ f$ and the data of $N, M'' := M \otimes f^*M', P$ and the composite

$$\tau'' : N \otimes M^{\otimes 2} \otimes f^*(M')^{\otimes 2} \xrightarrow{\tau \otimes f^*(M')^{\otimes 2}} f^!L \otimes f^*(M')^{\otimes 2} \xrightarrow{\simeq} f^!(L \otimes (M')^{\otimes 2}) \xrightarrow{f^!\tau'} f^!(f')^!P,$$

yielding a hermitian functor

$$(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_N) \xrightarrow{(f''_*(-\otimes M''), \eta_{f'',\tau''})} (\mathcal{D}^{\mathrm{p}}(Z), Ϙ^{\mathrm{s}}_P) .$$

Lemma 5.1.4. *The three hermitian functors just constructed fit into a commutative triangle*

$$\begin{array}{ccccc} & & (\mathcal{D}^{\mathrm{p}}(Y), \Qoppa^{\mathrm{s}}_{L}) & & \\ & \overset{(f_*(-\otimes M),\eta_{f,\tau})}{\nearrow} & & \overset{(f'_*(-\otimes M'),\eta_{f',\tau'})}{\searrow} & \\ (\mathcal{D}^{\mathrm{p}}(X), \Qoppa^{\mathrm{s}}_{N}) & & \xrightarrow[(f''_*(-\otimes M''),\eta_{f'',\tau''})]{} & & (\mathcal{D}^{\mathrm{p}}(Z), \Qoppa^{\mathrm{s}}_{P}) . \end{array}$$

Proof. We first note that the claim directly follows from the relevant definitions if either $M' = \mathcal{O}_Y$ or $f = \mathrm{id}$. It hence suffices to prove the claim in the case where $M = \mathcal{O}_X$ and $f' = \mathrm{id}$. We may then also assume without loss of generality that $N = f^!L$, $P = L \otimes (M')^{\otimes 2}$ and both τ and τ' are the respective identities. Now the projection formula gives an equivalence

$$f_*(-) \otimes M' \xrightarrow{\simeq} f_*(- \otimes f^*M')$$

which traces a commuting homotopy in the underlying triangle of exact functors. To lift this to a commuting homotopy on the level of hermitian functors we then need to construct, for $L, M \in \mathcal{P}ic(Y)^{\mathrm{BC}_2}$ and $f\colon X \to Y$ a qGqp map, a commuting homotopy in the diagram

$$\begin{array}{ccccc} \Qoppa^{\mathrm{s}}_{f^!L}(-) & \longrightarrow & \Qoppa^{\mathrm{s}}_{f^!L\otimes(f^*M')^{\otimes 2}}(- \otimes f^*M') & \longrightarrow & \Qoppa^{\mathrm{s}}_{f^!(L\otimes(M')^{\otimes 2})}(- \otimes f^*M') \\ \downarrow & & & & \downarrow \\ \Qoppa^{\mathrm{s}}_{L}(f_*(-)) & \longrightarrow & \Qoppa^{\mathrm{s}}_{L\otimes(M')^{\otimes 2}}(f_*(-) \otimes M') & \longleftarrow & \Qoppa^{\mathrm{s}}_{L\otimes(M')^{\otimes 2}}(f_*(- \otimes f^*M')) \end{array}$$

where the top right horizontal map is induced by the equivalence $f^!L \otimes (f^*M')^{\otimes 2} \xrightarrow{\simeq} f^!(L \otimes (M')^{\otimes 2})$ of Lemma A.6.2, the bottom right horizontal map is induced by the equivalence $f_*(-)\otimes M' \xrightarrow{\simeq} f_*(-\otimes f^*M')$ of the projection formula, and the rest of the maps are instances of the natural transformation described in Construction 5.1.3. Unwinding the definitions further, it will suffice to construct, naturally in $F \in \mathcal{D}^{\mathrm{p}}(X)$, a commuting homotopy in the external octagon of the diagram of spectra with C_2-action

(23)

$$\begin{array}{ccccc} & & \hom_X(F \otimes F \otimes f^*(M')^{\otimes 2}, f^!L \otimes f^*(M')^{\otimes 2}) & & \\ & \nearrow & & \searrow & \\ \hom_X(F \otimes F, f^!L) & & & & \hom_X(F \otimes F \otimes f^*(M')^{\otimes 2}, f^!(L \otimes (M')^{\otimes 2})) \\ \downarrow & & & & \downarrow \\ \hom_Y(f_*(F \otimes F), L) & & & & \hom_X(f_*(F \otimes F \otimes f^*(M')^{\otimes 2}), L \otimes (M')^{\otimes 2}) \\ & \searrow & & \swarrow & \\ \downarrow & & \hom_Y(f_*(F \otimes F) \otimes (M')^{\otimes 2}, L \otimes (M')^{\otimes 2}) & & \downarrow \\ & & \downarrow & & \\ \hom_Y(f_*F \otimes f_*F, L) & & & & \hom_Y(f_*(F \otimes f^*M') \otimes f_*(F \otimes f^*M'), L \otimes (M')^{\otimes 2}) \\ & \searrow & & \swarrow & \\ & & \hom_Y(f_*F \otimes f_*F \otimes (M')^{\otimes 2}, L \otimes (M')^{\otimes 2}) & & \end{array}$$

To construct this homotopy, it will suffice to fill in the two bottom squares and the top hexagon. Now the bottom left square is easily filled: its horizontal arrows are induced on mapping spectra by the functor $(-) \otimes (M')^{\otimes 2}$, its left vertical map is induced by pre-composition with $\sigma\colon f_*F \otimes f_*F \to f_*(F \otimes F)$ and its right vertical map is induced by pre-composition with the map $\sigma \otimes \mathrm{id}\colon f_*F \otimes f_*F \otimes (M')^{\otimes} \to f_*(F \otimes F) \otimes (M')^{\otimes 2}$. The right bottom square, in turn, can be filled by observing that the counit map $f^*f_* \Rightarrow \mathrm{id}$ is a lax symmetric monoidal natural transformation, and hence the same holds for the projection formula transformation $f_*(-) \otimes (-) \Rightarrow f_*(- \otimes f^*(-))$ (considered as a transformation between functors in two arguments).

We now proceed to construct a commuting homotopy in the top hexagon of (23). Consider, for $G \in \mathcal{D}^{\mathrm{p}}(X)$ and $L, Q \in \mathcal{D}^{\mathrm{p}}(Y)$, the natural commutative diagram

$$
\begin{tikzcd}
& \hom_Y(f_*G, f_*f^!L) \ar[dr] & \\
\hom_X(G, f^!L) \ar[ur] \ar[d] & & \hom_Y(f_*G \otimes Q, f_*(f^!L) \otimes Q) \ar[d, "\simeq"] \\
\hom_X(G \otimes f^*Q, f^!L \otimes f^*Q) \ar[dd, "\simeq"] \ar[dr] & & \hom_Y(f_*G \otimes Q, f_*(f^!L \otimes f^*Q)) \ar[dd, "\simeq"] \\
& \hom_Y(f_*(G \otimes f^*Q), f_*(f^!L \otimes f^*Q)) \ar[ur, "\simeq"] \ar[d, "\simeq"] & \\
\hom_X(G \otimes f^*Q, f^!(L \otimes Q)) \ar[r] \ar[dr] & \hom_X(f_*(G \otimes f^*Q), f_*f^!(L \otimes Q)) \ar[r, "\simeq"] \ar[d] & \hom_X(f_*G \otimes Q, f_*f^!(L \otimes Q)) \ar[d] \\
& \hom_X(f_*(G \otimes f^*Q), L \otimes Q) \ar[r, "\simeq"] & \hom_X(f_*G \otimes Q, L \otimes Q)\,.
\end{tikzcd}
$$

Here, the top commuting hexagon is induced on mapping spectra by the natural transformation

$$f_*(-) \otimes Q \Rightarrow f_*(- \otimes f^*Q)$$

underlying the projection formula, the three vertical maps in the middle row are induced by post-composition with $f_*(f^!L \otimes f^*Q) \xrightarrow{\simeq} f_*f^!(L \otimes Q)$, the two vertical maps in the bottom row are induced by post-composition with $f_*f^!(L \otimes Q) \to L \otimes Q$, and the top right vertical map is induced by post-composition with $f_*(f^!L) \otimes Q \to f_*(f^!L \otimes f^*Q)$. In addition, the three horizontal maps on the right side which are marked as equivalences are induced by pre-composition with $f_*G \otimes Q \xrightarrow{\simeq} f_*(G \otimes f^*Q)$, and all the other maps not marked as equivalences are induced on mapping spaces either by f_*, by $(-) \otimes Q$, or by $(-) \otimes f^!Q$. In particular, the right vertical total composite is induced by post-composition with the total composite

$$f_*(f^!L) \otimes Q \to f_*(f^!L \otimes f^*Q) \to f_*f^!(L \otimes Q) \to L \otimes Q.$$

We claim that this composite is homotopic to the map induced by the counit $f_*f^!L \to L$ after tensoring with Q. Indeed, the functor f_* is $\mathcal{D}^{\mathrm{p}}(Y)$-linear by the projection formula, and this structure induces a lax $\mathcal{D}^{\mathrm{p}}(Y)$-linear structure on the right adjoint $f^!$, which is an strong $\mathcal{D}^{\mathrm{p}}(Y)$-structure in the case at hand by Lemma A.6.2. In this situation the counit of $f_* \dashv f^!$ is automatically a $\mathcal{D}^{\mathrm{p}}(Y)$-linear transformation, and so the claim follows. We hence obtain a commutative diagram of the following form, natural in G, Q and L:

$$
\begin{tikzcd}
& \hom_X(G \otimes f^*Q, f^!L \otimes f^*Q) \ar[dr] & \\
\hom_X(G, f^!L) \ar[ur] \ar[d] & & \hom_X(G \otimes f^*Q, f^!(L \otimes Q)) \ar[d] \\
\hom_Y(f_*G, L) \ar[dr] & & \hom_X(f_*(G \otimes f^*Q), L \otimes Q) \ar[dl] \\
& \hom_Y(f_*G \otimes Q, L \otimes Q) &
\end{tikzcd}
$$

Substituting $G = F \otimes F$ and $Q = M' \otimes M'$ and reflecting we hence obtain the top hexagon in (23), as desired.

□

Lemma 5.1.5. *Let $f : X \to Y$ be a qGqp map (see Definition 5.1.1). Then the hermitian functor of Construction 5.1.3 is Poincaré for any choice of M, N and L.*

Proof. By Lemma 5.1.4 it will suffice to prove separately the case where $M = \mathcal{O}_X$ and the case where $f = \mathrm{id}$. Now when $f = \mathrm{id}$, or, more generally, when f is an equivalence, we have that the underlying exact functor $f_*(- \otimes M)$ is an equivalence and also that the only possibly non-invertible natural transformation (22) in the definition of $\eta_{f,\tau}$ is an equivalence, so that $\eta_{f,\tau}$ is a natural equivalence. The hermitian functor $(f_*(- \otimes M), \eta_{f,\tau})$ is then an equivalence, and in particular a Poincaré functor.

We now treat the case where $M = \mathrm{id}$. Here, we may as well suppose that $N = f^!L$ and τ is the identity. We need to show that the induced map $f_*\mathrm{D}_{f^!L}(M) \to \mathrm{D}_L(f_*M)$ is an equivalence for every $M \in \mathcal{D}^{\mathrm{p}}(X)$. Mapping a test object $N \in \mathcal{D}^{\mathrm{p}}(Y)$ into this map and using the adjunction $f^* \dashv f_*$, it will suffice to verify that the induced map

$$\mathrm{B}_{f^!L}(f^*N, M) = \hom(N, f_*\mathrm{D}_{f^!L}(M)) \to \hom(N, \mathrm{D}_L(f_*M)) = \mathrm{B}_L(N, f_*M)$$

is an equivalence. Unwinding the definitions (see [CDH+I, Proof of Lemma 1.2.4]), this last map is is simply the composite

$$\mathrm{B}_{f^!L}(f^*N, M) \xrightarrow{(\eta_{f,\mathrm{id}})_*} \mathrm{B}_L(f_*f^*N, f_*M) \to \mathrm{B}_L(N, f_*M).$$

Finally, by the very definition of $\eta_{f,\mathrm{id}}$, this composite is given by the composite

$$\hom(f^*N \otimes M, f^!L) \simeq \hom(f_*(f^*N \otimes M), L) \to \hom(f_*f^*N \otimes f_*M, L) \to \hom(N \otimes f_*M, L).$$

It will hence suffice to show that for every $M \in \mathcal{D}^{\mathrm{p}}(X)$ and $N \in \mathcal{D}^{\mathrm{p}}(Y)$, the composite

$$N \otimes f_*M \to f_*f^*N \otimes f_*M \to f_*(f^*N \otimes M)$$

is an equivalence. Indeed, this is exactly the projection formula says. ☐

Proposition 5.1.6. *Any proper local complete intersection map $f : X \to Y$ is qGqp.*

Corollary 5.1.7. *For any proper local complete intersection map $f : X \to Y$ and any line bundle $L \in Y$ the hermitian functor*

$$(f_*, \eta) : (\mathcal{D}^{\mathrm{p}}(X), \mathfrak{P}^{\mathrm{s}}_{f^!L}) \to (\mathcal{D}^{\mathrm{p}}(Y), \mathfrak{P}^{\mathrm{s}}_L)$$

is Poincaré.

For the proof of Proposition 5.1.6 we first verify that being a qGqp map is a local property on the codomain:

Lemma 5.1.8. *Let $f : X \to Y$ be a map of qcqs schemes, $Y = \cup_i V_i$ an open covering of Y. Then f is qGqp if and only if its base change $f_i : U_i = X \times_Y V_i \to V_i$ to V_i is qGqp for every i. In addition, when these equivalent conditions hold we have that $(f^!\mathcal{O}_Y|)_{U_i} \simeq f_i^!(\mathcal{O}_{V_i})$ for every i.*

Proof. By Corollary A.6.5 we have that f is quasi-perfect if and only if each f_i is quasi-perfect, and that when these equivalent conditions hold we have that $(f^!\mathcal{O}_Y|)_{U_i} \simeq f_i^!(\mathcal{O}_{V_i})$. It will hence suffice to show that $M \in \mathcal{D}^{\mathrm{qc}}(X)$ is invertible it and only if $M|_{U_i}$ is invertible in $\mathcal{D}^{\mathrm{qc}}(U_i)$ for every i. The only if direction is clear since the pullback functor $\mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(U_i)$ is monoidal. On the other hand, if each $M|_{U_i}$ is invertible then each $M|_{U_i}$ is in particular dualisable, hence compact, hence perfect. By Remark A.5.5 we have that M is perfect, hence dualisable with some dual $\mathrm{D}M$. Since each $M|_{U_i}$ is invertible the coevaluation map $M \otimes \mathrm{D}M \to \mathcal{O}_X$ maps to an equivalence in $\mathcal{D}^{\mathrm{qc}}(U_i)$ for each i and is hence an equivalence by Zariski descent (Proposition A.2.3). We conclude that M is invertible. ☐

Proof of Proposition 5.1.6. By Lemma 5.1.8, it suffices to prove the claim when f is a (proper) global complete intersection, that is, when f is the composite of a regular closed embedding and a proper smooth map. By Remark 5.1.2, it suffices to show this when f is either a regular closed embedding or a proper smooth map.

If f is a regular embedding, then in particular it is affine, and by replacing Y by sufficiently small affine open subschemes, we may assume that f is of the form $\mathrm{spec}(A/I) \to \mathrm{spec}(A)$, where $I \subseteq A$ is an ideal generated by a regular sequence $a_1, \dots, a_r \in A$. The functor $f_* : \mathcal{D}(A/I) \to \mathcal{D}(A)$ is then the forgetful (or inflation) functor obtained by precomposing an A/I-action with $A \to A/I$, and its right adjoint $f^!: \mathcal{D}(A) \to \mathcal{D}(A/I)$ sends an A-module M to the A/I-module $\mathrm{RHom}_A(A/I, M)$. The regularity assumption then implies that the Koszul complex

$$K_\bullet = \bigotimes_{i=1}^{r} [A \xrightarrow{a_i} A]$$

is a finite free resolution of the A-module A/I, where $A \xrightarrow{a_i} A$ is considered as a complex sitting in degrees $1, 0$. In particular, A/I is a perfect A-complex, and so f is quasi-perfect. In addition, for every A/I-module M we have that $f^! M = \mathrm{RHom}_A(A/I, M)$ can be modelled by the complex $\bigotimes_{i=1}^{r}[M \xrightarrow{a_i} M]$, where $[M \xrightarrow{a_i} M]$ is considered as a complex sitting in degree $0, -1$. Taking $M = A$ we obtain that $f^! A \simeq A/I[-r]$ is invertible.

We now consider the case where $f: X \to Y$ is smooth and proper. In particular, since f is proper it is by definition separated, and so the diagonal map

$$\Delta: X \to X \times_Y X$$

is a closed embedding. In addition, since f is smooth, we have by [SP, Tag 067U] that the closed embedding Δ is a regular embedding. By the first part of the theorem, we deduce that Δ is a qGqp morphism, so that $\Delta^!(\mathcal{O}_{X \times_Y X})$ is an invertible perfect complex on X. Now, consider the diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{\Delta} & & & \\ & & X \times_Y X & \xrightarrow{p_2} & X \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{f} & Y \end{array}$$

Since f is smooth, it is in particular flat. Combining Lemma A.6.4 and Lemma A.6.2, we then have that for $M \in \mathcal{D}^{\mathrm{qc}}(Y)$ we have

$$\begin{aligned} f^* M &= \Delta^! p_1^! f^* M \\ &= \Delta^! p_2^* f^! M \\ &= \Delta^!(\mathcal{O}_{X \times_Y X}) \otimes \Delta^* p_2^* f^! M \\ &= \Delta^!(\mathcal{O}_{X \times_Y X}) \otimes f^! M \end{aligned}$$

and so the functors $f^!$ and f^* differ by an invertible factor $\Delta^!(\mathcal{O}_{X \times_Y X})$. We conclude that $f^!$ preserves colimits and hence that f is quasi-perfect. In addition, for $M = \mathcal{O}_X$ we get that $\mathcal{O}_X = f^*\mathcal{O}_Y = \Delta^!(\mathcal{O}_{X \times_Y X}) \otimes f^!\mathcal{O}_Y$ and so $f^!\mathcal{O}_Y$ is an invertible object. We conclude that f is invertible, as desired. □

Remark 5.1.9. In the proof of Proposition 5.1.6, we show in particular that if $f: X \to Y$ is either a closed regular embedding or a proper smooth map, then f is qGqp. Elaborating slightly on the arguments presented above, one can show more precisely that if X is a closed regular embedding with rank r normal bundle $\mathcal{N}$, then $f^!\mathcal{O}_Y \simeq \det \mathcal{N}[-r]$ (see also [SP, Tag 0BR0]), and if f is a proper smooth map with rank r relative cotangent bundle $\Omega_{X/Y}$, then $f^!\mathcal{O}_Y \simeq (\Delta^!\mathcal{O}_{X \times_Y X})^{-1} \simeq \det \Omega_{X/Y}[r]$, where $\Delta: X \to X \times_Y X$ is the diagonal map.

5.2. **Global dévissage.** Let us now consider a closed embedding $i: Z \to X$ of finite dimensional regular Noetherian schemes with open complement $j: U \hookrightarrow X$. In particular, i is automatically a regular embedding and U is regular Noetherian. Given a line bundle with C_2-action L on X we have by Proposition 5.1.6 an associated Poincaré functor

$$(i_*, \eta): \left(\mathcal{D}^{\mathrm{p}}(Z), Ϙ^{\mathrm{s}}_{f^!L}\right) \to \left(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L}\right).$$

The functor $i_*: \mathcal{D}^{\mathrm{p}}(Z) \to \mathcal{D}^{\mathrm{p}}(X)$ takes values in the duality invariant full subcategory $\mathcal{D}^{\mathrm{p}}_Z(X) \subseteq \mathcal{D}^{\mathrm{p}}(X)$ spanned by the perfect complexes supported on Z. We consequently obtain that the Poincaré functor (i_*, η)

above induces a Poincaré functor

$$\left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\mathrm{s}}_{i^!L}\right) \to \left(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\mathrm{s}}_{L}|_Z\right),$$

where we have denoted by $\Omega^{\mathrm{s}}_{L}|_Z$ the restriction of Ω^{s}_{L} to $\mathcal{D}^{\mathrm{p}}_{Z}(X)$.

Theorem 5.2.1 (Dévissage)**.** *Let $i\colon Z \hookrightarrow X$ be a closed embedding of finite dimensional regular Noetherian schemes (automatically a regular embedding, see* [SP, Tag 0E9J]*). Let L be a line bundle with C_2-action on X and $n \in \mathbb{Z}$ an integer. Then the induced maps*

$$\mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\mathrm{s}}_{i^!L[n]}\right) \to \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\mathrm{s}}_{L[n]}|_Z\right) \quad \text{and} \quad \mathrm{L}\left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\mathrm{s}}_{i^!L[n]}\right) \to \mathrm{L}\left(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\mathrm{s}}_{L[n]}|_Z\right)$$

are equivalences.

Recall that on any regular Noetherian scheme the collection of perfect complexes coincides with the coherent ones, and so the stable subcategory of perfect complexes inherits from its embedding in quasi-coherent complexes a t-structure whose heart is the abelian category of coherent sheaves. In addition, any perfect complexes has only finitely many non-trivial homotopy sheaves, so that this t-structure is bounded. Now since $j\colon U \to X$ is flat the functor $j^*\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$ is t-exact, and hence the bounded t-structure of $\mathcal{D}^{\mathrm{p}}(X)$ restricts to a bounded t-structure on $\mathcal{D}^{\mathrm{p}}_{Z}(X)$, with $\mathcal{D}^{\mathrm{p}}_{Z}(X)^{\heartsuit} \subseteq \mathrm{Coh}(X)$ the full subcategory spanned by the coherent sheaves supported on Z. Since closed embeddings are in particular affine, the functor $i_*\colon \mathcal{D}^{\mathrm{p}}(Z) \to \mathcal{D}^{\mathrm{p}}_{Z}(X)$ is t-exact. We consequently obtain that for every $m \in \{-\infty\} \cup \mathbb{Z}$ the Poincaré functor (i_*, η) above induces a Poincaré functor

$$(i_*, \eta^{\geq m})\colon \left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\geq m}_{i^!L}\right) \to \left(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\geq m}_{L}|_Z\right).$$

Theorem 5.2.2 (Genuine dévissage)**.** *Let X be a regular Noetherian scheme of finite Krull dimension d, and $i\colon Z \hookrightarrow X$ a closed embedding with Z regular. Fix a line bundle with C_2-action $L \in \mathcal{D}^{\mathrm{p}}(X)$ and an $m \in \{-\infty, \infty\} \cup \mathbb{Z}$. Then the map*

$$\mathrm{L}_n\left(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\geq m}_{i^!L}\right) \to \mathrm{L}_n\left(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\geq m}_{L}|_Z\right)$$

is an isomorphism for and $n \geq 2m - 1 + d$ and injective for $n = 2m - 2 + d$.

Remark 5.2.3. In the situation of Theorem 5.2.1, one may also consider the corresponding map

$$\mathrm{K}\left(\mathcal{D}^{\mathrm{p}}(Z)\right) \to \mathrm{K}\left(\mathcal{D}^{\mathrm{p}}_{Z}(X)\right)$$

on the level of K-theory. Applying the theorem of the heart for K-theory (see [Bar15]) to the t-structures described above this map can be identified with the map

$$\mathrm{K}(\mathrm{Coh}(Z)) \to \mathrm{K}(\mathrm{Coh}_Z(X)),$$

which is an equivalence by Quillen's dévissage theorem for abelian categories, see [Qui06, Theorem 4].

Theorem 5.2.2 implies the L-theory part of Theorem 5.2.1 by taking $m = -\infty$ (where we note that shifting the line bundle translates to suspending L-theory by bordism invariance). At the same time, by Remark 5.2.3 and the fundamental fibre sequence (see [CDH⁺II, Corollary 4.4.14]) we have that the L-theory equivalence and GW-theory equivalence in Theorem 5.2.1 imply each other. We hence conclude that Theorem 5.2.2 implies Theorem 5.2.1. The following two subsections are dedicated to the proof of Theorem 5.2.2.

Remark 5.2.4. In the situation of Theorem 5.2.1, the natural transformations $\mathrm{GW} \Rightarrow \mathbb{GW}$ and $\mathrm{L} \Rightarrow \mathbb{L}$ are equivalences when evaluated on both $(\mathcal{D}^{\mathrm{p}}(Z), \Omega^{\mathrm{s}}_{i^!L})$ and $(\mathcal{D}^{\mathrm{p}}_{Z}(X), \Omega^{\mathrm{s}}_{L}|_Z)$. To see this, note that using the exact squares (3) (Karoubi cofinality), it suffices to show that the natural transformation $\mathrm{K} \Rightarrow \mathbb{K}$ is an equivalence when evaluated on both $\mathcal{D}^{\mathrm{p}}(Z)$ and $\mathcal{D}^{\mathrm{p}}_{Z}(X)$. This, in turn, follows from the fact that both these (idempotent complete) stable ∞-categories carry bounded t-structures with Noetherian hearts, and such stable ∞-categories have vanishing negative K-groups, see [AGH19, Theorem 1.2]. We conclude that the dévissage equivalence expressed in Theorem 5.2.1 equally holds if one replaces all appearances of GW by $\mathbb{GW}$ and all appearances of L by $\mathbb{L}$.

5.3. **Local dévissage.** Our goal in this section is to prove Theorems 5.2.2 in the case where $X = \mathrm{spec}(R)$ for R a regular Noetherian local ring with maximal ideal $\mathfrak{m}$ residue field $k := R/\mathfrak{m}$, and $Z = \mathrm{spec}(k)$ is the associated unique closed point. We denote by d the Krull dimension of R.

Recall that a finitely generated R-module M is said to have *finite length* if there is a finite filtration $0 = M_0 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is annihilated by $\mathfrak{m}$. A perfect R-complex M is then supported on $\mathrm{spec}(k)$ if and only if its homologies have finite length. In this case, we will also say that the R-complex M itself has finite length. We then let $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R) \subseteq \mathcal{D}^{\mathrm{p}}(R)$ denote the full stable subcategory spanned by the perfect R-complexes of finite length. As discussed in §5.2, the t-structure on $\mathcal{D}^{\mathrm{p}}(R)$ then restricts to a t-structure on $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)$ with heart $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)^{\heartsuit} = \mathrm{Coh}_{\mathfrak{m}}(R) \subseteq \mathrm{Coh}(R)$ the abelian category of finite length R-modules.

Let us fix an invertible perfect R-complex L equipped with an R-linear involution. Let Ω^{s}_L be the associated symmetric Poincaré structure and write $\Omega^{\mathrm{s}}_L|_{\mathfrak{m}} = \Omega^{\mathrm{s}}_R|_{\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)}$ for its restriction to $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)$. As in §5.2 we then consider the associated Poincaré functor

$$(i_*, \eta)\colon (\mathcal{D}^{\mathrm{p}}(k), \Omega^{\mathrm{s}}_{i^!L}) \to (\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\mathrm{s}}_L|_{\mathfrak{m}}). \tag{24}$$

Remark 5.3.1. Since R is local the invertible perfect complex L is equivalent to a shift $R[r]$ for some r. This isomorphism is however not unique, and it will be convenient to avoid choosing a particular one. In a similar manner, since R is regular of Krull dimension d we have that $i^!L$ is equivalent to $(L \otimes_R k)[-d] \simeq k[r-d]$ (see Lemma 5.3.5 below), though not canonically.

Theorem 5.3.2 (Dévissage for local rings)**.** *Let r be the unique integer for which $\pi_r L \neq 0$. Then for $-\infty \leq m < \infty$ the map*

$$\mathrm{L}_n(\mathcal{D}^{\mathrm{p}}(k), \Omega^{\geq m}_{i^!L}) \xrightarrow{\simeq} \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\geq m}_L|_{\mathfrak{m}})$$

induced by (24) is an isomorphism for $n \geq 2m - 1 + d - r$ and injective for $n = 2m - 2 + d - r$.

Remark 5.3.3. In the situation of Theorem 5.3.2 we have equivalences

$$\Omega^{\geq m-1}_{i^!L[-1]} \simeq (\Omega^{\geq m}_{i^!L})^{[-1]} \quad \text{and} \quad \Omega^{\geq m-1}_{L[-1]} \simeq (\Omega^{\geq m}_L)^{[-1]}.$$

Replacing L with $L[d-r]$, m with $m + d - r$ and n with $n + d - r$ we see that to prove Theorem 5.3.2 it will suffice to prove the case $r = d$.

The remainder of this subsection is devoted to the proof of Theorem 5.3.2.

Lemma 5.3.4. *The $(d-r)$-shifted counit map $i_*i^!L[d-r] \to L[d-r]$ factors as*

$$i_*i^!L[d-r] \to L_{\mathfrak{m}} \to L[d-r]$$

where $L_{\mathfrak{m}} \in \mathrm{Mod}(R) = \mathcal{D}(R)^{\heartsuit}$ is an injective R-module which satisfies the following property: for every $M \in \mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)$ the map

$$\hom_R(M, L_{\mathfrak{m}}) \to \hom_R(M, L[d-r]) = \Sigma^d \mathrm{D}_L(M)$$

is an equivalence in $\mathcal{D}(R)$.

Proof. Let $k \subseteq I_k \in \mathrm{Mod}(R)$ be an injective envelop of the R-module k. By [Gil02, Lemma 3.3 (4)] there exists a map of R-module $\phi\colon I_k[-d] \to R$ such that for every $M \in \mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)$ the induced map

$$\hom(M, I_k[-d]) \to \hom(M, R)$$

is an equivalence. Since the underlying R-complex of L is equivalent to $R[r]$ the same holds for the map $L_{\mathfrak{m}} \to L[d-r]$, where $L_{\mathfrak{m}} = L[-r] \otimes_R I_k$. Since $i_*i^!L[d-r]$ belongs to $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)$ we can factor in particular the counit map via $L_{\mathfrak{m}}$. We now finish by noting that Hom into the injective R-module is t-exact. □

Lemma 5.3.5. *The underlying k-complex of i^*L is concentrated in degree $r - d$ and $\pi_{r-d} i^*L$ is a 1-dimensional k-vector space. In particular, there exists a (non-canonical) equivalence $i^*L \simeq k[r-d]$.*

Proof. Since $i\colon \mathrm{spec}(k) \to \mathrm{spec}(R)$ is a closed embedding and R is regular we have that $i^!L$ is a dualising complex on $\mathrm{spec}(k)$, and hence an invertible perfect complex. In particular, $i^!L$ can have at most one non-trivial homotopy group, and the non-trivial homotopy group must be an 1-dimensional k-vector space. It will hence suffice to show that

$$\pi_n i^!L = \pi_n \hom_R(k, L) = 0$$

for $n \neq r-d$. Indeed, by Lemma 5.3.4 we have

$$\pi_n \mathrm{hom}_R(k, L) = \pi_n \mathrm{hom}_R(k, L_{\mathfrak{m}}[r-d]) = \mathrm{Ext}_R^{-n+r-d}(k, L_{\mathfrak{m}})$$

where $L_{\mathfrak{m}} \in \mathcal{D}(R)^{\heartsuit} = \mathrm{Mod}(R)$ is an injective R-module, so that the last Ext-group vanishes whenever $n \neq r-d$. □

We summarize a key conclusion of the previous two lemmas:

Corollary 5.3.6. *Suppose that $r = d$. Then, with respect to the standard t-structures, we have that*

(1) The duality of $(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), Ϙ^{\geq m}_{L}|_{\mathfrak{m}})$ sends connective objects to coconnective objects and vice-versa. The induced duality on $\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R)^{\heartsuit} = \mathrm{Coh}_{\mathfrak{m}}(R)$ is given by $M \mapsto \mathrm{Hom}_R(M, L_{\mathfrak{m}})$.
(2) The duality of $(\mathcal{D}^{\mathrm{p}}(k), Ϙ^{\geq m}_{i^!L})$ sends connective objects to coconnective objects and vice-versa. The induced duality on $\mathcal{D}^{\mathrm{p}}(k)^{\heartsuit} = \mathrm{Coh}(k)$ is given by $M \mapsto \mathrm{Hom}_k(M, \pi_{-d} i^! L)$ (in particular, $\pi_{-d} i^! L$ is a k-vector space of dimension 1).

Proof. The first statement follows from Lemma 5.3.4 and the fact that $L_{\mathfrak{m}}$ is injective, and the second statement from Lemma 5.3.5. □

Proposition 5.3.7. *Let $\mathcal{C}$ be a stable ∞-category with equipped with a duality functor $\mathrm{D}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and $Ϙ^{\mathrm{s}}_{\mathrm{D}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ the associated symmetric Poincaré structure on $\mathcal{C}$. Suppose that $\mathcal{C}$ admits a t-structure such that* D *sends connective objects to coconnective objects and vice-versa and let* $\mathrm{D}^{\heartsuit} \colon \mathcal{C}^{\heartsuit} \to \mathcal{C}^{\heartsuit}$ *be the induced duality on $\mathcal{C}^{\heartsuit}$. We then write $-\mathrm{D}^{\heartsuit}$ for the sign twist of* D*. For $-\infty \leq m < \infty$ we may consider the truncated Poincaré structure $Ϙ^{\geq m}_{\mathrm{D}}$ on $\mathcal{C}$, defined using the induced t-structure on* $\mathrm{Ind}(\mathcal{C})$*. Then the following holds:*

(1) The groups $\mathrm{L}_n(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}})$ vanish for odd n, and for $n = 2k$ the duality preserving map

$$(\mathcal{C}^{\heartsuit}, (-1)^k \mathrm{D}^{\heartsuit}) \to (\mathcal{C}, \Omega^{2k}\mathrm{D}) \qquad x \mapsto x[-k]$$

induces an isomorphism

$$\mathrm{W}(\mathcal{C}^{\heartsuit}, \mathrm{D}^{\heartsuit}) \xrightarrow{\cong} \mathrm{L}_{2k}(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}}).$$

In addition, every stably metabolic Poincaré object in $(\mathcal{C}^{\heartsuit}, \mathrm{D}^{\heartsuit})$ is metabolic.
(2) For $n \geq 2m-3$ odd the group $\mathrm{L}_n(\mathcal{C}, Ϙ^{\geq m}_{\mathrm{D}})$ vanishes.
(3) For n even the map

$$\mathrm{L}_n(\mathcal{C}, Ϙ^{\geq m}_{\mathrm{D}}) \to \mathrm{L}_n(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}})$$

is an isomorphism for $n \geq 2m$ and injective for $n = 2m-2$.

Proof. For (1) apply [CDH⁺III, Proposition 1.3.1], with $d = 0, r = \infty, b = 0$ and $a = -1$ to deduce that $\mathrm{L}_n(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}})$ vanishes for odd n and with $d = 0, r = 2-m, b = 0$ and $a = -1$ to deduce that $\mathrm{L}_n(\mathcal{C}, Ϙ^{\geq m}_{\mathrm{D}})$ vanishes for odd $n \geq 2r+1 = 2m-3$. This covers all statement involving odd symmetric or genuine L-groups. For even n-groups, we first reduce for simplicity to the case $n = 0$ as follows. For a given $k \in \mathbb{Z}$, if D interchanges $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ then $\Omega^{2k}\mathrm{D}$ interchanges $\mathcal{C}_{\geq -k}$ and $\mathcal{C}_{\leq -k}$, and so the assumptions of the theorem also hold when D is replaced with the shifted duality $\Omega^{2k}\mathrm{D}$ and $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ with the shifted t-structure $(\mathcal{C}_{\geq -k}, \mathcal{C}_{\leq -k})$. We that the t-structure on $\mathrm{Ind}(\mathcal{C})$ induced by the shifted t-structure on $\mathcal{C}$ is just the one obtained by shifting the t-structure on $\mathrm{Ind}(\mathcal{C})$ induced by $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We then have that

$$Ϙ^{\mathrm{s}}_{\Omega^{2k}\mathrm{D}} = \Omega^{2k} Ϙ^{\mathrm{s}}_{\mathrm{D}} \quad \text{and} \quad Ϙ^{\geq m-k}_{\Omega^{2k}\mathrm{D}} = \Omega^{2k} Ϙ^{\geq m}_{\mathrm{D}},$$

where the truncation in $Ϙ^{\geq m-k}_{\Omega^{2k}\mathrm{D}}$ is calculated with respect to the t-structure $(\mathcal{C}_{\geq -k}, \mathcal{C}_{\leq k})$ and the truncation in $\Omega^{2k} Ϙ^{\geq m}_{\mathrm{D}}$ is calculated with respect to the t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Replacing the duality and t-structures by their shifts we may thus reduce to proving the statements involving even L-groups to the case $n = 0$. In particular, to prove (1) it will suffice to show that the map $\mathrm{W}(\mathcal{C}, \mathrm{D}) \to \mathrm{L}_0(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}})$ is an isomorphism and that every stably metabolic object in $\mathrm{W}(\mathcal{C}, \mathrm{D})$ is metabolic, and to prove (3) it will suffice to show that the map

$$\mathrm{L}_n(\mathcal{C}, Ϙ^{\geq m}_{\mathrm{D}}) \to \mathrm{L}_n(\mathcal{C}, Ϙ^{\mathrm{s}}_{\mathrm{D}})$$

is an isomorphism for $m \leq 0$ and injective for $m = 1$.

For (1), we first apply [CDH+III, Proposition 1.3.1], with $d = 0, r = \infty$ and $n = a = b = 0$ to deduce that the map

$$\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}}) \to \mathrm{L}_0(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$$

is an isomorphism. By definition, $\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$ is the quotient of the monoid of Poincaré objects (x, q) such that x is connective modulo the submonoid of those which admits a Lagrangian $w \to x$ such that w and $\mathrm{cof}[w \to x] \simeq \mathrm{D}w$ are connective. Since D maps connective objects to coconnective objects and vice verse we see that such Poincaré objects and Lagrangians are in fact contained in $\mathcal{C}^{\heartsuit}$, and Lagrangian inclusions $w \to x$ are monomorphisms in the abelian category $\mathcal{C}^{\heartsuit}$. We may consequently identify $\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$ with $\mathrm{W}(\mathcal{C}, \mathrm{D})$, establishing the even L-groups part of (1). To obtain final part of (1) note that every stably metabolic object (x, q) represents zero in $\mathrm{W}(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$, and hence $(x[0], q)$ represents zero in $\mathrm{L}_0(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$. It follows that $(x[0], q)$ is metabolic, and so by [CDH+III, Proposition 1.3.1 (ii)], with $d = 0, r = \infty$ and $n = a = b = 0$ it admits a Lagrangian $w \to x$ such that w and $\mathrm{cof}[w \to x] \simeq \mathrm{D}w$ are connective, so that w belongs to $\mathcal{C}^{\heartsuit}$ and $w \to x$ is a monomorphism in $\mathcal{C}^{\heartsuit}$. This exactly means that (x, q) is metabolic in $(\mathcal{C}^{\heartsuit}, \mathrm{D}^{\heartsuit})$.

Finally, let us prove (3) in the case of $n = 0$. For this, we apply [CDH+III, Proposition 1.3.1], with $d = 0, r = 2 - m$ and $n = a = b = 0$ to deduce that the map

$$\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) \to \mathrm{L}_0(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}})$$

is an isomorphism whenever $m \leq 2$. As above, $\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}})$ is the quotient of the monoid of Poincaré objects (x, q) such that x belongs to $\mathcal{C}^{\heartsuit}$ modulo the submonoid of those which admits a Lagrangian $w \to x$ such that $w \in \mathcal{C}^{\heartsuit}$ and $w \to x$ is a monomorphism in $\mathcal{C}^{\heartsuit}$. Now for $x \in \mathcal{C}^{\heartsuit}$ the fibre of

$$\Omega^{\leq m}_{\mathrm{D}}(x) \to \Omega^{\mathrm{s}}_{\mathrm{D}}(x)$$

is $(m-2)$-truncated (essentially by definition). Suppose first that $m \leq 0$, so that this fibre is (-2)-truncated. Then for $x \in \mathcal{C}^{\heartsuit}$ we have that the map

$$\Omega^{\infty} \Omega^{\leq m}(x) \to \Omega^{\infty} \Omega^{\mathrm{s}}_{\mathrm{D}}(X)$$

is an equivalence. Similarly, for a map $w \to x$ in $\mathcal{C}^{\heartsuit}$ we have that the total fibre of the square

$$\begin{array}{ccc} \Omega^{\leq m}(x) & \longrightarrow & \Omega^{\mathrm{s}}(x) \\ \downarrow & & \downarrow \\ \Omega^{\leq m}(w) & \longrightarrow & \Omega^{\mathrm{s}}(w) \end{array}$$

is (-2)-truncated, so that the map

$$\Omega^{\infty}\, \mathrm{fib}[\Omega^{\leq m}(x) \to \Omega^{\leq m}(w)] \to \Omega^{\infty}\, \mathrm{fib}[\Omega^{\mathrm{s}}(x) \to \Omega^{\mathrm{s}}(w)]$$

Combining this with the above we therefore conclude that when $m \leq 0$ the top horizontal and both vertical maps in the square

$$\begin{array}{ccc} \mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) & \longrightarrow & \mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{L}_0(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) & \longrightarrow & \mathrm{L}_0(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}}) \end{array}$$

are isomorphisms, and hence the bottom horizontal map is an isomorphism as well. Finally, suppose that $m = 1$. Then for every $x \in \mathcal{C}^{\heartsuit}$ the map $\Omega^{\geq 0}_{\mathrm{D}}(x) \to \Omega^{\mathrm{s}}_{\mathrm{D}}(x)$ is (-1)-truncated. Now as we saw above the map $\mathrm{L}_0^{0,0}(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) \to \mathrm{L}_0(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}})$ is still an isomorphism for $m = 1$, and so every class in $\mathrm{L}_0(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}})$ can be represented by a Poincaré object (x, q) such that $x \in \mathcal{C}^{\heartsuit}$. For such a Poincaré objects, if its image in $\mathrm{L}_0(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$ is zero then by [CDH+III, Proposition 1.3.1 (ii)], with $d = 0, r = \infty$ and $n = a = b = 0$ the image of (x, q) in $(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$ admits a Lagrangian $w \to x$ with $w \in \mathcal{C}^{\heartsuit}$. Now if we consider the associated commutative rectangle with exact rows

$$\begin{array}{ccccc} \Omega^{\geq m}_{\mathrm{Met}}(w \to x) & \longrightarrow & \Omega^{\leq m}(x) & \longrightarrow & \Omega^{\leq m}(w) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^{\mathrm{s}}_{\mathrm{Met}}(w \to x) & \longrightarrow & \Omega^{\mathrm{s}}(x) & \longrightarrow & \Omega^{\mathrm{s}}(w)\ , \end{array}$$

then the fibre of the right most vertical map is (-1)-truncated and hence the total fibre of the left square is (-2)-truncated. We hence conclude that the Lagrangian $w \to x$ lifts to a Lagrangian with respect to $\Omega^{\geq m}_{\mathrm{D}}$, so that $[x, q] = 0$ in $\mathrm{L}_0(\mathcal{C}, \Omega^{\geq 0}$. We conclude that the map

$$\mathrm{L}_0(\mathcal{C}, \Omega^{\geq m}_{\mathrm{D}}) \to \mathrm{L}_0(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$$

is injective when $m = 1$, as desired. □

Proof of Theorem 5.3.2. By Remark 5.3.3 we may, and will, assume that $r = d$, so that the underlying R-complex of L is equivalent to $R[d]$. By Corollary 5.3.6 we can apply Proposition 5.3.7 to both $(\mathcal{D}^{\mathrm{p}}(k), \Omega^{\geq m}_{i^!L})$ and $(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\geq m}_{L}|_{\mathfrak{m}})$. To finish the proof it will now suffice to show that the map

$$\mathrm{inf}_* :\ \mathrm{W}^{\pm}(\mathrm{Coh}(k)) \to \mathrm{W}^{\pm}(\mathrm{Coh}_{\mathfrak{m}}(R)) \tag{25}$$

of classical (symmetric and anti-symmetric) Witt groups induced by the inflation functor $\mathrm{inf} : \mathrm{Coh}(k) \to \mathrm{Coh}_{\mathfrak{m}}(R)$, is an isomorphism. Here, $\mathrm{Coh}(k)$ is endowed with the duality $V \mapsto \mathrm{Hom}_k(V, \pi_{-d} i^! L)$ and $\mathrm{Coh}_{\mathfrak{m}}(R)$ is endowed with the duality $M \mapsto \mathrm{Hom}_R(M, L_{\mathfrak{m}})$, see Corollary 5.3.6. We note that by Proposition 5.3.7 (1) any (anti-)symmetric Poincaré object in $\mathrm{Coh}(k)$ which maps to 0 in $\mathrm{W}^{\pm}(\mathrm{Coh}(k))$ contains a Lagrangian in $\mathrm{Coh}(k)$, and the same holds for Poincaré (anti-)symmetric objects in $\mathrm{Coh}_{\mathfrak{m}}(R)$. But if V is a finite dimensional k-vector space then $\mathrm{inf} : \mathrm{Coh}(k) \to \mathrm{Coh}_{\mathfrak{m}}(R)$ induces a bijection between k-vector subspaces of V and R-submodules of $\mathrm{inf}(V)$. It then follows that (25) is injective. We also note that a finite length R-module is in the image of inf if and only if it is annihilated by $\mathfrak{m} \subseteq R$. To finish the proof it will hence suffice to show that if M is a finite length R-module equipped with a non-degenerate (anti-)symmetric bilinear form then M contains a sub-Lagrangian $N \subseteq M$ such that $N^{\perp}/N$ is annihilated by $\mathfrak{m}$. Indeed, this is a special case of [QSS79, Theorem 6.10] for $\mathcal{M} = \mathrm{Coh}_{\mathfrak{m}}(R)$ and $\mathcal{M}_0 = \mathrm{Coh}(k)$. □

Corollary 5.3.8. *Let $g :\ \mathrm{spec}(R') \to \mathrm{spec}(R)$ be a closed embedding associated to a surjective ring homomorphism $R \to R'$ between regular Noetherian local rings of Krull dimensions d and d' and with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Let $r \in \mathbb{Z}$ be an integer and L an invertible perfect R-complex with C_2-action whose underlying R-complex is equivalent to an r-shift of a line bundle. Then the map*

$$\mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\geq m}_{L}|_{\mathfrak{m}}) \to \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}'}(R'), \Omega^{\geq m}_{g_!L}|_{\mathfrak{m}})$$

induced by push-forward along g is an isomorphism when $n \geq 2m - 1 + d - r$ and injective for $n = 2m - 2 + d - r$.

Proof. Write $k = R/\mathfrak{m} = R'/\mathfrak{m}'$ for the common residue field of R and R'. Since $\mathrm{spec}(R)$ is regular L is a dualising complex on $\mathrm{spec}(R)$ and since $\mathrm{spec}(R') \to \mathrm{spec}(R)$ is a closed embedding we have that $g^! L$ is a dualising complex on $\mathrm{spec}(R')$, see [SP, Tag 0BZI]. But since $\mathrm{spec}(R')$ is regular $g^! L$ must be an invertible perfect complex, hence of the form $R'[r']$ for some r'. Let $i' :\ \mathrm{spec}(k) \to \mathrm{spec}(R')$ be the inclusion of the unique closed point of $\mathrm{spec}(R')$, so that $i = r \circ i' :\ \mathrm{spec}(k) \to \mathrm{spec}(R)$ is the inclusion of the unique closed point of $\mathrm{spec}(R)$. By Lemma 5.3.5 we then have

$$k[r-d] \simeq i^! L = (i')^! g^! L \simeq (i')^! R'[r'] \simeq k[r' - d']$$

and so $r - d = r' - d'$. Replacing L with $L[d - r]$ and using Remark 5.3.3 we may hence assume without loss of generality that $r = d$ and $r' = d'$.

Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}(k), \Omega^{\geq m}_{i^!L}) & \longrightarrow & \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}'}(R'), \Omega^{\geq m}_{r^!L}|_{\mathfrak{m}'}) & \longrightarrow & \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\geq m}_{L}|_{\mathfrak{m}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}(k), \Omega^{\mathrm{s}}_{i^!L}) & \longrightarrow & \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}'}(R'), \Omega^{\mathrm{s}}_{r^!L}|_{\mathfrak{m}'}) & \longrightarrow & \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}_{\mathfrak{m}}(R), \Omega^{\mathrm{s}}_{L}|_{\mathfrak{m}})\ . \end{array}$$

By Theorem 5.3.2 and the 2-out-of-3 property the top horizontal arrows are isomorphisms for $n \geq 2m-1$ (recall that we have set $r = d$ and $r' = d'$) and the bottom horizontal arrows are isomorphisms for every n. In addition, for $n = 2m-2$ we have by Proposition 5.3.7 that the vertical maps are injective and hence the top horizontal maps are injective as well.

□

5.4. **End of the proof of global dévissage.** We are finally ready to conclude the proof of the dévissage theorem. As explained in §5.2, given dévissage for K-theory (Remark 5.2.3) and the fundamental fibre sequence [CDH⁺II, Corollary 4.4.14], Theorem 5.2.1 is a consequence of Theorem 5.2.2.

Proof of Theorem 5.2.2. We prove by descending induction on c that the map

$$\mathrm{L}(\mathcal{D}^{\mathrm{p}}(Z)^{\geq c}, \Qoppa^{\mathrm{s}}_{i^!L}) \to \mathrm{L}(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \Qoppa^{\mathrm{s}}_{L})$$

has a $(2m-2-d)$-truncated cofibre. For the base of the induction we note that for $c > \dim(Z)$ the domain and codomain of this map are zero. Now suppose that $c \geq 0$ is such that the claim holds for $c+1$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{L}(\mathcal{D}^{\mathrm{p}}(Z)^{\geq c+1}, \Qoppa^{\geq m}_{i^!L}) & \longrightarrow & \mathrm{L}(\mathcal{D}^{\mathrm{p}}(Z)^{\geq c}, \Qoppa^{\geq m}_{i^!L}) & \longrightarrow & \bigoplus_{x\in Z^{(c)}} \mathrm{L}(\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{Z,x}), \Qoppa^{\geq m}_{i^!L}|_x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}(\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c+1}, \Qoppa^{\geq m}_{L}) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}_Z(X)^{\geq c}, \Qoppa^{\geq m}_{L}) & \longrightarrow & \bigoplus_{x\in Z^{(c)}} \mathrm{L}(\mathcal{D}^{\mathrm{p}}_x(\mathcal{O}_{X,i(x)}), \Qoppa^{\geq m}_{L}|_x) \end{array} \tag{26}$$

whose rows are fibre sequences by Lemma 4.5.1. Then the cofibre of the left most vertical map is $(2m-2-d)$-truncated by the induction hypothesis and the cofibre of the right most vertical map is $(2m-2-d)$-truncated by Corollary 5.3.8 (applied with $r = 0$). We hence conclude that the cofibre of the middle vertical map has $(2m-2-d)$-truncated cofibre, as desired.

□

6. The projective bundle formula and $\mathbb{A}^1$-invariance

In this section we prove a projective bundle formula for symmetric Karoubi-Grothendieck-Witt theory (and more generally for invariants of schemes arising from Karoubi-localising invariants of Poincaré ∞-categories as in Notation 3.3.4), generalizing in particular results obtained in [Wal03], [Nen09] and [Roh22], which were proven under the assumption that 2 was invertible in the base scheme. This is achieved in §6.1, see Theorem 6.1.6. In §6.2 we investigate the analogue result for the genuine symmetric Poincaré structure, but only in the case of constant $\mathbb{P}^1$-bundles. Finally, in §6.3 we combine the projective bundle formula with dévissage to deduce $\mathbb{A}^1$-invariance of symmetric GW-theory over regular Noetherian schemes of finite Krull dimension.

6.1. **The projective bundle formula.** Let X be a qcqs scheme, L an invertible perfect complex over X equipped with C_2-action and V a vector bundle over X of rank $r+1$. Write $p\colon \mathbb{P}_X V = \mathrm{Proj}(\mathrm{Sym}^\bullet V) \to X$ for the corresponding projective bundle over X, with tautological line bundle $\mathcal{O}(1)$. For $n \in \mathbb{Z}$ we will write $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ and $p^*L(n) := p^*L \otimes \mathcal{O}(n)$. Our goal is to describe the symmetric GW-spectra $\mathrm{GW}^{\mathrm{s}}(\mathbb{P}_X V, p^*L(n))$ in terms of certain symmetric GW-spectra of X. In fact, we will provide a structural result for the Poincaré ∞-category $(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Qoppa^{\mathrm{s}}_{p^*L(n)})$ that will allow us to give a projective bundle formula for every additive invariant.

Remark 6.1.1. Suppose M is any line bundle on a scheme Y. Then the canonical C_2-action on $M \otimes_Y M$ is trivial. Indeed, since M is a line bundle we have that $M \otimes_Y M$ is a line bundle as well, and in particular belongs to the heart of $\mathcal{D}^{\mathrm{qc}}(Y)$. We may hence view it as an object in the ordinary category $\mathrm{Mod}^{\mathrm{qc}}(Y)$ of quasi-coherent sheaves on Y, so that the triviality of the C_2-action amounts to an equality between the action of the generator and the identity map inside the abelian group $\mathrm{Hom}_Y(M \otimes_Y M, M \otimes_Y M)$. By definition of mappings between two Zariski sheaves this equality can be checked Zariski-locally. We may hence assume that $M = \mathcal{O}_Y$. But then the multiplication map $\mathcal{O}_Y \otimes \mathcal{O}_Y \to \mathcal{O}_Y$ is a C_2-equivariant isomorphism for the trivial action on the target. In particular, this shows that

$$(-) \otimes \mathcal{O}(t)\colon\ (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Qoppa^{s}_{p^*L(n)}) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Qoppa^{s}_{p^*L(n-2t)})$$

is an equivalence of Poincaré ∞-categories.

By [Kha20, Theorem 3.3](i) we have that the functor

$$p^*(-) \otimes \mathcal{O}(i) : \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$$

is fully faithful for every i. Let us write $\mathcal{A}(i)$ for its image. Then by [Kha20, Theorem 3.3 (ii)]the full subcategories $\mathcal{A}(i+r), \mathcal{A}(i+r-1), \dots, \mathcal{A}(i)$ form a semi-orthogonal decomposition of $\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$. In particular, if $i, j \in \mathbb{Z}$ are such that $-r \le j - i \le -1$ then $\hom_{\mathbb{P}_X V}(M, N) = 0$ for every $M \in \mathcal{A}(i)$ and $N \in \mathcal{A}(j)$.

Lemma 6.1.2. *For every $i, n \in \mathbb{Z}$ we have*

$$\mathrm{D}_{p^*L(n)}(\mathcal{A}(i)) = \mathcal{A}(n-i)$$

as full subcategories of $\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$.

Proof. Since $\mathrm{D}^2_{p^*L(n)} \simeq \mathrm{id}$ it suffices to prove one inclusion, and in doing so we may restrict attention to the generators of $\mathcal{A}(i)$. For $P \in \mathcal{D}^{\mathrm{p}}(X)$ we then have

$$\begin{aligned} \mathrm{D}_{p^*L(n)}(p^*(P) \otimes \mathcal{O}(i)) &= \mathrm{D}_{\mathbb{P}_X V}(p^*(P) \otimes \mathcal{O}(i)) \otimes p^*L(n) \\ &= \mathrm{D}_{\mathbb{P}_X V}(p^*(P)) \otimes \mathrm{D}_{\mathbb{P}_X V}(\mathcal{O}(i)) \otimes p^*L(n) \\ &= p^*\mathrm{D}_X(P) \otimes \mathcal{O}(-i) \otimes p^*L(n) \in \mathcal{A}(n-i) \end{aligned}$$

where we have used that $\mathrm{D}_{\mathbb{P}_X V}$ is a symmetric monoidal duality (it coincides with the canonical duality of $\mathbb{P}_X V$ as a rigid symmetric monoidal ∞-category, see §3.1), and that p^* is duality preserving for D_X and $\mathrm{D}_{\mathbb{P}_X V}$. □

Write $r' = r$ if $n - r$ is even and $r' = r - 1$ if $n - r$ is odd. In particular $n - r'$ is always even. Let $\mathcal{C}$ be the stable subcategory of $\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ generated by

$$\mathcal{A}(\frac{n+r'}{2}) \cup \mathcal{A}(\frac{n+r'}{2} - 1) \cup \dots \cup \mathcal{A}(\frac{n-r'}{2}) .$$

Then $\mathcal{C}$ is stable under the duality, hence a Poincaré subcategory. Moreover, if $n - r$ is even then $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ by [Kha20, Theorem 3.3 (ii)]. When $n - r$ is odd $\mathcal{C} \ne \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ but one can nonetheless describe explicitly its Poincaré-Verdier quotient. For this, let $\mathcal{V} = p^*V$ be the pullback of V to $\mathbb{P}_X V$. By the construction of $\mathbb{P}_X V$ as $\mathrm{Proj}(\mathrm{Sym}^\bullet V)$ the vector bundle $\mathcal{V}$ comes with a tautological map $\mathcal{V} \to \mathcal{O}(1)$. The $\mathcal{O}(-1)$-twisted map $\mathcal{V}(-1) \to \mathcal{O}_{\mathbb{P}_X V}$ then fits in an exact complex

$$0 \to \Lambda^{r+1}\mathcal{V}(-r-1) \to \cdots \to \Lambda^1\mathcal{V}(-1) \to \mathcal{O}_{\mathbb{P}_X V} \to 0 .$$

of vector bundles on $\mathbb{P}_X V$. In particular, $\det \mathcal{V}(-r-1)[r] = \Lambda^{r+1}\mathcal{V}(-r-1)[r]$ is quasi-isomorphic to the complex

$$\Lambda^r\mathcal{V}(-r) \to \cdots \to \Lambda^1\mathcal{V}(-1) \to \mathcal{O}_{\mathbb{P}_X V} .$$

concentrated in degrees $[0, r]$. The inclusion of the bottom term gives a map

$$\mathcal{O}_{\mathbb{P}_X V} \to \det \mathcal{V}(-r-1)[r]. \tag{27}$$

Since $p_*\Lambda^i\mathcal{V}(-i) = 0$ for $i = 1, \dots, r$ this map induces an equivalence

$$p_*\mathcal{O}_{\mathbb{P}_X V} \xrightarrow{\simeq} p_* \det \mathcal{V}(-r-1)[r] = p_*\mathcal{O}(-r-1) \otimes \det V[r].$$

Now recall that the unit map $\mathcal{O}_X \to p_*p^*\mathcal{O}_X = p_*\mathcal{O}_{\mathbb{P}_X V}$ is an equivalence by [Kha20, Theorem 3.3 (i)]. Combining this with the above equivalence we obtain an induced equivalence

$$\tau : p_*\mathcal{O}(-r-1) \xrightarrow{\simeq} \det V^\vee[-r]. \tag{28}$$

Lemma 6.1.3. *Suppose $n - r$ is odd. Then there is a split Poincaré-Verdier sequence*

$$\big(\mathcal{C}, \Omega^{\mathrm{s}}_{p^*L(n)}|_{\mathcal{C}}\big) \to \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big) \xrightarrow{p_*(-\otimes\mathcal{O}(-\frac{n+r+1}{2}))} \big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes\det V^\vee[-r]}\big)$$

Remark 6.1.4. When $n = -r - 1$, the Poincaré-Verdier projection of Lemma 6.1.3 coincides with the Poincaré functor

$$(p_*, \eta)\colon \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^! L}\big) \to \big(\mathcal{D}^{\mathrm{p}}(X), L\big)$$

of Lemma 5.1.5 under the identification

$$p^! L = p^* L \otimes p^! \mathcal{O}_X = p^* L \otimes \det \Omega_{\mathbb{P}_X V/X} = p^* L(-r-1),$$

see Lemma A.6.2 and Remark 5.1.9.

Proof of Lemma 6.1.3. Since $(\mathcal{A}(\frac{n+r+1}{2}), \mathcal{C})$ is a semi-orthogonal decomposition we have a left split Verdier sequence

$$\mathcal{C} \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V) \to \mathcal{A}\Big(\frac{n+r+1}{2}\Big)\,.$$

At the same time, the functor $p_*(-\otimes\mathcal{O}(-\frac{n+r+1}{2}))$ is trivial on $\mathcal{C}$ and restricts to an equivalence from $\mathcal{A}(\frac{n+r+1}{2})$ to $\mathcal{D}^{\mathrm{p}}(X)$, and so determines an equivalence between the above sequence and the sequence

$$\mathcal{C} \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V) \xrightarrow{p_*(-\otimes\mathcal{O}(-\frac{n+r+1}{2}))} \mathcal{D}^{\mathrm{p}}(X)\,.$$

and so the latter is again a left-split Verdier sequence. As the first arrow is a Poincaré-Verdier inclusion by definition, we can upgrade it to a left-split Poincaré-Verdier sequence, of the form

$$(\mathcal{C}, \Omega^{\mathrm{s}}_{p^*L(n)}|_{\mathcal{C}}) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}) \xrightarrow{p_*(-\otimes\mathcal{O}(-\frac{n+r+1}{2}))} (\mathcal{D}^{\mathrm{p}}(X), \Omega)$$

which is then automatically split, see [CDH⁺II, Observation 1.2.1]. It remains to compute the Poincaré structure Ω. We know by [CDH⁺II, Proposition 1.2.2 (i)] that it is given by pre-composing $\Omega^{\mathrm{s}}_{p^*L(n)}$ with the left adjoint $p^*(-) \otimes \mathcal{O}(\frac{n+r+1}{2})$ of $p_*(- \otimes \mathcal{O}(-\frac{n+r+1}{2}))$, and so we calculate

$$\begin{aligned}
\Omega(P) &= \hom_{\mathbb{P}_X V}(p^* P \otimes p^* P \otimes \mathcal{O}(n+r+1), p^* L(n))^{\mathrm{hC}_2}\\
&= \hom_{\mathbb{P}_X V}(p^*(P \otimes P), p^* L(-r-1))^{\mathrm{hC}_2}\\
&= \hom_X(P \otimes P, L \otimes p_*\mathcal{O}(-r-1))^{\mathrm{hC}_2}\\
&\overset{\tau}{=} \hom_X(P \otimes P, L \otimes \det V^\vee[-r])^{\mathrm{hC}_2}
\end{aligned}$$

where in last step we have used the equivalence (28) above. □

Let $\mathcal{L} \subseteq \mathcal{C}$ be the stable subcategory generated by $\mathcal{A}(\frac{n+r'}{2}) \cup \cdots \cup \mathcal{A}(\lceil\frac{n+1}{2}\rceil)$. Note that $\mathcal{L}$ has a semi-orthogonal decomposition of the form $(\mathcal{A}(\frac{n+r'}{2}), \ldots, \mathcal{A}(\lceil\frac{n+1}{2}\rceil))$. In particular, for every additive invariant $\mathcal{F}$ there's an equivalence

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{L}) \simeq \mathcal{F}^{\mathrm{hyp}}(\mathcal{D}^{\mathrm{p}}(X))^{\oplus\lfloor\frac{r'+1}{2}\rfloor}$$

induced by the functors $\{p^*(-) \otimes \mathcal{O}(i)\}_{i=\frac{n+r'}{2},\ldots,\lceil\frac{n+1}{2}\rceil}$ (see [Kha20, Lemma 2.8]).

Lemma 6.1.5. *If n is odd then the stable subcategory $\mathcal{L} \subseteq \mathcal{C}$ is a Lagrangian with respect to the Poincaré structure $\Omega^{\mathrm{s}}_{p^*L}|_{\mathcal{C}}$. If n is even then $\mathcal{L}$ is isotropic with respect to the Poincaré structure and its homology inclusion can be identified with the Poincaré functor*

$$p^*(-) \otimes \mathcal{O}\Big(\frac{n}{2}\Big)\colon (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}).$$

Proof. Since $(\mathcal{A}(\frac{n+r'}{2}), \ldots, \mathcal{A}(\frac{n-r'}{2}))$ is a semi-orthogonal decomposition of $\mathcal{C}$ the inclusion $\mathcal{L} \subseteq \mathcal{C}$ has a right adjoint $r\colon \mathcal{C} \to \mathcal{L}$ whose kernel can be identified using Lemma 6.1.2 with either $\mathrm{D}_{p^*L(n)}\mathcal{L}$ if n is odd or with the stable subcategory generated by $\mathrm{D}_{p^*L(n)}\mathcal{L}$ and $\mathcal{A}(\frac{n}{2})$ if n is even. Hence by [CDH⁺II, Remark 3.2.3] and Lemma 6.1.2 the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$ is either $\mathcal{L}$ if n is odd or the subcategory generated by $\mathcal{L}$ and $\mathcal{A}(\frac{n}{2})$ if n is even. In particular, $\mathcal{L} \subseteq \mathcal{L}^\perp$, and since the Poincaré structure is a symmetric one this suffices to prove that $\Omega^{\mathrm{s}}_{p^*L(n)}|_{\mathcal{L}} = 0$. Thus $\mathcal{L}$ is an isotropic subcategory of $\mathcal{C}$ which is furthermore a Lagrangian when n is odd. Finally, when n is even $\mathcal{L}^\perp$ admits a semi-orthogonal decomposition of the form

$(\mathcal{L}, \mathcal{A}(\frac{n}{2})$ and so the duality invariant subcategory $\mathcal{L}^\perp \cap \mathrm{D}_{p^*L(n)}\mathcal{L}^\perp = \mathcal{L}^\perp \cap \ker(r)$ coincides with $\mathcal{A}(\frac{n}{2})$. We may then identify the associated homology inclusion with the resulting Poincaré functor

$$\big(\mathcal{A}(\frac{n}{2}), \Omega^{\mathrm{s}}_{p^*L(n)}|_{\mathcal{A}(\frac{n}{2})}\big) \hookrightarrow \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big).$$

Now since the functor $p^*(-) \otimes \mathcal{O}(\frac{n}{2})\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ is fully-faithful with image $\mathcal{A}(\frac{n}{2})$ the last Poincaré functor is equivalent to the Poincaré functor

$$(\mathcal{D}^{\mathrm{p}}(X), \Omega) \hookrightarrow \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big)$$

where Ω is obtained by restricting $\Omega^{\mathrm{s}}_{p^*L(n)}$ along $p^*(-) \otimes \mathcal{O}(\frac{n}{2})$. We then compute

$$\begin{aligned}\Omega(P) &= \hom_{\mathbb{P}_X V}(p^*P \otimes p^*P \otimes \mathcal{O}(n), p^*L(n))^{\mathrm{hC}_2}\\ &= \hom_{\mathbb{P}_X V}(p^*(P \otimes P), p^*L)^{\mathrm{hC}_2}\\ &= \hom_X(P \otimes P, L)^{\mathrm{hC}_2}\\ &= \Omega^{\mathrm{s}}_L(P)\,,\end{aligned}$$

as desired. □

Putting the two lemmas together we obtain the projective bundle formula.

Theorem 6.1.6 (The projective bundle formula)**.** *Let X be a qcqs scheme, let L be an invertible perfect complex over X with C_2-action and let V be a vector bundle over X of rank $r+1$. Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$ be an additive functor valued in some stable ∞-category $\mathcal{E}$. Then the following holds*

(1) If n and r are both odd then the exact functors $p^*(-)\otimes\mathcal{O}(i)\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ for $i = \frac{n+r}{2}, \dots, \frac{n+1}{2}$ induce an equivalence

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{D}^{\mathrm{p}}(X))^{\oplus(r+1)/2} \xrightarrow{\simeq} \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big).$$

(2) If n and r are both even then the exact functors $p^*(-)\otimes\mathcal{O}(i)\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ for $i = \frac{n+r}{2}, \dots, \frac{n+2}{2}$ together with the Poincaré functor $p^*(-)\otimes\mathcal{O}(\frac{n}{2})\colon (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)})$ induce an equivalence

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{D}^{\mathrm{p}}(X))^{\oplus r/2} \oplus \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \xrightarrow{\simeq} \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big).$$

(3) If n is odd and r is even then we have a fiber sequence

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{D}^{\mathrm{p}}(X))^{\oplus r/2} \to \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}\big) \xrightarrow{p_*((-)\otimes\mathcal{O}(-\frac{n+r+1}{2}))} \mathcal{F}\big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes\det V^\vee[-r]}\big)\,,$$

where the first arrow is induced by the functors $p^*(-)\otimes\mathcal{O}(i)\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ for $i = \frac{n+r-1}{2}, \dots, \frac{n+1}{2}$.

(4) If n is even and r is odd then we have a fiber sequence

$$\mathcal{F}^{\mathrm{hyp}}(\mathcal{D}^{\mathrm{p}}(X))^{\oplus(r-1)/2} \oplus \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \to \mathcal{F}(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)}) \xrightarrow{p_*((-)\otimes\mathcal{O}(-\frac{n+r+1}{2}))} \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes\det V^\vee[-r]})\,,$$

where the first arrow is induced by the functors $p^*(-)\otimes\mathcal{O}(i)\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ for $i = \frac{n+r-1}{2}, \dots, \frac{n+2}{2}$ together with the Poincaré functor $p^*(-) \otimes \mathcal{O}(\frac{n}{2})\colon (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_L) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^*L(n)})$.

Proof. Apply [CDH+II, Theorem 3.2.10] to the isotropic subcategory $\mathcal{L} \subseteq \mathcal{C}$, together with Lemma 6.1.3, Lemma 6.1.5 and [Kha20, Lemma 2.8]. □

Our goal now is to study the splitting of the fiber sequences ((4)) and ((3)). In order to do so we will need to compute a special case of the boundary operator. Suppose $r = \mathrm{rk}V - 1$ is odd. Then the non-degenerate symmetric bilinear form

$$\wedge\colon \Lambda^{\frac{r+1}{2}}V[\frac{r+1}{2}] \otimes \Lambda^{\frac{r+1}{2}}V[\frac{r+1}{2}] \to \det V[r+1]\,.$$

determines a class $e(V) \in \mathrm{L}^{\mathrm{s}}_0(X, \det V[r+1]) = \mathrm{L}^{\mathrm{s}}_{-r-1}(X, \det V)$. We call it the *Euler class* of V. Note that $e(V) = 0$ whenever V has a quotient bundle of odd rank [Wal03, Proposition 8.1], in particular whenever V is trivial.

Lemma 6.1.7. *Suppose that r is odd and n is even. Then the boundary map*

$$\partial\colon\ \mathrm{L}^{\mathrm{s}}_0(X) \to \mathrm{L}^{\mathrm{s}}_{-1}(X, \det V[r]) = \mathrm{L}^{\mathrm{s}}_{-r-1}(X, \det V)$$

of the fiber sequence of Theorem 6.1.6(4) for $\mathcal{F} = \mathrm{L}$ and $\mathrm{L} = \det V[r]$, sends 1 to $e(V)$.

Proof. We first note that since $\mathrm{L}^{\mathrm{hyp}} = 0$ the domain and target of this boundary map indeed as indicated. Second, the parity conditions on n and r imply that $n + r + 1$ is even, and hence by Remark 6.1.1 we may assume without loss of generality that $n = -r - 1$ (this assumption will simplify the notation in the following argument, though mathematically it makes no essential difference). In particular, our Poincaré-Verdier projection takes the form

$$p_*\colon\ \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{p^!\mathcal{O}_X}\big) = \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{\det \mathcal{V}(-r-1)[r]}\big) \to (\mathcal{D}^{\mathrm{p}}(X), \mathcal{O}_X),$$

see Remark 6.1.4. We now carry out the explicit procedure to produce the boundary map as described in [CDH⁺II, Proposition 4.4.8]. For this, we need to find a hermitian object in $\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{\det \mathcal{V}(-r-1)[r]}\big)$ lifting the unit Poincaré object $\mathcal{O}_X$ in $(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_X)$. Since p_* is a Poincaré-Verdier projection there is a natural candidate for such a lift, obtained by applying to $\mathcal{O}_X$ the left adjoint p^*, in which case the hermitian structure map

$$\Omega^{\mathrm{s}}_{\det \mathcal{V}(-r-1)[r]}(\mathcal{O}_{\mathbb{P}_X V}) = \Omega^{\mathrm{s}}_{\det \mathcal{V}(-r-1)[r]}(p^*\mathcal{O}_X) \to \Omega^{\mathrm{s}}_X(\mathcal{O}_X)$$

is an equivalence. Let $q \in \Omega^\infty \Omega^{\mathrm{s}}_{\det \mathcal{V}(-r-1)[r]}(\mathcal{O}_{\mathbb{P}_X V})$ be the hermitian form corresponding to $\mathrm{id} \in \Omega^\infty \Omega^{\mathrm{s}}_X(\mathcal{O}_X)$ under the above equivalence. To calculate the boundary of q, we first note that the induced map of spaces

$$\mathrm{Map}_{\mathbb{P}_X V}\big(\mathcal{O}_{\mathbb{P}_X V}, \mathrm{D}_{\det \mathcal{V}(-r-1)[r]}\mathcal{O}_{\mathbb{P}_X V}\big) \to \mathrm{Map}_X(\mathcal{O}_X, \mathrm{D}_X \mathcal{O}_X)$$

is already an equivalence before taking C_2-homotopy fixed points, and that the second, and hence also the first, are discrete spaces. In particular, q is determined up to homotopy by the associated map

$$q_\sharp\colon\ p^*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}_X V} \to \mathrm{D}_{\det \mathcal{V}(-r-1)[r]}\mathcal{O}_{\mathbb{P}_X V} = \det \mathcal{V}(-r-1)[r],$$

which in turn is determined up to homotopy by the condition that it maps to the identity map $\mathcal{O}_X \to \mathcal{O}_X$ by p_*, under the identifications $\mathcal{O}_X = p_*p^*\mathcal{O}_X$ and $p_*\det\mathcal{V}(-r-1)[r] = \mathcal{O}_X$ determines by the unit and the map τ of (28) used to define to Poincaré structure on p_*. By the construction of τ the map $q_\sharp$ must then be homotopy equivalent to the map

$$\mathcal{O}_{\mathbb{P}_X V} \to \det \mathcal{V}(-r-1)[r]$$

of (27), determined by the inclusion of the bottom term in the complex

$$\Lambda^r \mathcal{V}(-r) \to \cdots \to \Lambda^1 \mathcal{V}(-1) \to \mathcal{O}_{\mathbb{P}_X V}\,.$$

Unwinding the definitions, we then see that the boundary of the hermitian object $(\mathcal{O}_{\mathbb{P}_X V}, q)$ is the object $K \in \mathcal{C} \subseteq \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V)$ represented by the complex

$$\Lambda^r \mathcal{V}(-r) \to \cdots \to \Lambda^1 \mathcal{V}(-1)$$

concentrated in degrees $[1, r]$, with the Poincaré form $\beta \in \Omega^{\mathrm{s}}_{\det \mathcal{V}[r+1](-r-1)}(K)$ given by the collection of wedge pairings

$$\Lambda^i \mathcal{V}(-i) \otimes \Lambda^{r+1-i}\mathcal{V}(-r-1+i) \to \Lambda^{r+1}\mathcal{V}(-r-1).$$

Note that the subcomplex of K given by

$$\Lambda^{\frac{r-1}{2}}\mathcal{V}\big(-\frac{r-1}{2}\big) \to \cdots \to \Lambda^1 \mathcal{V}(-1)$$

lives in $\mathcal{L}$ and so it forms canonically a surgery datum (since $\mathcal{L}$ is isotropic). The result of the surgery is exactly $\Lambda^{\frac{r+1}{2}}\mathcal{V}[\frac{r+1}{2}](\frac{-r-1}{2}) = \Lambda^{\frac{r+1}{2}}\mathcal{V}[\frac{r+1}{2}](\frac{n}{2})$ with bilinear form given by the wedge pairing

$$\Lambda^{\frac{r+1}{2}}\mathcal{V}[\frac{r+1}{2}]\big(\frac{n}{2}\big) \otimes \Lambda^{\frac{r+1}{2}}\mathcal{V}[\frac{r+1}{2}]\big(\frac{n}{2}\big) \to \Lambda^{r+1}\mathcal{V}[r+1](n).$$

that is, it's the pullback of $e(V)$ along $p^*(-) \otimes \mathcal{O}\big(\frac{n}{2}\big)$, as required. ∎

Corollary 6.1.8. *Suppose $n+r$ is odd. If either r is even or r is odd and $e(V)=0$ then for every $L \in \mathcal{P}ic(X)^{\mathrm{hC}_2}$ the Poincaré functor*

$$(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{p^*L(n)}) \xrightarrow{p_*((-)\otimes \mathcal{O}(-\frac{n+r+1}{2}))} (\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]})$$

admits a Poincaré section. In particular, in these cases the fiber sequence in Items (3) and (4) of Theorem 6.1.6 split.

Proof. Assume that $n+r$ is odd. We claim that if either r is even or r is odd and $e(V)=0$ then $1 \in \mathrm{L}^{\mathrm{s}}_0(X)$ belongs to the image of the map

$$\mathrm{L}^{\mathrm{s}}_0(\mathbb{P}_X V, \det \mathcal{V}(n)[r]) \xrightarrow{p_*((-)\otimes \mathcal{O}(\frac{n+r+1}{2}))} \mathrm{L}^{\mathrm{s}}_0(X)\,.$$

Indeed when r is even this map is an equivalence by Theorem 6.1.6(3) and when r is odd the result follows from Lemma 6.1.7. This means that there exists a Poincaré object (x,q) in $(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{\det \mathcal{V}(n)[r]})$ whose image under $p_*((-)\otimes \mathcal{O}(-\frac{n+r+1}{2}))$ is cobordant to $(\mathcal{O}_x, \mathrm{id})$ (where $\mathrm{id} \in \Omega^\infty \mathfrak{Q}^{\mathrm{s}}_X(\mathcal{O}_X)$ is the identity form on $\mathcal{O}_X$). Now since $p_*((-)\otimes \mathcal{O}(-\frac{n+r+1}{2}))$ is a split Poincaré-Verdier projection [CDH+II, Theorem 2.5.3] tells that the induced functor on cobordism categories is bicartesian fibration, and hence we can lift any cobordism between $p_*(x\otimes \mathcal{O}(-\frac{n+r+1}{2}))$ (with its induced form) and $(\mathcal{O}_X, \mathrm{id})$ to a cobordism between (x,q) and some Poincaré object (x',q') in $\left(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{\det \mathcal{V}(n)[r]}\right)$ lying above $(\mathcal{O}_X, \mathrm{id})$. In particular, $p_*(x' \otimes \mathcal{O}(-\frac{n+r+1}{2})) \simeq \mathcal{O}_X$.

We are now going to use the Poincaré object (x',q') in order to construct the desired Poincaré section. For this, note that the symmetric monoidal structure maps of the top horizontal functor in (12) determine in particular a Poincaré functor

$$(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{\det \mathcal{V}(n)[r]}) \otimes_{(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_X)} \left(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\right) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{p^*L(n)})$$

which is an equivalence on underlying stable ∞-categories with duality. In particular, the Poincaré object (x',q') in $(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{\det \mathcal{V}(n)[r]})$ determines a $(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_X)$-linear Poincaré functor

$$F_{(x',q')} \colon \left(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\right) \to \left(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{p^*L(n)}\right),$$

whose underlying exact functor is $F_{x'}(-) = p^*(-)\otimes x'$. Consider the composite Poincaré functor

$$G_{(x',q')} \colon \left(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\right) \xrightarrow{F_{(x',q')}} \left(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \mathfrak{Q}^{\mathrm{s}}_{p^*L(n)}\right) \xrightarrow{p_*((-)\otimes \mathcal{O}(-\frac{n+r+1}{2}))} \left(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\right).$$

On the level of underlying exact functors this composite is given by

$$z \mapsto p_*\big(p^*(z)\otimes x' \otimes \mathcal{O}\big(-\frac{n+r+1}{2}\big)\big) = z \otimes p_*\big(x' \otimes \mathcal{O}\big(-\frac{n+r+1}{2}\big)\big) = z,$$

where the first equivalence is by the projection formula and the second by the construction of x'. We conclude that $G_{(x',q')}$ is an equivalence on underlying stable ∞-categories, and hence on the level of underlying stable ∞-categories with duality. Since $\mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}$ is the symmetric Poincaré structure associated to the underlying duality we conclude that $G_{(x',q')}$ is an equivalence of Poincaré ∞-categories. In fact, one can show that $G_{(x',q')}$ is homotopic to the identity Poincaré functor on $\left(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L\otimes \det V^\vee[-r]}\right)$, so that $F_{(x',q')}$ gives the desired section, though we will not have to prove this: even if $G_{(x',q')}$ is merely some equivalence, we get a section by taking $F_{(x',q')} \circ G^{-1}_{(x',q')}$.

□

6.2. **Genuine refinement for the projective line.** In §6.1 we have proven a general projective bundle formula for the symmetric Poincaré structure. In this section, we want to address the analogous question for the genuine Poincaré structures $\mathfrak{Q}^{\geq m}$. For simplicity, and since it is the only case we will need to use, we will consider only the case $V = \mathcal{O}^{\oplus 1}$ and $m=0$, so that $\mathbb{P}_X V = X \times \mathbb{P}^1$ and $r=1$. In this case, the projective bundle formula of Theorem 6.1.6 (4) provides a Poincaré-Verdier sequence

$$(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_L) \xrightarrow{p^*} (\mathcal{D}^{\mathrm{p}}(X\times \mathbb{P}^1), \mathfrak{Q}^{\mathrm{s}}_{p^*L}) \xrightarrow{p_*(-\otimes \mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{L[-1]})\,. \tag{29}$$

Our goal in this subsection is to prove the following refinement of this result:

Proposition 6.2.1. *For every $m \in \mathbb{Z} \cup \{-\infty, \infty\}$ the sequence (29) refines to a split Poincaré-Verdier sequence*

$$(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\geq m}_{L}) \xrightarrow{p^*} (\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1 \times X), \Omega^{\geq m}_{p^*L}) \xrightarrow{p_*(-\otimes \mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\geq m-1}_{L[-1]}).$$

Before going into the proof of Proposition 6.2.1, we note that the underlying Verdier sequence of (29) induces for every $M \in \mathcal{D}^{\mathrm{qc}}(X \times \mathbb{P}^1)$ a fiber sequence of the form

$$p^*p_*M \to M \to p^*p_*(M(-1)) \otimes \Sigma\mathcal{O}(-1)\ , \tag{30}$$

also known as the Beilinson sequence. The proof of Proposition 6.2.1 will then be based on the following lemma:

Lemma 6.2.2.

(1) There are natural equivalences $p_*\mathcal{E}_{p^*L} \simeq \mathcal{E}_L$ and $p_*\mathcal{E}_{p^*L}(-1) \simeq \Sigma^{-1}\mathcal{E}_L$. In particular, the Beilinson sequence (30) for $M = \mathcal{E}_{p^*L}$ has the form

$$p^*\mathcal{E}_L \to \mathcal{E}_{p^*L} \to p^*\mathcal{E}_L \otimes \mathcal{O}(-1).$$

(2) The above exact sequence splits Zariski locally on $\mathbb{P}^1 \times X$.

(3) For every $m \in \mathbb{Z}$ the associated sequence of m-connective covers

$$\tau_{\geq m}(p^*\mathcal{E}_L) \to \tau_{\geq m}(\mathcal{E}_{p^*L}) \to \tau_{\geq m}(p^*\mathcal{E}_L \otimes \mathcal{O}(-1))$$

remains exact in $\mathcal{D}^{\mathrm{qc}}(\mathbb{P}^1 \times X)$.

Proof. We first prove (1). In the split Poincaré-Verdier sequence (29), the Verdier inclusion p^* has a right adjoint p_* and a left adjoint $p_*((-) \otimes \Sigma\mathcal{O}(-2))$, and the Poincaré projection $p_*((-) \otimes \mathcal{O}(-1))$ has a left adjoint $p^*(-) \otimes \mathcal{O}(1)$ and a right adjoint $p^*(-) \otimes \Sigma\mathcal{O}(-1)$. For $M \in \mathcal{D}^{\mathrm{qc}}(X)$ we consequently have

$$\begin{aligned} \hom_X(M, p_*\mathcal{E}_{p^*L}) &= \hom_{\mathbb{P}^1\times X}(p^*M, \mathcal{E}_{p^*L}) \\ &= \hom_{\mathbb{P}^1\times X}(p^*M \otimes p^*M, p^*L)^{\mathrm{tC}_2} \\ &= \hom_X(M \otimes M, L)^{\mathrm{tC}_2} \\ &= \hom_X(M, \mathcal{E}_L)\ , \end{aligned}$$

and

$$\begin{aligned} \hom_X(M, p_*\mathcal{E}_{p^*L}(-1)) &= \hom_{\mathbb{P}^1\times X}(p^*M \otimes \mathcal{O}(1), \mathcal{E}_{p^*L}) \\ &= \hom_{\mathbb{P}^1\times X}(p^*M \otimes p^*M, p^*L \otimes \mathcal{O}(-2))^{\mathrm{tC}_2} \\ &= \hom_X(M \otimes M, L \otimes p_*\mathcal{O}(-2))^{\mathrm{tC}_2} \\ &= \hom_X(M, \Sigma^{-1}\mathcal{E}_L)\ . \end{aligned}$$

This establishes (1).

For Claim (2), the statement is local in X and hence we may as well assume that $X = \mathrm{spec}(R)$ is affine. We will prove that the sequence in question splits when restricted to $\mathbb{A}^1_R = \mathbb{A}^1 \times R$ for any embedding $j : \mathbb{A}^1_R \hookrightarrow \mathbb{P}^1_R$ over $\mathrm{spec}(R)$. Since $\mathbb{P}^1_R$ can be covered by (two) such copies of $\mathbb{A}^1_R$ this implies the desired result. Now for an open embedding $j : \mathbb{A}^1_R \hookrightarrow \mathbb{P}^1_R$ it follows from Corollary 4.2.12 (applied to the $\geq -\infty = \mathrm{s}$ decoration) and the fact that left Kan extensions commute with taking linear parts that $\mathcal{E}_{j^*p^*L} = j^*\mathcal{E}_{p^*L}$. Wring $f = pj : \mathbb{A}^1_R \to \mathrm{spec}(R)$ we are reduced to showing that the comparison map

$$f^*\mathcal{E}_L \to \mathcal{E}_{f^*T}$$

admits a retraction in $\mathcal{D}^{\mathrm{qc}}(\mathbb{A}^1_R)$. Making this map explicit, we note that L is an invertible R-module with an R-linear C_2-action and $\mathcal{E}_L$ is the object of $\mathcal{D}(R)$ corresponding to the $\mathrm{H}(R)$-module spectrum $\mathrm{H}(L)^{\mathrm{tC}_2}$, with $\mathrm{H}(R)$ acting via the Tate Frobenius $\mathrm{H}(R) \to \mathrm{H}(R)^{\mathrm{tC}_2}$ (the C_2 action on $\mathrm{H}(R)$ being the trivial one). On the level of $\mathrm{H}(R[t])$-module spectra the above comparison map can then be written as the $\mathrm{H}(R[t])$-linear map

$$\mathrm{H}(L)^{\mathrm{tC}_2} \otimes_{\mathrm{H}(R)} \mathrm{H}(R[t]) \to \mathrm{H}(L \otimes_R R[t])^{\mathrm{tC}_2}. \tag{31}$$

induced by the $\mathrm{H}(R)$-linear map $\mathrm{H}(L)^{\mathrm{tC}_2} \to \mathrm{H}(L \otimes_R R[t])^{\mathrm{tC}_2}$. We want to construct a retraction of (31). For this we first reduce to the case where $L = R$ with constant C_2-action. Consider the commutative square of E_∞-ring spectra

$$\begin{array}{ccc} \mathrm{H}(R) & \longrightarrow & \mathrm{H}(R[t]) \\ \downarrow & & \downarrow \\ \mathrm{H}(R)^{\mathrm{tC}_2} & \longrightarrow & \mathrm{H}(R[t])^{\mathrm{tC}_2} \end{array}$$

where the vertical maps are the corresponding Tate Frobenius maps. This square determines a map of E_∞-ring spectra

$$\mathrm{H}(R)^{\mathrm{tC}_2} \otimes_{\mathrm{H}(R)} \mathrm{H}(R[t]) \to \mathrm{H}(R[t])^{\mathrm{tC}_2}. \tag{32}$$

At the same time, the map (31) factors as a composite

$$\mathrm{H}(L)^{\mathrm{tC}_2} \otimes_{\mathrm{H}(R)} \mathrm{H}(R[t]) \to (\mathrm{H}L)^{\mathrm{tC}_2} \otimes_{\mathrm{H}(R)^{\mathrm{tC}_2}} \mathrm{H}(R[t])^{\mathrm{tC}_2} \to \mathrm{H}(L \otimes_R R[t])^{\mathrm{tC}_2},$$

where the first map is induced by the various Tate Frobenius maps and the second map by the lax monoidal structure of $(-)^{\mathrm{tC}_2}$. We then observe that the second map is an equivalence: indeed, since the Tate construction commutes with infinite direct sums of uniformly truncated objects we have that $\mathrm{H}(R[t])^{\mathrm{tC}_2}$ is an infinite direct sum of copies of $\mathrm{H}(R)^{\mathrm{tC}_2}$ indexed by the monomials t^i and $\mathrm{H}(L \otimes_R R[t])^{\mathrm{tC}_2}$ is an infinite direct sum of copies of $\mathrm{H}(L)^{\mathrm{tC}_2}$ indexed in a corresponding manner. We then see that the map (31) is obtained from (32) by tensoring with $(\mathrm{H}L)^{\mathrm{tC}_2}$ over $\mathrm{H}(R)^{\mathrm{tC}_2}$. It will hence suffice to show that the map (32) admits a $[\mathrm{H}(R)^{\mathrm{tC}_2} \otimes_{\mathrm{H}(R)} \mathrm{H}(R[t])]$-linear retraction.

Explicitly, $(\mathrm{H}R)^{\mathrm{tC}_2} \otimes_{\mathrm{H}R} \mathrm{H}(R[t])$ is an infinite direct sum of copies of $\mathrm{H}(R)^{\mathrm{tC}_2}$, indexed by the monomials t^i, where the multiplication is determines by the E_∞-ring structure of $\mathrm{H}(R)^{\mathrm{tC}_2}$ and the rule $t^i t^j = t^{i+j}$. As above, since $(-)^{\mathrm{tC}_2}$ commutes with uniformly truncated infinite direct sums the underlying spectrum of $\mathrm{H}(R[t])^{\mathrm{tC}_2}$ has the exact same form, and the exact same multiplication law. The map (32) hence corresponds to the endomorphism

$$\bigoplus_{i \geq 0} \mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle \to \bigoplus_{i \geq 0} \mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle \tag{33}$$

induced by the Tate Frobenius of $\mathrm{H}(R[t])$. We claim that this map identifies the $\mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle$ factor on the left with the $\mathrm{H}(R)^{\mathrm{tC}_2}\langle t^{2i} \rangle$ factor on the right. Indeed, since this map is $\mathrm{H}(R)^{\mathrm{tC}_2}$ linear it will suffice to check that it sends the component of $\mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle$ corresponding to the unit of $\mathrm{H}(R)^{\mathrm{tC}_2}$ to the component of $\mathrm{H}(R)^{\mathrm{tC}_2}\langle t^{2i} \rangle$ corresponding to the same unit. Now since the C_2-action on R is trivial we have that $\pi_0 \mathrm{H}(R[t])^{\mathrm{tC}_2} = \hat{H}^0(R[t]) = R[t]/2 = (R/2)[t]$, and the Tate Frobenius is the map

$$R[t] \to (R/2)[t] \qquad x \mapsto [x^2]$$

on the level of π_0. In particular, it sends $t^i \in R[t]$ to $t^{2i} \in (R/2)[t]$, as needed. We conclude that, as an $\bigoplus_{i \geq 0} \mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle$-module, the right hand side of (33) splits as a direct sum of two copies of $\bigoplus_{i \geq 0} \mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle$ (one spanned by the even powers of t and one by the odd powers), such that the map (32) is the inclusion of the even summand. It follows that this map admits an $\bigoplus_{i \geq 0} \mathrm{H}(R)^{\mathrm{tC}_2}\langle t^i \rangle$-linear retraction, as desired.

Finally, to prove (3), we note that for a sequence of quasi-coherent sheaves $M \to N \to P$ the comparison map $\eta_m : \operatorname{cof}[\tau_{\geq m} M \to \tau_{\geq m} N] \to \tau_{\geq m} P$ is an isomorphism on all homotopy sheaves π_i for $i \neq m$ by the 5-lemma, while the induced map on π_m is injective. But if the sequence splits Zariski locally then the map $N \to P$ is surjective on homotopy sheaves and hence the map η_m is an equivalence. □

Proof of Proposition 6.2.1. We first note that the exact functor p^* underlying the Poincaré-Verdier inclusion in (29) is t-exact, and in particular right t-exact, and hence the Poincaré-Verdier inclusion

$$(p^*, \eta^{\mathrm{s}}) \colon (\mathcal{D}^{\mathrm{qc}}(X), \Qoppa^{\mathrm{s}}_L) \to (\mathcal{D}^{\mathrm{qc}}(\mathbb{P}^1 \times X), \Qoppa^{\mathrm{s}}_{p^*L})$$

induces a Poincaré functor

$$(p^*, \eta^{\geq m}) \colon (\mathcal{D}^{\mathrm{qc}}(X), \Qoppa^{\geq m}_L) \to (\mathcal{D}^{\mathrm{qc}}(\mathbb{P}^1 \times X), \Qoppa^{\geq m}_{p^*L}),$$

see §3.3. To check that this Poincaré functor remains a Poincaré-Verdier inclusion one needs to check that the Poincaré structure map

$$\eta^{\geq m} \colon \Qoppa^{\geq m}_L \Rightarrow \Qoppa^{\geq m}_{p^*L}(p^*(-))$$

is an equivalence. This map is already known to be an equivalence on bilinear parts (since these are the same as those of the corresponding symmetric Poincaré structures), and hence it will suffice to verify that $\eta^{\geq m}$ is an equivalence on linear parts. Unwinding the definitions and using that p^* is t-exact this amounts to the composed map

$$\tau_{\geq m}(\mathcal{E}_L) \xrightarrow{\simeq} p_*p^*\tau_{\geq m}(\mathcal{E}_L) \xrightarrow{\simeq} p_*\tau_{\geq m}(p^*\mathcal{E}_L) \to p_*\tau_{\geq m}(\mathcal{E}_{p^*L})$$

being an equivalence in $\mathcal{D}^{\mathrm{qc}}(X)$, and hence by Lemma 6.2.2 to the vanishing of $p_*\tau_{\geq m}(p^*\mathcal{E}_L \otimes \mathcal{O}(-1))$. To see that this last term indeed vanishes, note that the natural map $\tau_{\geq m}(p^*\mathcal{E}_L) \otimes \mathcal{O}(-1) \to \tau_{\geq m}(p^*\mathcal{E}_L \otimes \mathcal{O}(-1))$ is an equivalence since it is an equivalence Zariski locally, and hence the push-forward of the latter quasi-coherent sheaf to X vanishes.

Now on the projection side, since $p_*((-) \otimes \mathcal{O}(-1))$ is a split Verdier projection it automatically refines to a split Poincaré-Verdier projection

$$(\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1 \times X), Ϙ^{\geq m}_{p^*L}) \to (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\geq m}_{p^*L}, \Phi)$$

where Φ is obtained by precomposing $Ϙ^{\geq m}_{p^*L}$ with the left adjoint of $p_*((-) \otimes \mathcal{O}(-1))$, which is given by $p^*(M) \otimes \mathcal{O}(1)$. The natural transformation $Ϙ^{\geq m}_{p^*L} \Rightarrow Ϙ^{\mathrm{s}}_{p^*L}$ then induces a natural transformation

$$\tau : \Phi \Rightarrow Ϙ^{\mathrm{s}}_{L[-1]},$$

which is an equivalence on bilinear parts. It will hence suffice to check that τ is an $(m-1)$-connective cover on linear parts. Now, by Lemma 6.2.2, we get that the map

$$\tau_{\geq m}(\mathcal{E}_{p^*L}) \to \tau_{\geq m}(p^*\mathcal{E}_L \otimes \mathcal{O}(-1)) = \tau_{\geq m}(p^*\mathcal{E}_L) \otimes \mathcal{O}(-1)$$

induces an equivalence on mapping spectra with domain of the form $p^*M \otimes \mathcal{O}(1)$ for $M \in \mathcal{D}^{\mathrm{p}}(X)$ (indeed, the fibre of this map is the constant sheaf $\tau_{\geq m}p^*\mathcal{E}_L = p^*\tau_{\geq m}\mathcal{E}_L$). We hence get that

$$\begin{aligned}
\Lambda_\Phi(M) &= \hom(p^*M \otimes \mathcal{O}(1), \tau_{\geq m}\mathcal{E}_{p^*L}) \\
&= \hom(p^*M \otimes \mathcal{O}(1), \tau_{\geq m}(p^*\mathcal{E}_L) \otimes \mathcal{O}(-1)) \\
&= \hom(M, p_*(\tau_{\geq m}p^*\mathcal{E}_L \otimes \mathcal{O}(-2))), \\
&= \hom(M, p_*(p^*\tau_{\geq m}\mathcal{E}_L \otimes \mathcal{O}(-2))), \\
&= \hom(M, \Sigma^{-1}\tau_{\geq m}\mathcal{E}_L),
\end{aligned}$$

and the map $\Sigma^{-1}\tau_{\geq m}\mathcal{E}_L \to \Sigma^{-1}\mathcal{E}_L$ is indeed an $(m-1)$-connective cover, as desired. □

6.3. **Homotopy invariance of symmetric Grothendieck-Witt.** As an application of the projective bundle formula (Theorem 6.1.6) and dévissage (Theorem 5.2.1), we show here that the symmetric Grothendieck-Witt spectrum is $\mathbb{A}^1$-invariant over a regular Noetherian base of finite Krull dimension. Here, it does not matter if we consider Grothendieck-Witt or Karoubi-Grothendieck-Witt theory: when X is regular the canonical map $\mathrm{GW}^{\mathrm{s}}(X) \to \mathbb{GW}^{\mathrm{s}}(X)$ is an equivalence. Indeed, by Karoubi cofinality (3), this follows from the map $\mathrm{K}(X) \to \mathbb{K}(X)$ being an equivalence, which is a well-known consequence of X being regular, see [TT90, Proposition 6.8], or in more modern language of [AGH19, Theorem 1.2].

Theorem 6.3.1 ($\mathbb{A}^1$-invariance)**.** *Let X be a regular Noetherian scheme of finite Krull dimension. Then the pullback map*

$$\begin{array}{ccc}
\mathrm{GW}^{\mathrm{s}}(\mathcal{D}^{\mathrm{p}}(X)) & \longrightarrow & \mathrm{GW}^{\mathrm{s}}(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{A}^1)) \\
\| & & \| \\
\mathbb{GW}^{\mathrm{s}}(\mathcal{D}^{\mathrm{p}}(X)) & \longrightarrow & \mathbb{GW}^{\mathrm{s}}(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{A}^1))
\end{array}$$

is an equivalence.

Proof. Write $p\colon X \times \mathbb{P}^1 \to X$ and $q\colon X \times \mathbb{A}^1 \to X$ for the corresponding projections. Consider the commutative diagram of Poincaré ∞-categories

$$\begin{array}{ccccc}
 & & (\mathcal{D}^{\mathrm{p}}(X \times \mathbb{A}^1), Ϙ^{\mathrm{s}}_{q^*L}) & & \\
 & \nearrow^{q^*} & \uparrow & & \\
(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L}) & \xhookrightarrow{p^*} & (\mathcal{D}^{\mathrm{p}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}) & \xrightarrow{p_*(-\otimes\mathcal{O}(-1))} & (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L[-1]}) \\
 & & \uparrow & \nearrow_{\varphi} & \\
 & & (\mathcal{D}^{\mathrm{p}}_{X\times\{\infty\}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}) & &
\end{array}$$

where the horizontal sequence is the fiber sequence of Theorem 6.1.6(4) for $V = \mathcal{O}^{\oplus 2}$, and the vertical sequence is the localisation sequence of Corollary 4.2.12 associated to the closed subscheme inclusion $X \times \{\infty\} \subseteq X \times \mathbb{P}^1$. Since both the vertical and horizontal sequences are Karoubi-Poincaré sequence we see that q^* induces an equivalence on $\mathbb{GW}$ if and only if φ induces an equivalence on $\mathbb{GW}$; indeed, both statements are equivalent to the map

$$\mathbb{GW}(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}) \to \mathbb{GW}(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{A}^1), Ϙ^{\mathrm{s}}_{q^*L}) \oplus \mathbb{GW}(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L[-1]})$$

induced by the right horizontal and top vertical arrows being an equivalence. Now to show that φ induces an equivalence on $\mathbb{GW}$ it will suffice to show that the composite Poincaré functor

$$(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{i^!p^*L}) \xrightarrow{i_*} (\mathcal{D}^{\mathrm{p}}_{X\times\{\infty\}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}) \xrightarrow{\varphi} (\mathcal{D}^{\mathrm{p}}(X), \Sigma^{-1}Ϙ^{\mathrm{s}}_{L})$$

is an equivalence, where the functor on the left is the dévissage Poincaré functor constructed in §5.4, and which is an equivalence by Theorem 5.2.1. Indeed, since φ is a composite

$$\big(\mathcal{D}^{\mathrm{p}}_{X\times\{\infty\}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}\big) \subseteq \big(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}\big) \xrightarrow{p_*((-)\otimes\mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L})$$

and hence it will suffice to show that the composite

$$(\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{i^!p^*L}) \xrightarrow{i_*} \big(\mathcal{D}^{\mathrm{p}}(X \times \mathbb{P}^1), Ϙ^{\mathrm{s}}_{p^*L}\big) \xrightarrow{p_*((-)\otimes\mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(X), Ϙ^{\mathrm{s}}_{L})$$

is an equivalence. Indeed, this now follows from Lemma 5.1.4. □

Let us finally build on a classical argument to show that on regular Noetherian schemes, outside of the symmetric case, neither L-theory nor GW-theory is homotopy invariant.

Example 6.3.2. For any $m \in \mathbb{Z} \cup \{+\infty\}$, the pull-back map $\mathrm{L}(\mathbb{F}_2, Ϙ^{\geq m}) \to \mathrm{L}(\mathbb{F}_2[t], Ϙ^{\geq m})$ is not an equivalence.

Proof. Let us start by the case $m = +\infty$, i.e. the quadratic case. The quadratic L-groups of rings coincide in non-negative degrees with Ranicki's periodic L-groups, by [CDH+I, Example 2.3.17]. The L_0-group is therefore the classical Witt group of non-degenerate quadratic forms over R.

For any scheme X, consider the sections $s_0, s_1 : X \to \mathbb{A}^1 \times X$ at 0 and 1 of the projection $p : \mathbb{A}^1 \times X \to X$. We have $s_0^* p^* = \mathrm{id} = s_1^* p^*$ on L-groups, so if p^* is an isomorphism, then $s_0^* = s_1^*$, both being inverse to p^*. We now show this cannot happen over $X = \mathrm{spec}(\mathbb{F}_2)$. The class of the non-singular quadratic form q of rank 2 given by $q(x, y) = x^2 + xy + ty^2$ is an element of $\mathrm{L}_0(\mathbb{F}_2[t], Ϙ^{\mathrm{q}})$, and it pulls back via s_i to the class of $q_i(x, y) = x^2 + xy + ixy$ in $\mathrm{L}_0(\mathbb{F}_2, Ϙ^{\mathrm{q}})$, whose Arf invariant is $i \in \mathbb{F}_2$. Thus $s_0^* \neq s_1^*$ on $\mathrm{L}_0(-, Ϙ^{\mathrm{q}})$. Actually, by [CDH+II, 4.5.6], the quadratic L-groups are 4-periodic by a natural isomorphism commuting to these pull-backs, so $s_0^* \neq s_1^*$ on $\mathrm{L}_{4n}(-, Ϙ^{\mathrm{q}})$ for any $n \in \mathbb{Z}$.

When $m \in \mathbb{Z}$, for $n << 0$, we have $\mathrm{L}_n(R, Ϙ^{\geq m}) \simeq \mathrm{L}_n(R, Ϙ^{\mathrm{q}})$ via the natural map, by [CDH+I, Corollary 1.2.12]. The quadratic case thus implies that $s_0^* \neq s_1^*$ on $\mathrm{L}_n(-, Ϙ^{\geq m})$ for some $n << 0$. □

In fact, we suspect that forcing homotopy invariance on the quadratic GW or any of the truncated versions might turn it into the symmetric GW.

7. Motivic realization of localising invariants

Our goal in this section is to construct, for a qcqs scheme S, a lax symmetric monoidal functor

$$\mathcal{R}^{\mathrm{s}}_S\colon \ \mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat^p}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p})),$$

which takes as input a Karoubi-localising functor $\mathcal{F}\colon \mathrm{Cat^p} \to \mathcal{S}\mathrm{p}$ on Poincaré ∞-categories and returns a pre-motivic spectrum over S, that is, a $\mathbb{P}^1$-spectrum object in Nisnevich sheaves of spectra. Taking $\mathcal{F}$ to be the Karoubi-Grothendieck-Witt functor then yields a hermitian K-theory pre-motivic spectrum, which is actually a motivic spectrum if S is regular Noetherian of finite Krull dimension. The study of this and related motivic spectra, such as the one associated to Karoubi L-theory, will be taken on in §8 below. The present section is, in turn, essentially dedicated to the construction of $\mathcal{R}^{\mathrm{s}}_S$, which is somewhat involved. In fact, it will be convenient to construct a slightly more flexible version of $\mathcal{R}^{\mathrm{s}}_S$, which takes as input a Karoubi-localising functor $\mathcal{F}\colon \mathrm{Cat^p} \to \mathcal{S}\mathrm{p}$ and an S-linear stable ∞-category with duality $(\mathcal{C}, \mathrm{D})$. We call the resulting pre-motivic spectrum $\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D}))$ the *pre-motivic realization of $\mathcal{F}$ with coefficients in* $(\mathcal{C}, \mathrm{D})$. A motivic realization functor is then obtained by post-composing with the $\mathbb{A}^1$-localisation functor $\mathrm{Loc}_{\mathbb{A}^1}$, which converts any pre-motivic spectrum to a motivic one. This is eventually achieved in the end of §7.3, see Construction 7.3.10 below. Then, in §7.4 we prove a technical lemma which says that, in a certain precise sense, the pre-motivic spectrum $\mathcal{R}^{\mathrm{s}}_S(\mathcal{F})$ only depends on the underlying $\mathrm{Grp}_{\mathrm{E}_\infty}$-valued localising functor $\Omega^\infty\mathcal{F}$.

7.1. **Recollections on motivic spectra.** Let S be a quasi-compact and quasi-separated scheme. Recall that we denote by Sm_S the category of smooth S-schemes $p\colon X \to S$. In the present section, we always consider Sm_S as endowed with the Nisnevich topology.

There are several equivalent ways to define the ∞-category of motivic spectra over S. To facilitate the manoeuvre between these, let us fix coefficient ∞-category $\mathcal{E}$ that is presentable. In practice, $\mathcal{E}$ will either be the ∞-category of spaces, pointed spaces, E_∞-groups, or spectra. Recall that a Nisnevich sheaf $\mathcal{F}\colon \mathrm{Sm}^{\mathrm{op}}_{/S} \to \mathcal{E}$ is said to be $\mathbb{A}^1$-*invariant* if the canonical map $\mathcal{F}(S) \to \mathcal{F}(\mathbb{A}^1_S)$ is an equivalence in $\mathcal{E}$. We then call $\mathrm{H}(S, \mathcal{E})$ the full subcategory $\mathrm{Sh}^{\mathrm{nis}}_{\mathbb{A}^1}(S, \mathcal{E}) \subseteq \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$ spanned by the $\mathbb{A}^1$-invariant Nisnevich sheaves. When $\mathcal{E} = \mathcal{S}$, these are known as *motivic spaces*, and one simply writes $\mathrm{H}(S)$ for $\mathrm{H}(S, \mathcal{S})$.

When $\mathcal{E}$ is pointed, it becomes canonically tensored and cotensored over the presentable ∞-category $\mathcal{S}_*$ of pointed spaces, in which case $\mathrm{Psh}(S, \mathcal{E})$ becomes tensored and cotensored over $\mathrm{Psh}(S, \mathcal{S}_*)$ by means of Day convolution with respect to the fibre product monoidal structure on Sm_S, which coincides in this case with the pointwise product (since the fibre product is the cartesian product in Sm_S). In particular, if $X \to S$ is a smooth S-scheme equipped with an S-point $s\colon S \to X$, then the pointed S-scheme (X, s) determines via representability a presheaf of pointed spaces, and for $\mathcal{F} \in \mathrm{Psh}(S, \mathcal{S}_*)$, the cotensor operation is given by

$$\mathcal{F}^{(X,s)}(T) = \mathrm{fib}[\mathcal{F}(X \times_S T) \xrightarrow{s^*} \mathcal{F}(T)].$$

When $X = \mathbb{P}^1_S$ and s is the section at ∞, we also write

$$\Omega_{\mathbb{P}^1}(\mathcal{F}) = \mathcal{F}^{(\mathbb{P}^1,\infty)}.$$

By [CDH+IV], the full subcategories $\mathrm{H}(S, \mathcal{E}) \subseteq \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}) \subseteq \mathrm{Psh}(S, \mathcal{E})$ are both $\mathrm{Psh}(S, \mathcal{S}_*)$-linear as accessible localisations of $\mathrm{Psh}(S, \mathcal{E})$. In particular, they both inherit the structure of being tensored and cotensored over $\mathrm{Psh}(S, \mathcal{S}_*)$, with the cotensor operation being given by the same formula.

Definition 7.1.1. Let $\mathcal{E}$ be a pointed presentable ∞-category. We define $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S, \mathcal{E})$ and $\mathrm{SH}(S, \mathcal{E})$ to be the stabilizations of $\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$ and $\mathrm{H}(S, \mathcal{E})$ respectively, with respect to the action of $(\mathbb{P}^1, \infty) \in \mathrm{Psh}(S, \mathcal{S}_*)$. In particular, $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S, \mathcal{E})$ and $\mathrm{SH}(S, \mathcal{E})$ are given by the homotopy limits

$$\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S, \mathcal{E}) = \lim \left(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}) \xleftarrow{\Omega_{\mathbb{P}^1}} \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}) \xleftarrow{\Omega_{\mathbb{P}^1}} \quad \cdots \quad \right)$$

and

$$\mathrm{SH}(S, \mathcal{E}) = \lim \left(\mathrm{H}(S, \mathcal{E}) \xleftarrow{\Omega_{\mathbb{P}^1}} \mathrm{H}(S, \mathcal{E}) \xleftarrow{\Omega_{\mathbb{P}^1}} \quad \cdots \quad \right).$$

The fully-faithful inclusion $\mathrm{H}(S,\mathcal{E}) \subseteq \mathrm{Sh}^{\mathrm{nis}}(S,\mathcal{E})$ then induces a fully-faithful inclusion $\mathrm{SH}(S,\mathcal{E}) \subseteq \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathcal{E})$. We refer to the objects of $\mathrm{SH}(S,\mathcal{E})$ as *$\mathcal{E}$-valued motivic spectra*, and to those of $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathcal{E})$ as *$\mathcal{E}$-valued pre-motivic spectra*. If $\mathcal{E} = \mathcal{S}\mathrm{p}$ then we drop the prefix "$\mathcal{E}$-valued".

Concretely, we may describe an $\mathcal{E}$-valued pre-motivic spectrum as a sequence $\mathcal{X}_0, \mathcal{X}_1, \dots,$ of $\mathcal{E}$-valued Nisnevich sheaves on S, together with, for every $n \geq 0$, and equivalence $f_n : \mathcal{X}_n \xrightarrow{\simeq} \Omega_{\mathbb{P}^1}\mathcal{X}_{n+1}$. Such an $\mathcal{E}$-valued pre-motivic spectrum is an $\mathcal{E}$-valued motivic spectrum exactly when all the Nisnevich sheaves $\mathcal{X}_n$ are $\mathbb{A}^1$-invariant.

Proposition 7.1.2. *Let $\mathcal{E}$ be a pointed presentable ∞-category. The ∞-category $\mathrm{SH}(S,\mathcal{E})$ is stable, and the functor*

$$\mathrm{SH}(S,\mathcal{S}\mathrm{p}(\mathcal{E})) \to \mathrm{SH}(S,\mathcal{E})$$

induced by $\Omega^\infty : \mathcal{S}\mathrm{p}(\mathcal{E}) \to \mathcal{E}$ is an equivalence. In particular, the functors

$$\mathrm{SH}(S,\mathcal{S}\mathrm{p}) \to \mathrm{SH}(S,\mathrm{Grp}_{\mathrm{E}_\infty}) \to \mathrm{SH}(S,\mathcal{S}_*)$$

are both equivalences.

We will consider any of the equivalent ∞-categories

$$\mathrm{SH}(S,\mathcal{S}\mathrm{p}) \simeq \mathrm{SH}(S,\mathrm{Grp}_{\mathrm{E}_\infty}) \simeq \mathrm{SH}(S,\mathcal{S}_*)$$

as a model for the ∞-category of *motivic spectra* over S.

Proof of Proposition 7.1.2. Now note that since $\mathrm{H}(S,\mathcal{E})$ is closed in $\mathrm{Psh}(S,\mathcal{E})$ under limits we have that limits in $\mathrm{H}(S,\mathcal{E})$ are computed levelwise. This applies in particular to taking loop objects, and so $\mathrm{SH}(S,\mathcal{S}\mathrm{p}(\mathcal{E})) = \mathcal{S}\mathrm{p}(\mathrm{SH}(S,\mathcal{E}))$ over $\mathrm{SH}(S,\mathcal{E})$. All claims hence follow once we show that $\mathrm{SH}(S,\mathcal{E})$ is stable. This, in turn, follows from the fact that in $\mathrm{SH}(S,\mathcal{S}_*)$ one has the equivalence $\mathrm{S}^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$. As a consequence, universally inverting the action of $\mathbb{P}^1$ inverts both the action of S^1 – thus stabilizing the category in the standard sense – and $\mathbb{G}_m$. ☐

The advantage of working with coefficients in $\mathcal{S}\mathrm{p}$ (or more generally in any stable $\mathcal{E}$) is that the localisation functor

$$\mathrm{Loc}_{\mathbb{A}^1} : \mathrm{Sh}^{\mathrm{nis}}(S,\mathcal{S}\mathrm{p}) \to \mathrm{H}(S,\mathcal{S}\mathrm{p})$$

is given by the explicit formula

$$(\mathrm{Loc}_{\mathbb{A}^1}\mathcal{X})_n(T) = |\mathcal{X}_n(T \times_S \Delta^\bullet_S)|,$$

where Δ^n_S is the algebraic n-simplex over S, that is, the closed subscheme of $\mathbb{A}^n \times S$ determined by the equation $\sum_i x_i = 1$. In fact, this formula always gives the reflection to the full subcategory spanned by $\mathbb{A}^1$-invariant presheaves, but for a general $\mathcal{E}$, it does not preserve the Nisnevich sheaf property. However, when $\mathcal{E}$ is stable the sheaf property is preserved under colimits, and hence $\mathrm{Loc}_{\mathbb{A}^1}$ sends Nisnevich sheaves to Nisnevich sheaves, which implies that its universal property persists to the sheaf context. In fact, when $\mathcal{E}$ is stable this formula also applies for $\mathbb{P}^1$-spectrum objects:

Lemma 7.1.3. *Suppose that $\mathcal{E}$ is stable. If $F : \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{E}$ is a Nisnevich sheaf, then the interchange map $\mathrm{Loc}_{\mathbb{A}^1}\Omega_{\mathbb{P}^1}F \to \Omega_{\mathbb{P}^1}\mathrm{Loc}_{\mathbb{A}^1}F$ is an equivalence. In particular, the localisation functor*

$$\mathrm{Loc}_{\mathbb{A}^1} : \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathcal{E}) \to \mathrm{SH}(S,\mathcal{E})$$

(which we denote by the same name) is computed $\mathbb{P}^1$-levelwise.

Proof. The cartesian products with $\mathbb{A}^n$ and $\mathbb{P}^1$ commute, and taking fibers commutes with colimits because in a stable ∞-category finite limits commute with arbitrary colimits. ☐

Remark 7.1.4. The above considerations also imply that the left adjoint to the inclusion

$$\mathrm{SH}(S,\mathrm{Grp}_{\mathrm{E}_\infty}) \subseteq \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathrm{Grp}_{\mathrm{E}_\infty})$$

is given by the composite

$$\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathrm{Grp}_{\mathrm{E}_\infty}) \xrightarrow{\iota_*} \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(S,\mathcal{S}\mathrm{p}) \xrightarrow{\mathrm{Loc}_{\mathbb{A}^1}} \mathrm{SH}(S,\mathcal{S}\mathrm{p}) = \mathrm{SH}(S,\mathrm{Grp}_{\mathrm{E}_\infty}),$$

where as in §7.2, the functor i_* is obtained by object-wise composing with the fully-faithful inclusion $\iota\colon \mathrm{Grp}_{\mathrm{E}_\infty} \hookrightarrow \mathcal{S}\mathrm{p}$ followed by the reflection functor

$$\mathcal{P}\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Psh}(S, \mathcal{S}\mathrm{p})) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p})).$$

7.2. **Bounded localising invariants.** Let S be a quasi-compact and quasi-separated scheme.

Construction 7.2.1. We denote by

$$\mathrm{Mod}^{\flat}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \subseteq \mathrm{Mod}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$$

the full subcategory spanned by those $\mathcal{D}^{\mathrm{p}}(S)$-linear ∞-categories with duality which are flat over $\mathcal{D}^{\mathrm{p}}(S)$ in the sense of Definition 4.1.4, see Notation 3.2.3. Our previous results establish the following:

- By Example 4.1.5, this full subcategory contains $\mathcal{D}^{\mathrm{p}}(X)$ for every flat qcqs $p\colon X \to S$, and so in particular for any smooth qcqs p.
- By Corollary 4.1.7, this full subcategory is closed under tensor products in $\mathrm{Mod}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$, and hence inherits from $\mathrm{Mod}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ its symmetric monoidal structure.

Combining these facts we find that the symmetric monoidal functor

$$(\mathrm{Sm}^{\mathrm{op}}_{/S})^{\otimes} \to (\mathrm{Mod}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}))^{\otimes} \qquad [X \to S] \mapsto (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{X})$$

of (13) then factors uniquely through a symmetric monoidal functor

$$(\mathrm{Sm}^{\mathrm{op}}_{/S})^{\otimes} \to \mathrm{Mod}^{\flat}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes}.$$

Now since the ∞-category $\mathrm{Sm}_{/S}$ of smooth S-schemes is essentially small, we may fix a large enough regular cardinal κ such that the image of this symmetric monoidal functor lies in the full subcategory

$$\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) = \mathrm{Mod}^{\flat}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \cap \mathrm{Mod}^{\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$$

spanned by the flat κ-compact $\mathcal{D}^{\mathrm{p}}(S)$-linear ∞-categories with duality. Since $\mathrm{Mod}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ is presentably symmetric monoidal, we can ensure that $\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ is closed under tensor products in $\mathrm{Mod}^{\flat}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ by taking κ sufficiently large. In particular, $\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ inherits a symmetric monoidal structure and the above symmetric monoidal functor further factors through a symmetric monoidal functor

$$(\mathrm{Sm}^{\mathrm{op}}_{/S})^{\otimes} \to \mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} \qquad [X \to S] \mapsto (\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{X}). \tag{34}$$

Let us now fix a presentable ∞-category $\mathcal{E}$.

Definition 7.2.2. We say that a functor $\mathcal{F}\colon \mathrm{Mod}^{\flat,\kappa}_{\mathcal{D}^{\mathrm{p}}(S)}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{E}$ is *bounded localising* if it is reduced and sends bounded Karoubi squares (see Definition 4.2.6) to fibre squares. We denote by

$$\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$$

the full subcategory spanned by the bounded localising functors.

Remark 7.2.3. If $\mathcal{E}$ is a stable then a reduced functor $\mathcal{F}\colon \mathrm{Mod}^{\flat,\kappa}_{\mathcal{D}^{\mathrm{p}}(S)}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{E}$ is bounded localising if and only if it sends bounded Karoubi sequences to exact sequences.

By [CDH+IV], the full subcategory $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ is an accessible localisation of $\mathrm{Fun}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ with associated localisation functor

$$\mathrm{Loc}_{\mathrm{b}}\colon \mathrm{Fun}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}).$$

Now, by Nisnevich descent (see Corollary 4.4.3), in the left-Kan-extension/restriction adjunction

$$\mathrm{Psh}(\mathrm{Sm}_{/S}, \mathcal{E}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{Fun}(\mathrm{Mod}^{\flat,\kappa}_{S}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}),$$

the right adjoint sends bounded localising functors to Nisnevich sheaves and so the left adjoint sends Nisnevich local equivalences to Loc_b-equivalences. By the universal property of localisations, we now obtain a commutative square of left adjoint functors

$$\begin{array}{ccc} \mathrm{Psh}(\mathrm{Sm}_{/S}, \mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \\ \big\downarrow{\scriptstyle (-)^{\mathrm{nis}}} & & \big\downarrow{\scriptstyle \mathrm{Loc}_\mathrm{b}} \\ \mathrm{Sh}^{\mathrm{nis}}(\mathrm{Sm}_{/S}, \mathcal{E}) & \xrightarrow{\mathcal{T}_S} & \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})\,. \end{array} \tag{35}$$

where $\mathcal{T}_S$ is given by left Kan extension followed by Loc_b.

7.3. **Motivic realization.** In this subsection we construct our motivic realization functor, see Constructions 7.3.4 and 7.3.10 below.

We begin by promoting the square (35) to a symmetric monoidal one, that which will enable us to eventually pass from Nisnevich sheaves to $\mathbb{P}^1$-spectrum objects therein, that is, to pre-motivic spectra (see Definition 7.1.1). For this note that if $\mathcal{E}$ is a presentably symmetric monoidal ∞-category, then the functor categories

$$\mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \quad \text{and} \quad \mathrm{Psh}(S, \mathcal{E}) = \mathrm{Fun}(\mathrm{Sm}^{\mathrm{op}}_{/S}, \mathcal{E})$$

inherit symmetric monoidal structures in the form of Day convolution, and the symmetric monoidal functor (34) induces by means of left Kan extension a symmetric monoidal functor

$$\mathrm{Psh}(S, \mathcal{E}) \to \mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}). \tag{36}$$

Note that the symmetric monoidal structure on $\mathrm{Sm}^{\mathrm{op}}_{/S}$ is the cocartesian one, and so Day convolution on $\mathrm{Psh}(S, \mathcal{E})$ amounts to pointwise products of presheaves.

Now, on the sheaf side, we have that $\mathrm{Psh}(S, \mathcal{E})$ is presentable and its full subcategory $\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}) \subseteq \mathrm{Psh}(S, \mathcal{E})$ spanned by the Nisnevich sheaves is an accessible localisation. Furthermore, this localisation is multiplicative with respect to Day convolution: indeed, the monoidal product on Sm_S is fibre product over S, and coverings families are closed under base change, see, e.g., the criterion in [CDH⁺I, Lemma 5.3.4]. In particular, $\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$ inherits a symmetric monoidal structure such that the sheafification functor

$$(-)^{\mathrm{nis}}\colon\ \mathrm{Psh}(S, \mathcal{E}) \to \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$$

refines to a symmetric monoidal functor. Explicitly, the symmetric monoidal product on $\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$ is given by taking pointwise products followed by sheafification.

As explained above, the top horizontal in this square refines to symmetric monoidal left adjoints. And since the localisation $(-)^{\mathrm{nis}}$ is multiplicative the same holds for the left vertical functor. The same holds for the right vertical functor:

Proposition 7.3.1. *The accessible localisation*

$$\mathrm{Loc}_\mathrm{b}\colon\ \mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$$

is multiplicative with respect to Day convolution. In particular, $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_{\mathcal{D}\mathrm{p}(S)}^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ *inherits a symmetric monoidal structure such that* Loc_b *refines to a symmetric monoidal functor.*

Proof. Proposition 4.2.15 and [CDH⁺I, Proposition 5.3.4]. □

By the universal property of multiplicative localisations (see [CDH⁺IV]), we now obtain a commutative square of symmetric monoidal left adjoint functors

$$\begin{array}{ccc} \mathrm{Psh}(\mathrm{Sm}_{/S}, \mathcal{E})^{\otimes} & \longrightarrow & \mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})^{\otimes} \\ \big\downarrow{\scriptstyle (-)^{\mathrm{nis}}} & & \big\downarrow{\scriptstyle \mathrm{Loc}_\mathrm{b}} \\ \mathrm{Sh}^{\mathrm{nis}}(\mathrm{Sm}_{/S}, \mathcal{E})^{\otimes} & \xrightarrow{\mathcal{T}_S} & \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})^{\otimes} \end{array} \tag{37}$$

whose horizontal functors are left adjoints to the corresponding restriction functors. This yields the desired symmetric monoidal refinement of the square (35).

Definition 7.3.2. Given an object $X \to S$ of $\mathrm{Sm}_{/S}$, let us denote by $\mathcal{E}[X]$ the $\mathcal{E}$-valued sheaf obtained by sheafifying the presheaf

$$Y \mapsto u\,\mathrm{Map}_{\mathrm{Sm}_{/S}}(Y, X),$$

where $u\colon \mathcal{S} \to \mathcal{E}$ is the unit of $\mathcal{E}$. If $x\colon S \to X$ is an S-point of X and $\mathcal{E}$ is pointed then we denote by $\mathcal{E}[X, x]$ the cofibre of the map $\mathcal{E}[S] \xrightarrow{x} \mathcal{E}[X]$ in $\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})$.

We can consider the functor

$$\mathrm{j}_{\mathrm{bloc}}\colon \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \tag{38}$$

sending $(\mathcal{C}, \mathrm{D})$ to $\mathrm{Loc}_{\mathrm{b}}\Sigma^\infty \mathrm{Map}((\mathcal{C}, \mathrm{D}), -)_+$ as a co-Yoneda embedding into bounded Karoubi-localising functors to $\mathcal{E}$.

Proposition 7.3.3. *Suppose that $\mathcal{E}$ is additive, so in particular pointed. Then the symmetric monoidal functor $\mathcal{T}_S$ sends the object $\mathcal{E}[\mathbb{P}^1_S, \infty]$ to $\mathrm{j}_{\mathrm{bloc}}(\mathcal{D}^{\mathrm{p}}(S), \Sigma^{-1}\mathrm{D}_S)$. In particular, it sends $\mathcal{E}[\mathbb{P}^1_S, \infty]$ to an* invertible *object in $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$, for the monoidal structure described above.*

Proof. Since left Kan extension sends representable functors to representable functors, we see that $\mathcal{T}_S\mathcal{E}[\mathbb{P}^1_S, \infty]$ is given by the cofibre of

$$\mathrm{Loc}_{\mathrm{b}} u\,\mathrm{Map}((\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S), -) \xrightarrow{(-)\circ\infty^*} \mathrm{Loc}_{\mathrm{b}} u\,\mathrm{Map}((\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_S), \mathrm{D}_{\mathbb{P}^1_S}), -)$$

where $\infty^*\colon (\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_S), \mathrm{D}_{\mathbb{P}^1_S}) \to (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S)$ is the $\mathcal{E}$-linear duality preserving functor induced by $\infty\colon S \to \mathbb{P}^1_S$. This functor admits a retraction which is induced by the terminal map $q\colon \mathbb{P}^1_S \to S$ in $\mathrm{Sm}_{/S}$, and since $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ inherits from $\mathcal{E}$ the property of being additive, it follows from the splitting lemma (see, e.g, [CDH+II, Lemma 1.5.12]) that the cofibre of $(-)\circ\infty^*$ is canonically equivalent to the fibre of the map

$$\mathrm{Loc}_{\mathrm{b}} u\,\mathrm{Map}((\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_S), \mathrm{D}_{\mathbb{P}^1_S}), -) \xrightarrow{(-)\circ q^*} \mathrm{Loc}_{\mathrm{b}} u\,\mathrm{Map}((\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S), -)$$

induced by the duality preserving $\mathcal{E}$-linear functor $q^*\colon (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_S), \mathrm{D}_{\mathbb{P}^1_S})$. By the bounded Karoubi sequence

$$(\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S) \xrightarrow{q^*} (\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_S), \mathrm{D}_{\mathbb{P}^1_S}) \to (\mathcal{D}^{\mathrm{p}}(S), \Sigma^{-1}\mathrm{D}_S)$$

underlying the split Poincaré-Verdier sequence (29) (see Example 4.2.7), we now conclude that the fibre of $(-)\circ q^*$ in the ∞-category $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ is equivalent to $\mathrm{Loc}_{\mathrm{b}} u\,\mathrm{Map}((\mathcal{D}^{\mathrm{p}}(S), \Omega\mathrm{D}_S), -)$, which is therefore invertible with respect to the localised Day convolution structure since $(\mathcal{D}^{\mathrm{p}}(S), \Sigma^{-1}\mathrm{D}_S)$ is invertible in $\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ and $\mathrm{Loc}_{\mathrm{b}}$, u and the Yoneda functor are all symmetric monoidal. □

By Proposition 7.3.3 and the universal property of $\mathbb{P}^1$-stabilization, if $\mathcal{E}$ is an additive presentably symmetric monoidal ∞-category, the symmetric monoidal functor $\mathcal{T}_S$ appearing in the lower horizontal row of (37) factors essentially uniquely via a symmetric monoidal left adjoint functor

$$\widetilde{\mathcal{T}}_S\colon \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}))^{\otimes} \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})^{\otimes},$$

whose right adjoint

$$\mathcal{R}_S\colon \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}))$$

then inherits an induced lax symmetric monoidal structure.

Construction 7.3.4 (Motivic realization for bounded localising functors)**.** Given a bounded localising functor $\mathcal{F}\colon \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{E}$ and a flat S-linear ∞-category with duality $(\mathcal{C}, \mathrm{D}) \in \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$, we will denote by

$$\mathcal{R}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D})) \in \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}))$$

the image under the right adjoint $\mathcal{R}_S$ above of the bounded localising functor $\mathcal{F}((-) \otimes (\mathcal{C}, \mathrm{D}))$ obtained by pre-composing $\mathcal{F}$ with $(-) \otimes (\mathcal{C}, \mathrm{D})$ (and operation which preserves bounded localising functors by Proposition 4.2.15). We refer to $\mathcal{R}_S(\mathcal{F}, \mathcal{C})$ as the *pre-motivic realization* of $\mathcal{F}$ with coefficients in $(\mathcal{C}, \mathrm{D})$, and to

$$\mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}_S(\mathcal{F}; (\mathcal{C}, \mathcal{D})) \in \mathrm{SH}(S, \mathcal{E})$$

as the *motivic realization* of $\mathcal{F}$ with coefficients in $(\mathcal{C}, \mathrm{D})$.

The association $(\mathcal{F}; (\mathcal{C}, \mathrm{D})) \mapsto \mathcal{R}_S(\mathcal{F}, (\mathcal{C}, \mathrm{D}))$ then assembles to form a functor

$$\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \times \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})).$$

This functor carries a canonical lax monoidal functor: indeed, it is the composite of the lax symmetric monoidal functor $\mathcal{R}_S(-)$ and the functor $(\mathcal{F}, (\mathcal{C}, \mathrm{D})) \mapsto \mathcal{F}((-) \otimes (\mathcal{C}, \mathrm{D}))$, which is nothing but the Day tensoring of $\mathcal{F}$ by $(\mathcal{C}, \mathrm{D})$, and is hence lax symmetric monoidal in its entries as a pair. In particular, if $\mathcal{F}\colon \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{E}$ is a commutative algebra object with respect to Day convolution, that is, $\mathcal{F}$ is a lax symmetric monoidal functor, then $\mathcal{R}_S(\mathcal{F})$ inherits the structure of a commutative algebra object in $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E}))$, and the same holds for $\mathcal{R}_S(\mathcal{F}, (\mathcal{C}, \mathrm{D}))$ for every commutative algebra object in $\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$. Similarly, if $\mathcal{F}$ is lax symmetric monoidal and $(\mathcal{C}, \mathrm{D})$ is any object of $\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ then $\mathcal{R}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D}))$ inherits the structure of a module over $\mathcal{R}_S(\mathcal{F})$.

Example 7.3.5. For every $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$ the S-linear ∞-category with duality $(\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_L)$ is a module over $(\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_S)$, and hence for every lax symmetric monoidal bounded localising functor $\mathcal{F}$ the pre-motivic realization $\mathcal{R}_S(\mathcal{F}; (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_L))$ is a module over $\mathcal{R}_S(\mathcal{F})$. This example is particularly important, and we will use the shorthand notation

$$\mathcal{R}_L(\mathcal{F}) := \mathcal{R}_S(\mathcal{F}; (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_L))$$

to denote it. We note that for every S-smooth scheme $p\colon X \to S$ we have $(\mathcal{D}^{\mathrm{p}}(X), \mathrm{D}_X) \otimes (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_L) = (\mathcal{D}^{\mathrm{p}}(X), \mathrm{D}_{p^*L})$, and so the underlying Nisnevich sheaf $\Omega^\infty_{\mathbb{P}^1}\mathcal{R}_L(\mathcal{F})$ is given by $[p\colon X \to S] \mapsto \mathcal{F}(\mathcal{D}^{\mathrm{p}}(X), \mathfrak{Q}^{\mathrm{s}}_{p^*L})$.

Construction 7.3.6. The functor $\widetilde{\mathcal{T}}_S$ being symmetric monoidal, we have for every $\mathcal{E}$-valued pre-motivic spectrum E a commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) & \xleftarrow{\widetilde{\mathcal{T}}_S} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \\ \downarrow{\scriptstyle (-)\otimes\widetilde{\mathcal{T}}_S(E)} & & \downarrow{\scriptstyle (-)\otimes E} \\ \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) & \xleftarrow{\widetilde{\mathcal{T}}_S} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \,. \end{array}$$

Passing to right adjoints, we obtain a commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) & \xrightarrow{\mathcal{R}_S} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \\ \downarrow{\scriptstyle (-)^{\widetilde{\mathcal{T}}_S(E)}} & & \downarrow{\scriptstyle (-)^E} \\ \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) & \xrightarrow{\mathcal{R}_S} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \,, \end{array}$$

where the vertical arrows denote the operation of cotensoring with E and $\widetilde{\mathcal{T}}_S(E)$, respectively. In particular, for $\mathcal{F} \in \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ we obtain a natural equivalence

$$\mathcal{R}_S(\mathcal{F})^E \simeq \mathcal{R}_S(\mathcal{F}^{\widetilde{\mathcal{T}}_S(E)}).$$

In addition, the symmetry of the monoidal structure on $\mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E})$ means that cotensors commute with cotensors, and hence for every $(\mathcal{C}, \mathrm{D}) \in \mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ we obtain a similar natural equivalence

$$\mathcal{R}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D}))^E \simeq \mathcal{R}_S(\mathcal{F}^{\widetilde{\mathcal{T}}_S(E)}; (\mathcal{C}, \mathrm{D})).$$

Using our notation for Poincaré ∞-categories, given an S-linear ∞-category with duality $(\mathcal{C}, \mathrm{D}) = (\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}})$, we denote by $(\mathcal{C}, \mathrm{D})^{[n]} := (\mathcal{C}, \Sigma^n \mathrm{D})$ the S-linear ∞-category obtained by shifting the duality/Poincaré structure by n. Given a functor $\mathcal{F}\colon \mathrm{Mod}^{\flat,\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathcal{E}$ we denote by $\mathcal{F}^{[n]}(-) := \mathcal{F}((-)^{[n]})$ the functor obtained by pre-composing with the n-shift operator. We note that if $\mathcal{F}$ is bounded localising then $\mathcal{F}^{[n]}$ is again bounded localising.

Example 7.3.7 (Bott periodicity). Applying Construction 7.3.6 in the case of $E = \mathcal{E}[\mathbb{P}^1_S, \infty]$ and using Proposition 7.3.3 we obtain a natural equivalence

$$\Omega_{\mathbb{P}^1}\mathcal{R}_S(\mathcal{F}) = \mathcal{R}_S(\mathcal{F}((-) \otimes (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_{\mathcal{O}_S[-1]}) = \mathcal{R}_S(\mathcal{F}((-)^{[-1]}) = \mathcal{R}_S(\mathcal{F}^{[-1]}).$$

Since the operations $\Omega_{\mathbb{P}^1}(-)$ and $(-)^{[-1]}$ are invertible this equivalence determines an equivalence

$$\Sigma^n_{\mathbb{P}^1}\mathcal{R}_S(\mathcal{F}) = \mathcal{R}_S(\mathcal{F}^{[n]})$$

for every integer $n \in \mathbb{Z}$. Similarly, for $(\mathcal{C}, \mathrm{D}) \in \mathrm{Mod}^{\flat,\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})$ we have a natural equivalence

$$\Sigma^n_{\mathbb{P}^1}\mathcal{R}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D})) \simeq \mathcal{R}_S(\mathcal{F}^{[n]}; (\mathcal{C}, \mathrm{D})).$$

In particular, in the case of $(\mathcal{C}, \mathrm{D}) = (\mathcal{D}^{\mathrm{p}}(S), \mathrm{D}_L)$ for $L \in \mathcal{P}ic^{\mathrm{BC}_2}(S)$ this equivalence can be rewritten using the notation of Example 7.3.5 as

$$\Sigma^n_{\mathbb{P}^1}\mathcal{R}_L(\mathcal{F}) = \mathcal{R}_L(\mathcal{F}^{[n]}) = \mathcal{R}_{L[n]}(\mathcal{F}).$$

Corollary 7.3.8. *For $p\colon X \to S$ in Sm_S, there is a natural equivalence*

$$\mathcal{R}_S(\mathcal{F}; (\mathcal{D}^{\mathrm{p}}(X), \mathrm{D}_X)) = p_* p^* \mathcal{R}_S(\mathcal{F}).$$

Proof. Apply Construction 7.3.6 to $E = \Sigma^\infty_{\mathbb{P}^1}\mathcal{E}[X]$. □

Corollary 7.3.9. *The functor $\mathcal{R}_S$ preserves all limits. If in addition $\mathcal{E}$ is stable, then $\mathcal{R}_S$ preserves all colimits.*

Proof. The first claim is because $\mathcal{R}_S$ is a right adjoint. To see the second claim, note that if $\mathcal{E}$ is stable, then the full subcategory of Nisnevich sheaves is closed under colimits inside all presheaves and the functor $\Omega_{\mathbb{P}^1}$ commutes with colimits. It then follows that colimits in $\mathbb{P}^1$-spectrum objects in Nisnevich sheaves are computed on the level of $\mathbb{P}^1$-prespectrum objects in presheaves. At the same time, when $\mathcal{E}$ is stable the full subcategory of bounded localising functors is closed under colimits inside all functors, and so colimits of bounded localising functors are computed levelwise. The desired result now follows from the fact that on the level of functors and $\mathbb{P}^1$-prespectra in presheaves, the functor $\mathcal{R}_S$ is given levelwise by suitable evaluations, see Example 7.3.7. □

Finally, we now obtain motivic realization functors for Karoubi-localising invariants of Poincaré ∞-categories:

Construction 7.3.10 (Motivic realization for Karoubi-localising functors)**.** By Proposition 4.2.8, the restriction along the lax symmetric monoidal functor

$$\mathrm{Mod}^{\flat,\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}) \to \mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}} \to \mathrm{Cat}^{\mathrm{p}} \qquad (\mathcal{C}, \mathrm{D}) \mapsto (\mathcal{C}, \Omega^{\mathrm{s}}_{\mathrm{D}}) \tag{39}$$

sends Karoubi-localising functors to bounded localising functors. By means of pre-composition we hence we obtain a lax symmetric monoidal functor

$$\mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat}^{\mathrm{p}}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}^{\flat,\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}), \mathcal{E}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \xrightarrow{\mathcal{R}_S} \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{E})) \tag{40}$$

which we denote by $\mathcal{R}^{\mathrm{s}}_S$. To avoid confusion, we note that since (39) is only lax symmetric monoidal, the above right adjoint is not part of a symmetric monoidal adjunction; more precisely, its left adjoint is only oplax symmetric monoidal, but not symmetric monoidal. More generally, for a Karoubi-localising functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$ and an S-linear stable ∞-category with duality $(\mathcal{C}, \mathrm{D})$, we write

$$\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}; (\mathcal{C}, \mathrm{D})) = \mathcal{R}_S(\mathcal{F}|_{\mathrm{Mod}^{\flat,\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})}; (\mathcal{C}, \mathrm{D}))$$

for the corresponding motivic realization with coefficients. As in Example 7.3.5, for an invertible perfect complex with C_2-action $L \in \mathcal{P}ic(S)^{BC_2}$, we write

$$\mathcal{R}^{s}_{L}(\mathcal{F}) = \mathcal{R}^{s}_{S}(\mathcal{F}; (\mathcal{D}^{p}(S), D_L)).$$

so that $\mathcal{R}^{s}_{L}$ is a module over $\mathcal{R}^{s}_{S}$ with respect to Day convolution. In particular, $\mathcal{R}_L(\mathcal{F})$ is a module over $\mathcal{R}_S(\mathcal{F})$ for any Karoubi-localising functor $\mathcal{F}$. When $L = \mathcal{O}_S$ we have $\mathcal{R}^{s}_{L} = \mathcal{R}^{s}_{S}$.

7.4. **The free delooping lemma.** Let us now consider the case where the target $\mathcal{E}$ is either the ∞-category $\mathcal{S}p$ of spectra or the ∞-category Grp_{E_∞} of E_∞-groups. These two ∞-categories are related via an adjunction

$$\iota\colon \mathrm{Grp}_{E_\infty} \rightleftarrows \mathcal{S}p : \Omega^\infty,$$

where Ω^∞ is the lift of the usual loop infinity functor from spaces to E_∞-groups and its left adjoint ι is the fully-faithful embedding of Grp_{E_∞} as the full subcategory of connective spectra. In fact, this adjunction also exhibits $\mathcal{S}p$ as the stabilization of Grp_{E_∞}, and so we could have called ι also Σ^∞, but we will stick with ι in order to avoid confusion with Σ^∞ of spaces (which is not fully-faithful). We then have an induced adjunction

$$\iota_*\colon \mathcal{S}p^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathrm{Grp}_{E_\infty})) \rightleftarrows \mathcal{S}p^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}p)) : \Omega^\infty_*,$$

where Ω^∞_* is given by object-wise post-composing with Ω^∞ and ι_* is given by first object-wise post-composing with ι and then applying the reflection functor

$$\mathcal{P}\mathcal{S}p^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}p)) \to \mathcal{S}p^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}p)).$$

Lemma 7.4.1 (Free delooping)**.** *Let $\mathcal{F}\colon \mathrm{Mod}^{\flat;\kappa}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,t}) \to \mathcal{S}p$ be a bounded localising functor. Then the counit map*

$$\iota_*\Omega^\infty_*\mathcal{R}_S(\mathcal{F}) \to \mathcal{R}_S(\mathcal{F})$$

is an equivalence.

Remark 7.4.2. Since $\Omega^\infty_*\mathcal{R}_S(\mathcal{F}) \simeq \mathcal{R}_S(\Omega^\infty\mathcal{F})$, Lemma 7.4.1 says in particular that $\mathcal{R}_S(\mathcal{F})$ only depends on the underlying Grp_{E_∞}-valued functor $\Omega^\infty\mathcal{F}$. Similarly, if $\mathcal{F}$ is a Karoubi-localising functor on $\mathrm{Cat}^{\mathrm{p}}$ then $\mathcal{R}^{s}_{S}(\mathcal{F})$ only depends on $\Omega^\infty\mathcal{F}$. This is a-priori not surprising, given that Karoubi-localising functors are uniquely determined by their connective covers, see [CDH+V]. Lemma 7.4.1 is however a bit more precise, giving the exact manner in which the eventual dependence on $\Omega^\infty\mathcal{F}$ is expressed.

Proof. Since the composite $\iota\circ\Omega^\infty$ is naturally equivalent to the connective cover functor this amounts to checking that the map of pre-spectrum objects in presheaves

$$\tau_{\geq 0}\mathcal{R}_S(\mathcal{F}) \to \mathcal{R}_S(\mathcal{F})$$

is a stable pre-motivic equivalence, that is, induces an equivalences on mapping spaces to any $\mathbb{P}^1$-spectrum object in sheaves. Here, the connective cover functor $\tau_{\geq 0}$ is applied to the spectrum object $\mathcal{R}_S(\mathcal{F})$ object-wise.

In what follows, let us consider $\mathbb{A}^1, \mathbb{G}_m$ and $\mathbb{P}^1 \smallsetminus \{0\}$ as pointed schemes over $\mathrm{spec}(\mathbb{Z})$ with a compatible base point given by $1 \in \mathbb{G}_m$ (and its images in $\mathbb{A}^1$ and $\mathbb{P}^1 \smallsetminus \{0\}$). We can then view them as representable sheaves on $\mathrm{Sm}_{\mathbb{Z}}$ taking values in pointed spaces, and we write

$$B = \mathbb{A}^1 \amalg_{\mathbb{G}_m} (\mathbb{P}^1 \smallsetminus \{0\}) \in \mathrm{Psh}(\mathrm{spec}(\mathbb{Z}), \mathcal{S}_*)$$

for the associated pushout calculated in the ∞-category of presheaves of pointed spaces. Since Grp_{E_∞} is tensored and cotensored over pointed spaces we can form the associated loop functor

$$\Omega_B\colon \mathrm{Psh}(S, \mathrm{Grp}_{E_\infty}) \to \mathrm{Sh}^{\mathrm{nis}}(S, \mathrm{Grp}_{E_\infty}),$$

and the obvious map $B \to \mathbb{P}^1$ induces a natural transformation

$$\Omega_{\mathbb{P}^1} \Rightarrow \Omega_B.$$

We now claim that for every pre-spectrum object $\mathcal{X}_\bullet \in \mathcal{P}\mathcal{S}p^{\mathbb{P}^1}(\mathrm{Psh}(S, \mathcal{S}p))$ the maps

$$\mathcal{X}_\bullet \to \Omega_{\mathbb{P}^1}\mathcal{X}_{\bullet+1} \to \Omega_B\mathcal{X}_{\bullet+1}$$

are both stable pre-motivic equivalences. To see this, we need to show that for every $\mathcal{Y} \in \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p}))$ the induced map

$$\lim_{\substack{\longleftarrow \\ n}} \mathrm{Map}(\Omega_B \mathcal{X}_{n+1}, \mathcal{Y}_n) \to \lim_{\substack{\longleftarrow \\ n}} \mathrm{Map}(\Omega_{\mathbb{P}^1} \mathcal{X}_{n+1}, \mathcal{Y}_n) \to \lim_{\substack{\longleftarrow \\ n}} \mathrm{Map}(\mathcal{X}_n, \mathcal{Y}_n)$$

are equivalences. Indeed, since the map $B \to \mathbb{P}^1$ is a Nisnevich local equivalence the maps $\mathcal{Y}_n \to \Omega_{\mathbb{P}^1}\mathcal{Y}_{n+1} \to \Omega_B \mathcal{Y}_{n+1}$ are both equivalences and so the map $\mathrm{Map}(\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}) \to \mathrm{Map}(\mathcal{X}_n, \mathcal{Y}_n)$ factors as a composite

$$\mathrm{Map}(\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}) \to \mathrm{Map}(\Omega_B \mathcal{X}_{n+1}, \mathcal{Y}_n) \to \mathrm{Map}(\Omega_{\mathbb{P}^1} \mathcal{X}_{n+1}, \mathcal{Y}_n) \to \mathrm{Map}(\Omega_B \mathcal{X}_{n+1}, \mathcal{Y}_n).$$

The three tower can hence be weaved into one big tower and the desired result follows from cofinality.

Having shown that $\mathcal{X}_\bullet \to \Omega_B \mathcal{X}_{\bullet+1}$ is a stable pre-motivic equivalence we may also conclude that the map

$$\mathcal{X}_\bullet \to L_B \mathcal{X}_\bullet = \operatorname*{colim}_n \Omega_B^n \mathcal{X}_{\bullet+n}$$

is again a stable pre-motivic equivalence. In particular, for $\mathcal{X} \in \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p})$ the left vertical map in the square

$$\begin{array}{ccc} \tau_{\geq 0}\mathcal{X}_\bullet & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \simeq \\ L_B(\tau_{\geq 0}\mathcal{X}_\bullet) & \xrightarrow{\eta} & L_B(X_\bullet) \end{array}$$

is a stable pre-motivic equivalence. To finish the proof we now show that in the case of $\mathcal{X}_\bullet = \mathcal{R}_S(\mathcal{F})$, the map

$$\eta\colon L_B \tau_{\geq 0} \mathcal{R}_S(\mathcal{F}) \to \mathcal{R}_S(\mathcal{F})$$

is a stable pre-motivic equivalence. More precisely, we claim that for every m, n and $T \in \mathrm{Sm}_{/S}$ the map

$$\eta\colon \Omega_B^n \tau_{\geq 0} \mathcal{R}_S(\mathcal{F})_m(T) \to \Omega_B^n R(\mathcal{F})_m(T)$$

is an equivalence on homotopy groups in degrees $\geq -n$. We note that the claim holds by construction for $n = 0$. Arguing by induction, it remains to show that if $\mathcal{Y}' \to \mathcal{R}_S(\mathcal{F})_m$ is a map from a presheaf $\mathcal{Y}'$ which exhibits $\mathcal{Y}'(T)$ as the $(-n)$-connective cover of $\mathcal{R}_S(\mathcal{F})_m(T)$ for every $T \in \mathrm{Sm}_{/S}$, then the induced map $\Omega_B \mathcal{Y}' \to \Omega_B R(\mathcal{F})_m$ exhibits $\Omega_B \mathcal{Y}'(T)$ as the $(-n-1)$-connective cover of $\Omega_B R(\mathcal{F})_m(T)$ for every $T \in \mathrm{Sm}_{/S}$. Indeed, for $T \in \mathrm{Sm}_{/S}$, consider the commutative diagram of spectra

$$\begin{array}{ccccc} \Omega_B \mathcal{Y}'(T) & \longrightarrow & \Omega_{\mathbb{A}^1} \mathcal{Y}'(T) \oplus \Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{Y}'(T) & \longrightarrow & \Omega_{\mathbb{G}_m} \mathcal{Y}'(T) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_B R(\mathcal{F})_m(T) & \longrightarrow & \Omega_{\mathbb{A}^1} \mathcal{R}_S(\mathcal{F})_m(T) \oplus \Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{R}_S(\mathcal{F})_m(T) & \longrightarrow & \Omega_{\mathbb{G}_m} \mathcal{R}_S(\mathcal{F})_m(T) \end{array}$$

whose rows are exact by construction. For every pointed scheme $U \in \mathrm{Sm}_{/\mathrm{spec}(\mathbb{Z})}$ and presheaf $\mathcal{Z}$, the spectrum $\Omega_U \mathcal{Z}(T)$ is a summand of $\mathcal{Z}(T \times U)$, and hence the middle and right most vertical maps in the above diagrams are isomorphisms on homotopy groups in degrees $\geq -n$ by the induction hypothesis. By the long exact sequence in homotopy groups and the five lemma, it will suffice to show that the square

$$\begin{array}{ccc} \pi_{-n-1}\Omega_{\mathbb{A}^1} \mathcal{Y}'(T) \oplus \pi_{-n-1}\Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{Y}'(T) & \longrightarrow & \pi_{-n-1}\Omega_{\mathbb{G}_m} \mathcal{Y}'(T) \\ \downarrow & & \downarrow \\ \pi_{-n-1}\Omega_{\mathbb{A}^1} \mathcal{R}_S(\mathcal{F})_m(T) \oplus \pi_{-n-1}\Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{R}_S(\mathcal{F})_m(T) & \longrightarrow & \pi_{-n-1}\Omega_{\mathbb{G}_m} \mathcal{R}_S(\mathcal{F})_m(T) \end{array}$$

induces an isomorphism on horizontal kernels. In the upper row, both terms are zero by the induction hypothesis (and the fact that $\Omega_U \mathcal{Z}(T)$ is a summand of $\mathcal{Z}(T \times U)$ as above), so this amounts to showing that the bottom horizontal arrow is injective. Since $\mathcal{R}_S(\mathcal{F})_m$ is a Nisnevich sheaf, this is equivalent by the long exact sequence in homotopy groups to the claim that the map

$$\pi_{-n-1}\Omega_{\mathbb{P}^1} \mathcal{R}_S(\mathcal{F})_m \to \pi_{-n-1}\Omega_{\mathbb{A}^1} \mathcal{R}_S(\mathcal{F})_m(T) \oplus \pi_{-n-1}\Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{R}_S(\mathcal{F})_m(T)$$

is the zero map. In fact, we claim that the map of spectra

$$\Omega_{\mathbb{P}^1} \mathcal{R}_S(\mathcal{F})_m(T) \to \Omega_{\mathbb{A}^1} \mathcal{R}_S(\mathcal{F})_m(T) \oplus \Omega_{\mathbb{P}^1 \setminus \{0\}} \mathcal{R}_S(\mathcal{F})_m(T)$$

is a null-homotopic. For this, note that the inclusions $i_0 \colon T \times \{0\} \hookrightarrow T \times \mathbb{P}^1$ and $i_\infty \colon T \times \{\infty\} \hookrightarrow T \times \mathbb{P}^1$ are regular codimension 1 embeddings (even if T and $T \times \mathbb{P}^1$ are not necessarily regular), and hence by Lemma 5.1.5 we have associated duality preserving push-forward functors

$$(i_0)_* \colon (\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \to (\mathcal{D}^{\mathrm{p}}(T\times\mathbb{P}^1), \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \quad \text{and} \quad (i_\infty)_* \colon (\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \to (\mathcal{D}^{\mathrm{p}}(T\times\mathbb{P}^1), \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}).$$

We now note that composing $(i_\infty)_*$ with pullback along $\mathbb{A}^1 \subseteq \mathbb{P}^1$ yields the zero functor, and similarly for composing $(i_0)_*$ with pullback along $\mathbb{P}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{P}^1$. In light of Example 7.3.7 to finish the proof we now show that the sequences

$$\mathcal{F}(\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \xrightarrow{(i_0)_*} \mathcal{F}(T \times \mathbb{P}^1, \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \xrightarrow{i_1^*} \mathcal{F}(T, \Sigma^m \mathrm{D}_T)$$

and

$$\mathcal{F}(\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \xrightarrow{(i_\infty)_*} \mathcal{F}(T \times \mathbb{P}^1, \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \xrightarrow{i_1^*} \mathcal{F}(T, \Sigma^m \mathrm{D}_T)$$

are fibre sequences, where i_1^* is pullback along the base point inclusion $T \times \{1\} \hookrightarrow T \times \mathbb{P}^1$ we have fixed above (and through which $\Omega_{\mathbb{P}^1}$ is defined). For this final part we use the fact that $\mathcal{F}$ is in particular additive by Example (4.2.7), and hence the split Poincaré-Verdier sequence (29) yields an exact sequence

$$\mathcal{F}(\mathcal{D}^{\mathrm{p}}(T), \Sigma^m \mathrm{D}_T) \xrightarrow{p^*} \mathcal{F}(\mathcal{D}^{\mathrm{p}}(T \times \mathbb{P}^1), \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \xrightarrow{p_*(-\otimes\mathcal{O}(-1))} \mathcal{F}(\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T),$$

where $p \colon T \times \mathbb{P}^1 \to T$ is the projection. Since $i_1 \colon T \to T \times \mathbb{P}^1$ is a section of p it will now suffice to show that the composites

$$(\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \xrightarrow{(i_0)_*} (T \times \mathbb{P}^1, \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \xrightarrow{p_*(-\otimes\mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T)$$

and

$$(\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T) \xrightarrow{(i_\infty)_*} (T \times \mathbb{P}^1, \Sigma^m \mathrm{D}_{T\times\mathbb{P}^1}) \xrightarrow{p_*(-\otimes\mathcal{O}(-1))} (\mathcal{D}^{\mathrm{p}}(T), \Sigma^{m-1}\mathrm{D}_T)$$

are both equivalences of stable ∞-categories with duality. In fact, since forgetting the duality is a conservative functor we may as well just verify these equivalences on the level of underlying stable ∞-categories. Indeed, by the projection formula, we have

$$p_*((i_0)_*(-) \otimes \mathcal{O}(-1)) \simeq p_*((i_0)_*(- \otimes (i_0)^*\mathcal{O}(-1)) \simeq (-) \otimes (i_0)^*\mathcal{O}(-1) \simeq \mathrm{id}$$

and

$$p_*((i_\infty)_*(-) \otimes \mathcal{O}(-1)) \simeq p_*((i_\infty)_*(- \otimes (i_\infty)^*\mathcal{O}(-1)) \simeq (-) \otimes (i_\infty)^*\mathcal{O}(-1) \simeq \mathrm{id}. \qquad \square$$

8. Hermitian K-theory as a motivic spectrum

In this section we build on the results proven so far to generalize the construction of the hermitian K-theory motivic spectrum KQ_S over a qcqs scheme S, without assuming that 2 is invertible in S (as is done in previous constructions, such as [Hor05] or [HJNY21]). Given a perfect invertible complex with C_2-action $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$ we also construct a motivic spectrum KQ_L of hermitian K-theory with coefficients in L, as well as corresponding versions KW_S and KW_L of motivic Witt spectra. We spend the majority of the section in order to establish the key fundamental properties of these motivic spectra. In particular, we prove:

(1) KQ_S and KW_S are motivic E_∞-ring spectra, and the natural map $\mathrm{KQ}_S \to \mathrm{KW}_S$ is an E_∞-map whose kernel is $\mathrm{KGL}_{\mathrm{hC}_2}$, where KGL is the motivic K-theory spectrum. The motivic spectra KQ_L and KW_L are modules over KQ_S and KW_S.
(2) The forgetful map $\mathrm{KQ}_S \to \mathrm{KGL}_S$ is also one of E_∞-ring spectra. It fits in the Wood fibre sequence
$$\Omega_{\mathbb{P}^1} \mathrm{KQ}_S \to \mathrm{KGL}_S \to \mathrm{KQ}_S .$$
(3) The hermitian K-theory spectrum is absolute: for any map $T \to S$ of qcqs schemes the natural map $f^* \mathrm{KQ}_S \to \mathrm{KQ}_T$ is an equivalence, see Proposition 8.2.1.
(4) The Thom isomorphism: $\Omega^V \mathrm{KQ}_L \simeq \mathrm{KQ}_{L\otimes \det V[-r]}$ for any rank r vector bundle $V \to S$, see Proposition 8.3.1.
(5) If S is Noetherian of finite Krull dimension then the motivic ring spectrum KQ_S is *absolutely pure*, see Theorem 8.4.2.

8.1. **Motivic hermitian K-theory spectra.** Recall the lax symmetric monoidal functor

$$\mathcal{R}^{\mathrm{s}}_S\colon \mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat}^{\mathrm{p}}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p}))$$

of (40), its line bundle generalisation $\mathcal{R}^{\mathrm{s}}_L$ from Construction 7.3.10, and the fact that $\mathbb{K}$, $\mathbb{GW}$ and $\mathbb{L}$ are E_∞-algebra objects in $\mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat}^{\mathrm{p}}, \mathcal{S}\mathrm{p})$ with respect to Day convolution, that is, each carries a canonical lax symmetric monoidal structure, see [CDH+IV], and both the universal natural transformations $\mathbb{GW} \to \mathbb{K}$ and $\mathbb{GW} \to \mathbb{L}$ are lax symmetric monoidal.

Definition 8.1.1. We define the motivic E_∞-ring spectra KGL_S, KQ_S and $\mathrm{KW}_S, \in \mathrm{SH}(S)$ by the formulas

$$\mathrm{KGL}_S = \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_S(\mathbb{K}), \qquad \mathrm{KQ}_S = \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW}) \quad \text{and} \quad \mathrm{KW}_S = \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_S(\mathbb{L}).$$

Given an invertible perfect complex with C_2-action $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$, we also define the motivic spectra $\mathrm{KQ}_L, \mathrm{KW}_L \in \mathrm{SH}(S)$ by the formulas

$$\mathrm{KQ}_L = \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_L(\mathbb{GW}) \quad \text{and} \quad \mathrm{KW}_L = \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_L(\mathbb{L}).$$

By construction of $\mathcal{R}^{\mathrm{s}}_L$, these are naturally modules over the E_∞-ring spectra KQ_S and KW_S, respectively.

Remark 8.1.2. When $L = \mathcal{O}_S$, we have $\mathrm{KQ}_L = \mathrm{KQ}_S$ and $\mathrm{KW}_L = \mathrm{KW}_S$.

Since $\mathbb{K}$ only depends on the underlying stable ∞-category, it is invariant under shifting the Poincaré structure in the input. By Example 7.3.7, this translates to a distinguished equivalence

$$\beta\colon \mathrm{KGL}_S \xrightarrow{\simeq} \Omega_{\mathbb{P}^1}\mathrm{KGL}_S,$$

which we call the Bott map. Similarly, by Ranicki periodicity the motivic spectra KQ_S and KW_S are equipped with equivalences of the form

$$\widetilde{\beta}\colon \mathrm{KQ}_S \to \Omega^4_{\mathbb{P}^1}\mathrm{KQ}_S \quad \text{and} \quad \overline{\beta}\colon \mathrm{KW}_S \to \Omega^4_{\mathbb{P}^1}\mathrm{KW}_S\,.$$

Remark 8.1.3. Let $\mathcal{GW}^\natural\colon \mathrm{Cat}^{\mathrm{p}} \to \mathrm{Grp}_{\mathrm{E}_\infty}$ be the functor given by

$$\mathcal{GW}^\natural(\mathcal{C}, \mathfrak{Q}) = \mathcal{GW}((\mathcal{C}, \mathfrak{Q})^\natural) = \Omega^\infty\mathbb{GW}(\mathcal{C}, \mathfrak{Q}),$$

see [CDH+V]. Then $\mathcal{GW}^\natural$ is a Karoubi-localising functor, and by Lemma 7.4.1, the canonical map

$$\iota_*\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural) \to \mathcal{R}^{\mathrm{s}}_S(\mathbb{GW})$$

is an equivalence in $\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p}))$. In particular, we may equally well have defined KQ_S by the formula $\mathrm{KQ}_S = \mathrm{Loc}_{\mathbb{A}^1}\iota_*\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural)$, that is, as the image of $\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural)$ under the reflection functor

$$\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathrm{Grp}_{\mathrm{E}_\infty})) \to \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}_{\mathbb{A}^1}(S, \mathcal{S}\mathrm{p})).$$

A similar statement holds for KGL and KW.

Given a scheme X and an invertible object $L \in \mathcal{D}^{\mathrm{p}}(X)$, let

$$\mathbb{HGW}^{\mathrm{s}}(X, L) = |\mathbb{GW}^{\mathrm{s}}(X \times \Delta^\bullet, p_\bullet^* L)|, \quad \mathbb{HL}^{\mathrm{s}}(X, L) = |\mathbb{HL}^{\mathrm{s}}(X \times \Delta^\bullet, p_\bullet^* L)|,$$

and

$$\mathbb{HK}(X) = |\mathbb{K}(X \times \Delta^\bullet)|,$$

where Δ^n is the algebraic n-simplex and $p_n\colon X \times \Delta^n \to X$ is the projection. When L is the structure sheaf, we also abbreviate $\mathbb{HGW}^{\mathrm{s}}(X, \mathcal{O})$ as $\mathbb{HGW}^{\mathrm{s}}(X)$ and $\mathbb{HL}^{\mathrm{s}}(X, \mathcal{O})$ as $\mathbb{HL}^{\mathrm{s}}(X)$. In particular, $\mathbb{HK}$ is homotopy K-theory, and similarly $\mathbb{HGW}^{\mathrm{s}} = \mathrm{Loc}_{\mathbb{A}^1}\mathbb{GW}^{\mathrm{s}}$ and $\mathbb{HL}^{\mathrm{s}} = \mathrm{Loc}_{\mathbb{A}^1}\mathbb{L}^{\mathrm{s}}$ are the $\mathbb{A}^1$-invariant replacements of the Nisnevich sheaves $\mathbb{GW}^{\mathrm{s}}$ and $\mathbb{L}^{\mathrm{s}}$, respectively. We refer to them as *homotopy* $\mathbb{GW}$*-theory* and *homotopy* $\mathbb{L}$*-theory*, respectively.

Theorem 8.1.4. *Let X be a smooth S-scheme and $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$. Then there are natural equivalences*

$$\begin{aligned}
\hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1}X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KGL}_S) &\simeq \mathbb{HK}(X),\\
\hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1}X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KQ}_L) &\simeq \mathbb{HGW}^{\mathrm{s}}(X, L[n]|_X), \quad \textit{and}\\
\hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1}X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KW}_L) &\simeq \mathbb{HL}^{\mathrm{s}}(X, L[n]|_X).
\end{aligned}$$

In particular, KGL_S *coincides (as a* $\mathbb{P}^1$*-spectrum object in Nisnevich sheaves, and hence as a motivic spectrum) with the usual algebraic* K*-theory motivic spectrum in a manner that identifies* β *with the usual motivic Bott map.*

Combining Theorem 8.1.4 with Theorem 6.3.1, we furthermore obtain:

Corollary 8.1.5. *In the situation of Theorem 8.1.4, if* X *is regular Noetherian of finite Krull dimension then the equivalences of that corollary become*

$$\begin{aligned} \hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KGL}_S) &\simeq \mathrm{K}(X), \\ \hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KQ}_L) &\simeq \mathrm{GW}^{\mathrm{s}}(X, L[n]|_X), \quad \textit{and} \\ \hom_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^n_{\mathbb{P}^1}\mathrm{KW}_L) &\simeq \mathrm{L}^{\mathrm{s}}(X, L[n]|_X). \end{aligned}$$

Moving on, let us exploit a bit more the properties of the motivic realization functor $\mathcal{R}^{\mathrm{s}}_S$. In particular, applying it to the lax symmetric monoidal transformation $\mathbb{GW} \to \mathbb{L}$ we obtain a map of E_∞-motivic spectra

$$w\colon\ \mathrm{KQ}_S \to \mathrm{KW}_S\,.$$

Similarly, we write

$$\mathrm{KGL}_S \xrightarrow{h} \mathrm{KQ}_S \xrightarrow{f} \mathrm{KGL}_S$$

for the maps of motivic spectra obtained by applying $\mathcal{R}^{\mathrm{s}}_S$ to the natural transformations

$$\mathbb{K} \overset{\mathrm{hyp}}{\to} \mathbb{GW} \overset{\mathrm{fgt}}{\to} \mathbb{K}.$$

In particular, since fgt is a lax symmetric monoidal transformation we have that f is a map of motivic E_∞-ring spectra, and similarly h is a map of $\mathbb{GW}$-modules. In addition, since the functor $\mathbb{K}$ carries a C_2-action in $\mathrm{Fun}^{\mathrm{kloc}}(\mathrm{Cat}^{\mathrm{p}}, \mathcal{S}\mathrm{p})$ such that the natural transformations hyp and fgt are C_2-equivariant (with respect to the trivial action on $\mathbb{GW}$), we have that the same structure is inherited at the level of motivic spectra, that is, KGL_S carries a C_2-structure such that f and h are C_2-equivariant. In particular, h and f induce maps

$$(\mathrm{KGL}_S)_{\mathrm{hC}_2} \xrightarrow{h_{\mathrm{hC}_2}} \mathrm{KQ}_S \xrightarrow{f^{\mathrm{hC}_2}} (\mathrm{KGL}_S)_{\mathrm{hC}_2}\,.$$

Corollary 8.1.6 (Tate sequence)**.** *The maps* w *and* h_{hC_2} *fit into a fibre sequence*

$$(\mathrm{KGL}_S)_{\mathrm{hC}_2} \xrightarrow{h_{\mathrm{hC}_2}} \mathrm{KQ}_S \xrightarrow{w} \mathrm{KW}_S\,.$$

Proof. Apply $\mathcal{R}^{\mathrm{s}}_S$ to the Karoubi-localising fundamental fibre sequence $\mathbb{K}_{\mathrm{hC}_2} \Rightarrow \mathbb{GW} \Rightarrow \mathbb{L}$ (see (2)) and use the fact that $\mathcal{R}^{\mathrm{s}}_S$ preserves colimits when the target is stable by Lemma 7.3.9. □

Corollary 8.1.7 (Wood sequence)**.** *The maps* $f\colon\ \mathrm{KQ}_S \to \mathrm{KGL}_S$ *and* $\mathrm{KGL}_S \simeq \Sigma^\infty_{\mathbb{P}^1}\mathrm{KGL}_S \xrightarrow{\Sigma^\infty_{\mathbb{P}^1} h} \Sigma^\infty_{\mathbb{P}^1}\mathrm{KQ}_S$ *fit into a fibre sequence*

$$\mathrm{KQ}_S \to \mathrm{KGL}_S \to \Sigma^\infty_{\mathbb{P}^1}\mathrm{KQ}_S\,.$$

Proof. Apply $\mathcal{R}^{\mathrm{s}}_S$ to the Karoubi-localising Bott-Genauer sequence $\mathbb{GW} \Rightarrow \mathbb{K} \Rightarrow \mathbb{GW}((-)^{[1]})$ (see (1)) and use Bott periodicity (Example 7.3.7) to identify the fibre. □

Remark 8.1.8. It follows from the projective bundle formula of Theorem 6.1.6 and the Bott periodicity of Example 7.3.7 that the motivic spectra $\Omega_{\mathbb{P}^2}\mathrm{KQ}_S = \mathrm{fib}[\mathrm{KQ}_S^{\mathbb{P}^2} \to \mathrm{KQ}_S]$ and KGL_S coincide as $\mathbb{P}^1$-spectra in Nisnevich sheaves, and hence also as motivic spectra. Unwinding the definitions in Theorem 6.1.6, we see that under this equivalence, the map

$$\Omega_{\mathbb{P}^2}\mathrm{KQ}_S \to \Omega_{\mathbb{P}^1}\mathrm{KQ}_S$$

induced by a linear hyperplane inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ coincides with the composite

$$\mathrm{KGL}_S \underset{\simeq}{\overset{\beta}{\to}} \Omega_{\mathbb{P}^1}\mathrm{KGL}_S \xrightarrow{\Omega_{\mathbb{P}^1} h} \Omega_{\mathbb{P}^1}\mathrm{KQ}_S\,.$$

It is known that the cofibre of $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \hookrightarrow \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^2$ in $\mathrm{SH}(\mathbb{Z})$ is equivalent to $\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1 \wedge \mathbb{P}^1)$, and the boundary map $\Omega\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1 \wedge \mathbb{P}^1) \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1$ of this cofibre sequence determines, after $\mathbb{P}^1$ delooping, a map

$$\eta\colon\ \Sigma^\infty_{\mathbb{P}^1}\mathbb{G}_m \to \mathbb{S},$$

often denoted by η. We hence conclude that the boundary map

$$\mathrm{KQ}_S \to \Sigma\Omega_{\mathbb{P}^1}\mathrm{KQ}_S = \Omega_{\mathbb{G}_m}\mathrm{KQ}_S$$

of the Wood fibre sequence is induced by η.

8.2. **Pullback invariance of hermitian K-theory.** Given a map $f : T \to S$ of qcqs schemes, the symmetric monoidal functor of (13) determines a commutative square of symmetric monoidal functors

$$\begin{array}{ccc} (\mathrm{Sch}_S^{\mathrm{op}})^{\otimes} & \xrightarrow{T\times_S(-)} & (\mathrm{Sch}_T^{\mathrm{op}})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} & \xrightarrow{(\mathcal{D}^{\mathrm{p}}(T),\mathrm{D}_T)\otimes_{(\mathcal{D}^{\mathrm{p}}(S),\mathrm{D}_S)}(-)} & \mathrm{Mod}_T(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}})^{\otimes} \end{array} \tag{41}$$

where the vertical functors are given by the composites in (13) for S and T, respectively. Running through the constructions of §7.2 and §7.3, we obtain a commutative square of symmetric monoidal functors

$$\begin{array}{ccc} \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S,\mathcal{S}\mathrm{p}))^{\otimes} & \xrightarrow{f^*} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(T,\mathcal{S}\mathrm{p}))^{\otimes} \\ {\scriptstyle\widetilde{\mathcal{F}}_S}\downarrow & & \downarrow{\scriptstyle\widetilde{\mathcal{F}}_T} \\ \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{S}\mathrm{p})^{\otimes} & \longrightarrow & \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_T(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{S}\mathrm{p})^{\otimes} \end{array}$$

where the bottom horizontal map is given by left Kan extension along the bottom horizontal arrow of (41) followed by the localisation functor

$$\mathrm{Loc_b}\colon\ \mathrm{Fun}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{E}) \to \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S^{\flat,\kappa}(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{E}).$$

Passing to right adjoints we obtain a commutative square of lax symmetric monoidal functors

$$\begin{array}{ccc} \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S,\mathcal{S}\mathrm{p}))^{\otimes} & \xleftarrow{f_*} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(T,\mathcal{S}\mathrm{p}))^{\otimes} \\ {\scriptstyle\mathcal{R}_S}\uparrow & & \uparrow{\scriptstyle\mathcal{R}_T} \\ \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_S(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{S}\mathrm{p})^{\otimes} & \longleftarrow & \mathrm{Fun}^{\mathrm{bloc}}(\mathrm{Mod}_T(\mathrm{Cat}^{\mathrm{ps}}_{\natural,\mathrm{t}}),\mathcal{S}\mathrm{p})^{\otimes} \end{array}$$

where the bottom horizontal arrow is now given by pre-composition with $(\mathcal{D}^{\mathrm{p}}(T),\mathrm{D}_T)\otimes_{(\mathcal{D}^{\mathrm{p}}(S),\mathrm{D}_S)}(-)$. Given a Karoubi-localising functor $\mathcal{F}\colon\ \mathrm{Cat}^{\mathrm{p}} \to \mathcal{S}\mathrm{p}$ the commutativity of this square and the duality preserving $\mathcal{D}^{\mathrm{p}}(S)$-linear functor $f^*\colon\ (\mathcal{D}^{\mathrm{p}}(S),\mathrm{D}_S) \to (\mathcal{D}^{\mathrm{p}}(T),\mathrm{D}_T)$ then determine a map

$$\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}) \to \mathcal{R}^{\mathrm{s}}_S(\mathcal{F};(\mathcal{D}^{\mathrm{p}}(T),\mathrm{D}_T)) = \mathcal{R}^{\mathrm{s}}_S(\mathcal{F}((\mathcal{D}^{\mathrm{p}}(T),\mathrm{D}_T)\otimes_{(\mathcal{D}^{\mathrm{p}}(S),\mathrm{D}_S)}(-)) = f_*\mathcal{R}^{\mathrm{s}}_T(\mathcal{F}),$$

and consequently an adjoint map

$$f^*\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}) \to \mathcal{R}^{\mathrm{s}}_T(\mathcal{F}).$$

Applying the $\mathbb{A}^1$-localisation functor $\mathrm{Loc}_{\mathbb{A}^1}$ and its commutativity with pullbacks we finally get a natural map

$$f^*\mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_S(\mathcal{F}) \to \mathrm{Loc}_{\mathbb{A}^1}\mathcal{R}^{\mathrm{s}}_T(\mathcal{F}).$$

relating the motivic realization of $\mathcal{F}$ over T with the pullback of the motivic realization of $\mathcal{F}$ over S. Taking $\mathcal{F} = \mathbb{GW}$ we write

$$\eta_f\colon\ f^*\mathrm{KQ}_S \to \mathrm{KQ}_T$$

for the resulting map. Our goal in this section is the prove the following:

Proposition 8.2.1. *Let $f : T \to S$ be a map of quasi-compact quasi-separated schemes. Then the map*

$$f^*\mathrm{KQ}_S \to \mathrm{KQ}_T$$

is an equivalence of motivic spectra.

Though the motivic spectrum KQ is built out of symmetric GW-theory, the proof will require passing through genuine GW-theory. For this, recall from Notation 3.3.4 that for a functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}} \to \mathcal{E}$, an invertible perfect complex with C_2-action $L \in \mathcal{P}ic(S)^{\mathrm{BC}_2}$ and integers $n, m \in \mathbb{Z}$, we define $\mathcal{F}_L^{\geq m,[n]}\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{E}$ by the formula

$$\mathcal{F}_L^{\geq m,[n]}(X) = \mathcal{F}\left(\mathcal{D}^{\mathrm{p}}(X), (\mathcal{Q}_L^{\geq m})^{[n]}\right).$$

For $m \in \{-\infty, +\infty\}$ we replace the superscripts $\geq -\infty$ and $\geq +\infty$ with s and q, and for $L = \mathcal{O}_S$ we use the subscript S instead of $\mathcal{O}_S$. In addition, when $n = 0$ we often drop it from the notation.

Using the genuine refinement of the projective line formula (Proposition 6.2.1), we now observe that Bott periodicity (Example 7.3.7) can be extended to the case of $\mathcal{F}_L^{\geq m,[n]}$, if one accepts to shift m:

Proposition 8.2.2 (Genuine Bott periodicity)**.** *If $\mathcal{F}$ is additive, for any $k \in \mathbb{N}$ and $m \in \mathbb{Z}$, there is a canonical equivalence*

$$\Omega^{2k}_{\mathbb{P}^1}\mathcal{F}_L^{\geq m} \simeq \mathcal{F}^{\geq m-k}_{(-1)^k L}.$$

Proof. By Proposition 6.2.1 and the additivity of $\mathcal{F}$, we have

$$\Omega_{\mathbb{P}^1}\mathcal{F}_L^{\geq m,[n]}(X) \simeq \mathcal{F}_L^{\geq m,[n-1]}$$

and hence

$$\Omega^k_{\mathbb{P}^1}\mathcal{F}_L^{\geq m,[n]}(X) \simeq \mathcal{F}_L^{\geq m,[n-k]}.$$

Combining this with Ranicki periodicity we get

$$\Omega^{2k}_{\mathbb{P}^1}\mathcal{F}_L^{\geq m}(X) \simeq \mathcal{F}_L^{\geq m,[-2k]} = \mathcal{F}^{\geq m-k}_{(-1)^k L},$$

as desired. □

Proof of Proposition 8.2.1. Since $f_*\colon \mathrm{Sh}^{\mathrm{nis}}(T, \mathcal{S}\mathrm{p}) \to \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p})$ sends $\mathbb{A}^1$-invariant sheaves to $\mathbb{A}^1$-invariant sheaves we have that $f^*\colon \mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p}) \to \mathrm{Sh}^{\mathrm{nis}}(T, \mathcal{S}\mathrm{p})$ commutes with $\mathrm{Loc}_{\mathbb{A}^1}$. Similarly, since f_* commutes with post-composition with $\mathcal{S}\mathrm{p} \to \mathrm{Grp}_{\mathrm{E}_\infty}$ we have after passing to left adjoint a commutative square

$$\begin{array}{ccc}
\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathrm{Grp}_{\mathrm{E}_\infty})) & \xrightarrow{f^*} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(T, \mathrm{Grp}_{\mathrm{E}_\infty})) \\
\downarrow{\iota_*} & & \downarrow{i_*} \\
\mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(S, \mathcal{S}\mathrm{p})) & \xrightarrow{f^*} & \mathcal{S}\mathrm{p}^{\mathbb{P}^1}(\mathrm{Sh}^{\mathrm{nis}}(T, \mathcal{S}\mathrm{p}))\,.
\end{array}$$

Combining these two facts with Remark 8.1.3 we conclude that to prove the desired claim it is enough to show that the map

$$f^*\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural) \to \mathcal{R}^{\mathrm{s}}_T(\mathcal{GW}^\natural)$$

is an equivalence.

By Example 7.3.7 we then have that the i'th object in the $\mathbb{P}^1$-spectrum $\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural)$ is given by $\mathcal{GW}_S^{\mathrm{s},[i]}$. The pullback functor f^* in this context is implemented by applying f^* to $\mathcal{R}^{\mathrm{s}}_S(\mathcal{GW}^\natural)$ object-wise and then applying to the resulting $\mathbb{P}^1$-prespectrum object the $\mathbb{P}^1$-spectrification operation (which itself preserves Nisnevich sheaves, since the Nisnevich site is finitary). What we need to show is consequently that for every $m \geq 0$ the map

$$\operatorname*{colim}_n \Omega^n_{\mathbb{P}^1} f^*\,\mathcal{GW}_S^{\mathrm{s},[i+n]} \to \mathcal{GW}_T^{\mathrm{s},[i]}$$

is an equivalence of Nisnevich sheaves on T. Now by cofinality the colimit on n can be taken in jumps of 4, that is, we can write this map as

$$\operatorname*{colim}_n \Omega^{4n}_{\mathbb{P}^1} f^*\,\mathcal{GW}_S^{\mathrm{s},[i+4n]} \to \mathcal{GW}_T^{\mathrm{s},[i]}\,.$$

At the same time, by Ranicki periodicity we have that $\mathcal{GW}_S^{\mathrm{s},[i+4n]} = \mathcal{GW}_S^{\mathrm{s},[i]}$ and so we can rewrite this as

$$\operatorname*{colim}_n \Omega^{4n}_{\mathbb{P}^1} f^*\,\mathcal{GW}_S^{\mathrm{s},[i]} \to \mathcal{GW}_T^{\mathrm{s},[i]}\,.$$

Finally, to check an equivalence on spectrum object it is enough to check any infinite family of indices. Taking only the i's that are divisible by 4 and using again Ranicki periodicity we can simply forget about the i and show that the map

$$\operatorname{colim}_n \Omega^{4n}_{\mathbb{P}^1} f^* \mathcal{GW}^{\mathrm{s}}_S \to \mathcal{GW}^{\mathrm{s}}_T$$

is an equivalence of Nisnevich sheaves on T.

Let Q be the poset whose objects are pairs (n, m) with n, m non-negative integers equipped with the order relation such that $(n, m) \leq (n', m')$ if and only if $n \leq n'$ and $n + m \leq n' + m'$. In particular, the projection $Q \to \mathbb{N}$ sending (n, m) to n is a cartesian fibration such that for each n' the transition functor $\tau\colon Q_{n+1} = \mathbb{N} \to \mathbb{N} = Q_n$ is given by $\tau(x) = x + 1$. Specifying a functor out of Q then amounts to specifying for each n a functor f_n out of $Q_n = \mathbb{N}$ together with natural transformations $\eta_n\colon \tau^* f_{n-1} \Rightarrow f_{n+1}$ for $n \geq 1$. In particular, we may consider the functor $\theta\colon Q \to \mathrm{Sh}^{\mathrm{nis}}(S, \mathrm{Grp}_{\mathrm{E}_\infty})$ given by

$$\theta(n, m) = \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq -2m}_S,$$

where the maps

$$\eta_n\colon \Omega^{4(n-1)}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq -2m-2}_S = \Omega^{4(n-1)}_{\mathbb{P}^1_T} f^* \Omega^{4n}_{\mathbb{P}^1_T} \mathcal{GW}^{\geq -2m}_S \to \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq -2m}_S$$

are given by the Beck-Chevalley map $f^* \circ \Omega^4_{\mathbb{P}^1_T} \Rightarrow \Omega^4_{\mathbb{P}^1_T} \circ f^*$. Here, $\mathcal{GW}^{\geq -2m}_S$ is defined to be the Nisnevich sheafification of the presheaf $[X \to S] \mapsto \mathcal{GW}(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\geq -2m}_X)$. Now consider the commutative diagram in $\mathrm{Sh}^{\mathrm{nis}}(T, \mathrm{Grp}_{\mathrm{E}_\infty})$ given by

$$\begin{array}{ccccc}
\operatorname{colim}_n \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq 0}_S & \longrightarrow & \operatorname{colim}_n \Omega^{4n}_{\mathbb{P}^1_T} \mathcal{GW}^{\geq 0}_T & = & \operatorname{colim}_n \mathcal{GW}^{\geq -2n}_T \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\operatorname{colim}_{(n,m)\in Q} \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq -2m}_S & \longrightarrow & \operatorname{colim}_{(n,m)\in Q} \Omega^{4n}_{\mathbb{P}^1_T} \mathcal{GW}^{\geq -2m}_T & = & \operatorname{colim}_{(n,m)\in Q} \mathcal{GW}^{\geq -2n-2m}_T \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
\operatorname{colim}_{(n,m)\in \mathbb{N}\times\mathbb{N}} \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\geq -2m}_S & \longrightarrow & \operatorname{colim}_{(n,m)\in \mathbb{N}\times\mathbb{N}} \Omega^{4n}_{\mathbb{P}^1_T} \mathcal{GW}^{\geq -2m}_T & = & \operatorname{colim}_{(n,m)\in \mathbb{N}\times\mathbb{N}} \mathcal{GW}^{\geq -2n-2m}_T \\
\downarrow \simeq & & & & \downarrow \simeq \\
\operatorname{colim}_n \Omega^{4n}_{\mathbb{P}^1_T} f^* \mathcal{GW}^{\mathrm{s}}_S & & \longrightarrow & & \mathcal{GW}^{\mathrm{s}}_T
\end{array}$$

where the vertical maps in the top row are induced by restriction to the sub-poset $\mathbb{N} \times \{0\} = \{(n, m) \in Q | m = 0\} \subseteq Q$, while the vertical maps in the middle row are induced by restriction along the poset map $\mathbb{N} \times \mathbb{N} \to Q$ which is the identity on objects. It is straightforward to verify that that both these functors are cofinal, so that the associated vertical maps are equivalences. In addition, the bottom vertical maps are equivalences as well, as the sequence of Poincaré structures $\Omega^{\geq 0}$ converges to Ω^{s}. To show that the bottom horizontal map is an equivalence it will hence suffice to show that the top horizontal map is an equivalence. We have thus reduced to proving that the induced map

$$f^* \mathcal{GW}^{\geq 0}_S \to \mathcal{GW}^{\geq 0}_T$$

is an equivalence of Nisnevich sheaves. Now as a Nisnevich sheaf valued in E_∞-groups, $\mathcal{GW}^{\geq 0}_T$ is the group completion of the $\mathrm{Mon}_{\mathrm{E}_\infty}$-valued sheaf $\mathrm{Vect}^{\mathrm{s}}_S$ of vector bundles equipped with perfect symmetric bilinear forms: indeed, this statement can be verified on the level of stalks, and for affine schemes the group completion statement is proven in [HS21, Theorem A]. Since f^* commutes with group completion (again as above this follows from the fact that f_* preserves the group-like property) it will now suffice to show that the map

$$f^* \mathrm{Vect}^{\mathrm{s}}_S \to \mathrm{Vect}^{\mathrm{s}}_T$$

is a motivic equivalence of $\mathrm{Mon}_{\mathrm{E}_\infty}$-valued sheaves; in fact it is a Zariski-local equivalence by [EHK+20, Proposition A.0.4], since $\mathrm{Vect}^{\mathrm{s}}$ is an algebraic stack with affine diagonal. □

8.3. **Thom isomorphism.** Let X be a smooth S-scheme and $p\colon V \to X$ a vector bundle over X of rank r with zero section $s\colon X \to V$. Recall [MV99, Definition 2.16] that the Thom space $\mathrm{Th}(V)$ of V is the quotient $V/V \smallsetminus \{0\}$, considered as a pointed motivic space over X. Since $\mathrm{SH}(X)$ is tensored and cotensored over pointed motivic spaces over X we may consider for every $E \in \mathrm{SH}(X)$ the tensor and cotensor

$$\Sigma^V E = \mathrm{Th}(V) \otimes E \quad \text{and} \quad \Omega^V E = E^{\mathrm{Th}(V)}$$

in $\mathrm{SH}(X)$. We note that we can also describe $\Sigma^V E$ and $\Omega^V E$ by the formulas

$$\Sigma^V E = p_\sharp s_* E \quad \text{and} \quad \Omega^V E = s^! p^* E$$

using the six-functor formalism.

Proposition 8.3.1 (Thom isomorphism)**.** *Let X be a smooth S-scheme, let L be a perfect invertible complex with C_2-action on X and let $p\colon V \to X$ be a vector bundle over X of rank r with zero section $s\colon X \to V$. Then there is a natural equivalence of motivic spectra over X*

$$\Omega^V \mathrm{KQ}_L \simeq \mathrm{KQ}_{L \otimes \det V[-r]}$$

and hence natural equivalences of spectra

$$\hom(\Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(V), \Sigma^n_{\mathbb{P}^1} \mathrm{KQ}_L) \simeq \hom(\Sigma^\infty_{\mathbb{P}^1} X, \Sigma^n_{\mathbb{P}^1} \Omega^V \mathrm{KQ}_L) \simeq \mathbb{HGW}^{\mathrm{s}}(X, L \otimes \det V[n-r]).$$

In particular, if X is regular Noetherian of finite Krull dimension then

$$\hom(\Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(V), \Sigma^n_{\mathbb{P}^1} \mathrm{KQ}_L) \simeq \mathbb{GW}^{\mathrm{s}}(X, L \otimes \det V[n-r]).$$

Proof. We use the equivalence of motivic spaces $\mathrm{Th}(V) \simeq \mathbb{P}_X(\mathcal{O} \oplus V^\vee)/\mathbb{P}_X(V^\vee)$ from [MV99, Proposition 2.17, 3]. In addition, using the equivalence $\mathrm{Th}(\mathcal{O} \oplus V) \simeq \Sigma_{\mathbb{P}^1}\mathrm{Th}(V)$ from loc. cit. and the Bott periodicity statement of Example 7.3.7 we may assume without loss of generality that r is odd. Applying Construction 7.3.6, it will suffice to provide a natural equivalence of cotensors

$$\mathbb{GW}^{\widetilde{\mathcal{T}}_X(\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}_X(\mathcal{O}\oplus V^\vee)/\mathbb{P}_X(V^\vee)))} \simeq \mathbb{GW}^{\mathrm{j}_{\mathrm{bloc}}(\mathcal{D}^{\mathrm{p}}(X), \mathrm{D}_{\det V[-r]})}$$

(where $\mathrm{j}_{\mathrm{bloc}}$ is defined in (38)). In particular, we no longer need to keep track of L. As in §6.1, let $\mathcal{C}$ be the stable subcategory of $\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(\mathcal{O} \oplus V^\vee))$ generated by

$$\mathcal{A}(\frac{r-1}{2}) \cup \mathcal{A}(\frac{r-2}{2} - 1) \cup \cdots \cup \mathcal{A}(\frac{-r+1}{2}),$$

where $\mathcal{A}(i)$ is the image of the fully-faithful functor $p^*(-) \otimes \mathcal{O}(i)\colon \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(\mathcal{O} \oplus V^\vee))$, so that $\mathcal{C}$ is invariant under the duality $\mathrm{D}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}$ by Lemma 6.1.2. Since the restriction functor

$$i^*\mathbb{P}_X(\mathcal{O} \oplus V^\vee) \to \mathbb{P}_X(V^\vee)$$

sends $\mathcal{O}(i)$ to $\mathcal{O}(i)$, Lemma 6.1.5 and Example 4.2.7 together imply that the composite

$$(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}|_{\mathcal{C}}) \hookrightarrow \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(\mathcal{O} \oplus V^\vee), \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}\big) \xrightarrow{i^*} \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(V^\vee), \Omega^{\mathrm{s}}_{\mathbb{P}_X(V^\vee)}\big)$$

is sent to an equivalence by any bounded localising functor on $\mathrm{Mod}^{\flat,\kappa}_X(\mathrm{Cat}^{\mathrm{ps}}_{\natural,t})$. But by Lemma 6.1.3 the inclusion on the left participates in a split Poincaré-Verdier sequence

$$\big(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}|_{\mathcal{C}}\big) \to \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}\big) \xrightarrow{p_*(-\otimes\mathcal{O}(-\frac{r+1}{2}))} \big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes\det V^\vee[-r]}\big).$$

We hence conclude that

$$\mathrm{j}_{\mathrm{bloc}}\big(\mathcal{D}^{\mathrm{p}}(X), \Omega^{\mathrm{s}}_{L\otimes\det V^\vee[-r]}\big) = \mathrm{fib}\,\big[\mathrm{j}_{\mathrm{bloc}}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X V), \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}\big) \to \mathrm{j}_{\mathrm{bloc}}\big(\mathcal{C}, \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}|_{\mathcal{C}}\big)\big] =$$

$$\mathrm{cof}\,\big[\mathrm{j}_{\mathrm{bloc}}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(V^\vee), \Omega^{\mathrm{s}}_{\mathbb{P}_X(V^\vee)}\big) \to \mathrm{j}_{\mathrm{bloc}}\big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}_X(\mathcal{O}\oplus V^\vee), \Omega^{\mathrm{s}}_{\mathbb{P}_X(\mathcal{O}\oplus V^\vee)}\big)\big] = \widetilde{\mathcal{T}}_X(\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}_X(\mathcal{O}\oplus V^\vee)/\mathbb{P}_X(V^\vee))),$$

as desired. □

8.4. **Purity of the hermitian K-theory spectrum.** In their work [DJK21], Déglise, Jin and Khan constructed what they call a *system of fundamental classes* in motivic homotopy theory. Given a fixed quasi-compact quasi-separated base scheme S and a motivic spectrum $E \in \mathrm{SH}(S)$, these consists of a compatible collection of maps

$$\eta_f : \mathbb{S}_X \to \mathrm{Th}(T_f)^{-1} \otimes f^! \mathbb{S}_Y$$

for every smoothable lci map $f : X \to Y$ of (separated finite-type) S-schemes with tangent complex $T_f \in \mathcal{D}^{\mathrm{p}}(X)$, where $\mathbb{S}_X$ and $\mathbb{S}_Y$ are the sphere spectra in $\mathrm{SH}(X)$ and $\mathrm{SH}(Y)$, respectively. Here, smoothable lci means that f can be factored as a composite $X \xrightarrow{i} Y' \xrightarrow{p} Y$ where i is a regular embedding and p is smooth. In this case, the tangent complex T_f is perfect and sits in an exact sequence

$$T_f \to i^* T_p \to N_X Y'$$

where T_p is the relative tangent bundle of p and $N_X Y'$ is the normal bundle of X in Y'. The associated Thom object $\mathrm{Th}(T_f) \in \mathrm{SH}(X)$ is then by definition the ratio $\mathrm{Th}(i^* T_p) \otimes \mathrm{Th}(N_X Y')^{-1}$ (which does not depend on the choice of factorization). Given a motivic spectrum $E \in \mathrm{SH}(S)$ and a smoothable lci map $f : X \to Y$ one may then define an E-valued variant η^E_f of η_f as the composite

$$E_X \xrightarrow{E_X \otimes \eta_f} E_X \otimes \mathrm{Th}(T_f)^{-!} \otimes f^! \mathbb{S}_Y \to \mathrm{Th}(T_f)^{-!} \otimes f^! E_Y.$$

The maps η^E_f are by construction natural in E. In addition, the satisfy two types of compatibility conditions:

(1) Composition: if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are a composable pair of smoothable lci morphisms among separated finite type S-schemes then $h := g \circ f$ is smoothable lci with $\mathrm{Th}(T_h) \simeq \mathrm{Th}(T_f) \otimes f^* \mathrm{Th}(T_g)$ and $\eta^E_{g \circ f}$ is given by the composite

$$\begin{aligned} E_X \xrightarrow{\eta^E_f} \mathrm{Th}(T_f)^{-1} \otimes f^! E_Y \xrightarrow{\mathrm{Th}(T_f)^{-1} \otimes f^!(\eta^E_g)} & \mathrm{Th}(T_f) \otimes f^!(\mathrm{Th}(T_g) \otimes g^! E|_Z) \\ = & \mathrm{Th}(T_f) \otimes f^* \mathrm{Th}(g) \otimes f^! g^! E|_Z \\ = & \mathrm{Th}(T_h) \otimes h^! E_Z \end{aligned}$$

where the second equivalence is because $\mathrm{Th}(T_f)$ is tensor invertible.

(2) Transverse base change: for every Tor-independent square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ W & \xrightarrow{f'} & Z \end{array}$$

among separated finite type S-schemes for which the horizontal maps are smoothable lci the maps η^E_f and $\eta^E_{f'}$ fit into a commutative diagram

$$\begin{array}{ccc} g^* E_W & \xrightarrow{g^* \eta^E_{f'}} & g^* \mathrm{Th}(T_{f'})^{-1} \otimes g^*(f')^! E_Z \\ \| & & \uparrow \simeq \\ g^* E_W & \longrightarrow & \mathrm{Th}(T_f)^{-1} \otimes g^*(f')^! E_Z \\ \| & & \downarrow \\ E_X & \xrightarrow{\eta^E_f} & \mathrm{Th}(T_f)^{-1} \otimes f^! E_Y \end{array}$$

where the right vertical arrows are induced by the map $T_f \to g^* T_{f'}$ (which is an equivalence for Tor independent squares) and the Beck Chevalley map $g^*(f')E_Z \to f^!(g')^* E_Z = f^! E_Y$.

Given a smoothable lci map $X \to Y$ among separated finite type S-schemes and a motivic spectrum $E \in \mathrm{SH}(S)$, we say that E is *pure at* f if the map η^E_f is an equivalence. We say that E is *absolutely pure* if it is pure at f for every smoothable lci morphism of finite type separated S-schemes

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

such that X and Y are both regular. We note that for any E, the collection P_E of maps at which E is pure satisfies the following properties:

(1) The collection P_E contains all smooth maps (see [DJK21, Example 2.3.4]).

(2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are a pair of composable smoothable lci morphisms among separated finite type S-schemes and g belongs to P_E then f belongs to P_E if and only if $g \circ f$ belongs to P_E. This follows immediately from Property ((1)) concerning compatibility with composition. Combined with the previous point this means in particular that any smoothable lci morphism between *smooth* S-schemes is in P_E.

(3) If $f : X \to Y$ and $Y = \cup_i U_i$ is a Zariski open covering then f belongs to P_E if and only if the base change $f_i : V_i := X \times_Y U_i \to U_i$ belongs to P_E for every i. To see this, note that, on the one hand, equivalences in $\mathrm{SH}(X)$ are detected locally, and on the other hand, the square

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & U_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is Tor independent and the Beck-Chevalley map $f^!(-)|_{V_i} \Rightarrow f_i^!((-)|_{U_i})$ is an equivalence, and so Property ((2)) implies that $\eta^E_{f_i} = (\eta^E_f)|_{V_i}$.

Combining the above we deduce the following:

Lemma 8.4.1. *A motivic spectrum $E \in \mathrm{SH}(S)$ is pure if and only if it is pure at f whenever*

$$\begin{array}{ccc} \mathrm{spec}(A/a) & \xrightarrow{\quad f \quad} & \mathrm{spec}(A) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

is a closed embedding of regular affine schemes of finite type over S corresponding to a principal ideal $(a) \subseteq A$ generated by a non-zero divisor a.

Proof. The only if direction is clear since the map $\mathrm{spec}(A/a) \to \mathrm{spec}(A)$ appearing in the lemma is a closed embedding among regular schemes and hence in particular smoothable lci among such. For the if direction, note first that the closure properties ((1)) and ((2)) above together imply that E is absolutely pure if and only if it is pure at every (automatically regular) closed embedding of regular separated finite type S-schemes. Let now $f : X \to Y$ be such a closed embedding. Since f is regular there exists an open cover by affines $Y = \cup_i \mathrm{spec}(A_i)$ and for each i a regular sequence $a_{1,i}, ..., a_{n_i,i} \in A_i$ such that $X \times_Y \mathrm{spec}(A_i) = \mathrm{spec}(A_i/(a_{1,i}, ..., a_{n,i}))$. In addition, since Y is regular we have that each $\mathrm{spec}(A_i)$ and each $\mathrm{spec}(A_i/(a_{1,i}, ..., a_{n_i,i}))$ are regular. By ((3)) we then have that E is absolutely pure if and only if it is pure at closed embeddings of the form $\mathrm{spec}(A/(a_1, ..., a_n)) \to \mathrm{spec}(A)$ where A is a regular ring and $a_1, ..., a_n \in A$ is a regular sequence. Using again the closure under composition property ((2)) it will suffice to prove this for the case where the regular sequence has length one, and so the desired result follows. □

Theorem 8.4.2 (Purity)**.** *Let S be a Noetherian scheme of finite Krull dimension. Then the motivic spectrum $\mathrm{KQ}_S \in \mathrm{SH}(S)$ constructed in §8.1 is absolutely pure.*

In order to prove Theorem 8.4.2 we first express the fundamental classes η_S^{KQ} in Poincaré categorical terms. This is done via the following construction.

Construction 8.4.3. Let A be a regular ring, $a \in A$ a non-zero divisor and $i\colon \operatorname{spec}(A/a) \to A$ the associated regular embedding. Since the ideal (a) is principal and non-zero we have an equivalence $i^! A[1] \simeq A/a$. The push-forward Poincaré functor of Corollary 5.1.7 for $L = A[1]$ then takes the form

$$(i_*, \eta)\colon (\mathcal{D}^{\mathrm{p}}(A/a), \Qoppa^{\mathrm{s}}_{A/a}) \to (\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\mathrm{s}}_{A[1]}).$$

Taking motivic realizations as in Construction 7.3.4 and using Corollary 7.3.8, we then obtain an induced map

$$\mu_i\colon i_* \mathrm{KQ}_{\operatorname{spec}(A/a)} = \mathcal{R}^{\mathrm{s}}_{\operatorname{spec}(A)}(\mathbb{GW}; (\mathcal{D}^{\mathrm{p}}(A/a), \mathrm{D}_{A/a})) \to \mathcal{R}^{\mathrm{s}}_{\operatorname{spec}(A)}(\mathbb{GW}; (\mathcal{D}^{\mathrm{p}}(A), \mathrm{D}_{A[1]})) = \mathrm{KQ}_{A[1]}\,.$$

Let us now consider the compatibility of the above construction with respect to a ring maps $B \to A$ sending a non-zero divisor element $b \in B$ to a non-zero divisor element $a \in A$. We note that the condition of being non-zero divisors implies that the multiplication maps $a\colon A \to A$ and $b\colon B \to B$ are injective and the map

$$A \otimes^L_B B/b \to A/a$$

is an equivalence (where on the left we have the derived tensor product in $\mathcal{D}^{\mathrm{p}}(B)$). In particular, the cartesian square of affine schemes

$$\begin{array}{ccc} \operatorname{spec}(B/b) & \xrightarrow{j} & \operatorname{spec}(B) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f} \\ \operatorname{spec}(A/a) & \xrightarrow{i} & \operatorname{spec}(A) \end{array}$$

is Tor-independent and hence determines a commutative square of $(\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\mathrm{s}}_A)$-module Poincaré ∞-categories

$$\begin{array}{ccc} (\mathcal{D}^{\mathrm{p}}(B/b), \Qoppa^{\mathrm{s}}_{B/b}) & \xrightarrow{j_*} & (\mathcal{D}^{\mathrm{p}}(B), \Qoppa^{\mathrm{s}}_{B[1]}) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ (\mathcal{D}^{\mathrm{p}}(A/a), \Qoppa^{\mathrm{s}}_{A/a}) & \xrightarrow{i_*} & (\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\mathrm{s}}_{A[1]})\,. \end{array}$$

Applying the motivic realization functor $\mathcal{R}^{\mathrm{s}}_B(\mathbb{GW}; -)$ and arguing as above we then obtain a commutative square

$$\begin{array}{cccc} & j_* \mathrm{KQ}_{B/b} & \xrightarrow{\mu_j} & \mathrm{KQ}_{B[1]} \\ \swarrow & & & \\ j_* g_* \mathrm{KQ}_{A/a} & \downarrow & & \downarrow \\ \searrow{\scriptstyle \simeq} & & & \\ & f_* i_* \mathrm{KQ}_{A/a} & \xrightarrow{f_*\mu_i} & f_* \mathrm{KQ}_{A[1]} \end{array}$$

in $\mathrm{SH}(A)$, which transforms via the adjunction $f^* \dashv f_*$ into a commutative square

$$\begin{array}{cccc} & f^* j_* \mathrm{KQ}_{B/b} & \xrightarrow{f^*\mu_j} & f^* \mathrm{KQ}_{B[1]} \\ \swarrow{\scriptstyle \simeq} & & & \\ i_* g^* \mathrm{KQ}_{B/b} & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \searrow{\scriptstyle \simeq} & & & \\ & i_* \mathrm{KQ}_{A/a} & \xrightarrow{\mu_i} & \mathrm{KQ}_{A[1]} \end{array}$$

whose vertical maps are equivalences by Proposition 8.2.1 and proper base change (for motivic spectra). In summary, the outcome of this procedure can be viewed as a natural homotopy

$$\mu_i \sim f^* \mu_j$$

of maps $i_* \mathrm{KQ}_{\operatorname{spec}(A/a)} \to \mathrm{KQ}_{A[1]}$.

Note that for a regular embedding of the form $i\colon \operatorname{spec}(A/a) \to \operatorname{spec}(A)$ associated to a regular ring A and non-zero divisor $a \in A$, the projective A/a-module associated to the normal bundle is the dual of the

A/a-module $(a)/(a^2)$, which is free of rank one with generator the class of a. In particular, the normal bundle $N_{\mathrm{spec}(A/a)}\mathrm{spec}(A)$ is equipped with a distinguished trivialization, which determined in particular an identification of functors

$$\mathrm{Th}(T_i) \otimes (-) \simeq \Sigma_{\mathbb{P}^1}.$$

We may hence view the fundamental class η_i^{KQ} as a map

$$\eta_i^{\mathrm{KQ}} \colon \mathrm{KQ}_{\mathrm{spec}(A/a)} \to \mathrm{Th}(T_i) \otimes i^! \mathrm{KQ}_A = \Sigma_{\mathbb{P}^1} i^! \mathrm{KQ}_A .$$

Lemma 8.4.4. *For any regular embedding of the form* $i\colon \mathrm{spec}(A/a) \to \mathrm{spec}(A)$ *associated to a regular ring* A *(of finite type over* S*) and non-zero divisor* $a \in A$*, the map*

$$\mu_i \colon i_* \mathrm{KQ}_{\mathrm{spec}(A/a)} \to \mathrm{KQ}_{A[1]},$$

of Construction 8.4.3 corresponds, under the adjunction $i_* \dashv i^!$ *and the equivalence* $\mathrm{KQ}_{A[1]} = \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A$ *arising from Bott periodicity (Example 7.3.7), to the fundamental class map*

$$\eta_i^{\mathrm{KQ}} \colon \mathrm{KQ}_{\mathrm{spec}(A/a)} \to \Sigma_{\mathbb{P}^1} i^! \mathrm{KQ}_A = i^!(\Sigma_{\mathbb{P}^1} \mathrm{KQ}_A).$$

Proof. Without loss of generality, we may assume that $S = \mathrm{spec}(A)$. The element $a \in A$ determines an A-algebra homomorphism $A[t] \to A$ sending t to a. Since $t \in A[t]$ is never a zero divisor we obtain, as described in Construction 8.4.3, a Tor independent square of affine schemes

$$\begin{array}{ccc} \mathrm{spec}(A/a) & \xrightarrow{i} & \mathrm{spec}(A) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathrm{spec}(A[t]/t) & \xrightarrow{j} & \mathrm{spec}(A[t]) \end{array}$$

and compatibility homotopy $\mu_i \sim f^*\mu_j$ of maps $i_*\colon \mathrm{KQ}_{\mathrm{spec}(A/a)} \to \mathrm{KQ}_{A[1]}$. At the same time, the construction of the fundamental classes is also compatible with Tor independent squares, as recalled above. In particular, at the case at hand we get that η_i^{KQ} is given as the composite

$$\mathrm{KQ}_{A/a} = g^* \mathrm{KQ}_{B/b} \xrightarrow{f^*\eta_j^{\mathrm{KQ}}} g^*\Sigma_{\mathbb{P}^1} j^! \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} g^* j^! \mathrm{KQ}_B \to \Sigma_{\mathbb{P}^1} i^! f^* \mathrm{KQ}_B = i^! \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A .$$

We thus get that the adjoint of η_i^{KQ} along $i_* \dashv i^!$ is given as the composite

$$\begin{aligned} i_* \mathrm{KQ}_{A/a} = i_* g^* \mathrm{KQ}_{B/b} &\xrightarrow{i_* g^* \eta_j^{\mathrm{KQ}}} i_* g^* \Sigma_{\mathbb{P}^1} j^! \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} i_* g^* j^! \mathrm{KQ}_B \\ &\longrightarrow \Sigma_{\mathbb{P}^1} i_* i^! f^* \mathrm{KQ}_B \longrightarrow \Sigma_{\mathbb{P}^1} f^* \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A . \end{aligned} \tag{42}$$

On the other hand, the adjoint of η_j^{KQ} along $j_* \dashv j^!$ is given by the composite

$$j_* \mathrm{KQ}_{B/b} \xrightarrow{j_* \eta_j^{\mathrm{KQ}}} j_* \Sigma_{\mathbb{P}^1} j^! \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} j_* j^! \mathrm{KQ}_B \longrightarrow \Sigma_{\mathbb{P}^1} \mathrm{KQ}_B,$$

and if we note apply f^* to this we get the composite

$$\begin{aligned} i_* \mathrm{KQ}_{A/a} = i_* g^* \mathrm{KQ}_{B/b} = f^* j_* \mathrm{KQ}_{B/b} &\xrightarrow{f^* j_* \eta_j^{\mathrm{KQ}}} f^* j_* \Sigma_{\mathbb{P}^1} j^! \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} f^* j_* j^! \mathrm{KQ}_B \\ &\longrightarrow \Sigma_{\mathbb{P}^1} f^* \mathrm{KQ}_B = \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A \end{aligned} \tag{43}$$

We claim that the composite (42) is homotopic to the composite (43). Unwinding both constructions, this amounts to constructing a commuting homotopy in the diagram

$$\begin{array}{ccc} i_* g^* j^! & \overset{\simeq}{\Longleftarrow} & f^* j_* j^! \\ \Downarrow & & \Downarrow \\ i_* i^! f^* & \Longrightarrow & f^* . \end{array}$$

Here, the two arrows with target f^* are induced by the counits of $j_* \dashv j^!$ and $i_* \dashv i^!$, respectively, while the two other arrows are induced by the Beck-Chevalley transformations $f^* j_* \Rightarrow i_* g^*$ and $g^* j^! \Rightarrow i^! f^*$.

These two Beck-Chevalley transformations are by construction the mates of one another, and so we may write either one as a composite involving the other and (co)units. In particular, we may extend the above a-priori non-commutative diagram to a larger diagram as

$$\begin{array}{ccccc} i_*i^!i_*g^*j^! & \Longleftarrow & i_*g^*j^! & \overset{\simeq}{\Longleftarrow} & f^*j_*j^! \\ \Downarrow & & \Downarrow & & \Downarrow\simeq \\ i_*i^!f^*j_*j^! & \Longrightarrow & i_*i^!f^* & \Longrightarrow & f^* . \end{array}$$

where the left half commutes. We then observe that the external part of the diagram commutes as well, yielding a commuting homotopy for the right half. To summarize so far, we have thus shown that the adjoint of η_i^{KQ} along $i_* \dashv i^!$ is homotopic to the image under f^* of the adjoint of η_j^{KQ} along $j_* \dashv j^!$. Otherwise put, these adjoint satisfy the same compatibility as μ_i and μ_j. It will hence suffice to show that μ_j is homotopic to the adjoint along $j_* \dashv j^!$ of η_j^{KQ}. In other words, we may replace the pair $(A, A/a)$ with the pair $(A[t], A)$ and prove the claim for the regular embedding $j\colon \operatorname{spec}(A) \to \operatorname{spec}(A[t])$ of A-schemes.

Now in this case the fundamental class η_j^{KQ} is defined by noting that of we write $p\colon \operatorname{spec}(A[t]) \to \operatorname{spec}(A)$ for the structure map, then the functor $j^!p^*\colon \mathrm{SH}(A) \to \mathrm{SH}(A)$ is equivalent to $\Omega_{\mathbb{P}^1}$ and

$$\eta_j^{\mathrm{KQ}}\colon \mathrm{KQ}_A \to \Sigma_{\mathbb{P}^1} j^! \mathrm{KQ}_{A[t]} = \Sigma_{\mathbb{P}^1} j^! p^* \mathrm{KQ}_A = \Sigma_{\mathbb{P}^1}\Omega_{\mathbb{P}^1} \mathrm{KQ}_A$$

is the canonical equivalence exhibiting $\Sigma_{\mathbb{P}^1}$ and $\Omega_{\mathbb{P}^1}$ as inverses to each other. In particular, if we post-compose this with the equivalence $\Sigma_{\mathbb{P}^1} j^! p^* \mathrm{KQ}_A = j^! p^* \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A$ then the map η^{KQ} becomes the unit

$$\eta_j^{\mathrm{KQ}}\colon \mathrm{KQ}_A \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1} \mathrm{KQ}_A .$$

Now on the side of μ, we have that the map

$$\mu_j\colon j_* \mathrm{KQ}_A \to \mathrm{KQ}_{(A[t])[1]} = p^* \mathrm{KQ}_{A[1]}$$

determined an adjoint map

$$\mathrm{KQ}_A \to j^! p^* \mathrm{KQ}_{A[1]} = \Omega_{\mathbb{P}^1} \mathrm{KQ}_{A[1]},$$

and then again an adjoint map

$$\iota\colon \Sigma_{\mathbb{P}^1} \mathrm{KQ}_A \to \mathrm{KQ}_{A[1]} .$$

In light of the above, to finish the proof we now need to check that this map is homotopic to the equivalence $\Sigma_{\mathbb{P}^1} \mathrm{KQ}_A \simeq \mathrm{KQ}_{A[1]}$ given by Bott periodicity of Example 7.3.7. We point out that this claim now only concerns Construction 8.4.3, and does not need any further input concerning the motivic fundamental classes. Now the Bott periodicity equivalence is obtained from the split Poincaré-Verdier sequence

$$(\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\mathrm{s}}_A) \xrightarrow{q^*} \big(\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_A), \Qoppa^{\mathrm{s}}_{\mathbb{P}^1_A}\big) \xrightarrow{q_*((-)\otimes\mathcal{O}(1))} (\mathcal{D}^{\mathrm{p}}(A), \Qoppa^{\mathrm{s}}_{A[-1]})$$

where $q\colon \mathbb{P}^1_A \to \operatorname{spec}(A)$ is the structure map and $i_\infty\colon \operatorname{spec}(A) \to \mathbb{P}^1_A$ the inclusion of the point at ∞, pull-back along which provides a Poincaré retraction to q^*. Applying the motivic realization functor $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW}; -)$ yields a split exact sequence

$$\mathrm{KQ}_A \longrightarrow q_*q^* \mathrm{KQ}_A \longrightarrow \mathrm{KQ}_{A[-1]}$$

where the retraction is induced by the inclusion i_∞ of the point at ∞. In particular, this gives an identification

$$\Omega_{\mathbb{P}^1} \mathrm{KQ} = \operatorname{fib}[q_*q^* \mathrm{KQ}_A \xrightarrow{\infty^*} \mathrm{KQ}_A] \xrightarrow{\simeq} \mathrm{KQ}_{A[-1]} .$$

Let $U, V \subseteq \mathbb{P}^1_A$ be the complements of the zero and infinity sections, respectively, so that both U and V are isomorphic to the affine line over X. Given a smooth A-scheme $X \to \operatorname{spec}(A)$ let us write $\mathbb{P}^1_X := \mathbb{P}^1_A \times_A X$ and consider the full subcategory $\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_X) \subseteq \mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_X)$ given by those perfect complexes which are supported at the zero section $0\colon X \to \mathbb{P}^1_X$. Similarly, we write $V_X := V \times_A X$ and $U_X := U \times_A X$ and consider the full subcategory $\mathcal{D}^{\mathrm{p}}_0(V\times_A X)$ of $\mathcal{D}^{\mathrm{p}}(V\times_A X)$ spanned by the complexes supported at the zero section. Then in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_X),\Qoppa^{\mathrm{s}}_{\mathbb{P}^1_X}\right) & \longrightarrow & \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_X),\Qoppa^{\mathrm{s}}_{\mathbb{P}^1_X}\right) & \xrightarrow{i_\infty^*} & \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(X),\Qoppa^{\mathrm{s}}_A\right) \\
\| & & \| & & \uparrow\simeq \\
\mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_X),\Qoppa^{\mathrm{s}}_{\mathbb{P}^1_X}\right) & \longrightarrow & \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(\mathbb{P}^1_X),\Qoppa^{\mathrm{s}}_{\mathbb{P}^1_X}\right) & \longrightarrow & \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(U),\Qoppa^{\mathrm{s}}_U) \\
\downarrow\simeq & & \downarrow & & \downarrow \\
\mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}_0(V_X),\Qoppa^{\mathrm{s}}_{V_X}\right) & \longrightarrow & \mathrm{GW}\left(\mathcal{D}^{\mathrm{p}}(V_X),\Qoppa^{\mathrm{s}}_{V_X}\right) & \longrightarrow & \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(U_X\cap V_X),\Qoppa^{\mathrm{s}}_{U_X\cap V_X})
\end{array}$$

the top right vertical arrow is an equivalence by $\mathbb{A}^1$-invariance of GW (Theorem 6.3.1) and the bottom left vertical arrow is an equivalence by excision (Corollary A.2.6). The same remains true if one shifts all the Poincaré structures by some $n\in\mathbb{Z}$. Taking into account Example 7.3.7 we thus obtain a distinguished equivalences

$$\mathcal{R}^{\mathrm{s}}_S\left(\mathbb{GW};\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_A),\mathrm{D}_{\mathbb{P}^1_A}\right)\simeq\mathcal{R}^{\mathrm{s}}_S\left(\mathrm{GW};\mathcal{D}^{\mathrm{p}}_0(V),\mathrm{D}_V\right)\simeq\mathrm{fib}[q_*q^*\,\mathrm{KQ}\xrightarrow{\infty^*}\mathrm{KQ}_A]=\Omega_{\mathbb{P}^1}\,\mathrm{KQ}_A\,.$$

Under this equivalence, the identification

$$\Omega_{\mathbb{P}^1}\,\mathrm{KQ}_A\xrightarrow{\simeq}\mathrm{KQ}_{A[-1]}$$

of Example 7.3.7 becomes the image under $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW};-)$ of the Poincaré functor

$$q^*((-)\otimes\mathcal{O}(1))\colon\ \left(\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_A),\Qoppa^{\mathrm{s}}_{\mathbb{P}^1_A}\right)\to\left(\mathcal{D}^{\mathrm{p}}(A),\Qoppa^{\mathrm{s}}_{A[-1]}\right).$$

On the other hand, the map

$$\mathrm{KQ}_{A[-1]}\xrightarrow{\simeq}\Omega_{\mathbb{P}^1}\,\mathrm{KQ}_A$$

resulting from the map μ_j is the image under $\mathcal{R}^{\mathrm{s}}_S(\mathbb{GW};-)$ of the Poincaré functor

$$(i_0)_*\colon\ \left(\mathcal{D}^{\mathrm{p}}(A),\Qoppa^{\mathrm{s}}_{A[-1]}\right)\to\left(\mathcal{D}^{\mathrm{p}}_0(V),\Qoppa^{\mathrm{s}}_V\right).$$

Since the Poincaré functor $(i_0)_*$ factors through $(\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_A),\Qoppa^{\mathrm{s}}_A)$, to show the compatibility of the two it will hence suffice to show that the composed Poincaré functor

$$(\mathcal{D}^{\mathrm{p}}(A),\Qoppa^{\mathrm{s}}_{A[-1]})\xrightarrow{(i_0)_*}(\mathcal{D}^{\mathrm{p}}_0(\mathbb{P}^1_A),\Qoppa^{\mathrm{s}}_A)\xrightarrow{q_*((-)\otimes\mathcal{O}(-1))}(\mathcal{D}^{\mathrm{p}}(A),\Qoppa^{\mathrm{s}}_{A[-1]})$$

is homotopic to the identity Poincaré functor. Indeed, this last statement follows from Lemma 5.1.4 since $q\circ i_0=\mathrm{id}$ and $i_0^*\mathcal{O}(-1)\simeq\mathcal{O}_X$. □

Proof of Theorem 8.4.2. By Lemmas 8.4.1 and 8.4.4 it will suffice to show that for every regular embedding of the form $i\colon\ \mathrm{spec}(A/a)\to\mathrm{spec}(A)$ for A a regular Noetherian ring of finite Krull dimension, the map

$$\mu_i\colon\ i_*\,\mathrm{KQ}_{\mathrm{spec}(A/a)}\to\mathrm{KQ}_{A[1]},$$

is adjoint to an equivalence $\mathrm{KQ}_{\mathrm{spec}(A/a)}\to i^!(\mathrm{KQ}_{A[1]})$. Since i_* is fully-faithful, this is equivalent to saying that the functor $i^!$ sends μ_i to an equivalence, and also equivalent to saying that the functor $i_*i^!$ sends μ_i to an equivalence. Let $U:=\mathrm{spec}(A[a^{-1}])\subseteq\mathrm{spec}(A)$ be the open complement of $\mathrm{spec}(A/a)$ in $\mathrm{spec}(A)$ and write $j\colon U\hookrightarrow\mathrm{spec}(A)$ for the inclusion. Then we have fibre sequence of functors

$$i_*i^!\Rightarrow\mathrm{id}\Rightarrow j_*j^*,$$

and so to prove that $i_*i^!$ sends μ_i to an equivalence it will be enough to prove that the sequence

$$i_*\,\mathrm{KQ}_{\mathrm{spec}(A/a)}\xrightarrow{\mu_i}\mathrm{KQ}_{A[1]}\to j_*j^*\,\mathrm{KQ}_{A[1]}=j_*\,\mathrm{KQ}_{U[1]},$$

whose composite admits a unique null homotopy since $j^*i_*\,\mathrm{KQ}_{\mathrm{spec}(A/a)}=0$, is exact. Here we have made implicit use of the equivalence $j^*\,\mathrm{KQ}_{A[1]}=\mathrm{KQ}_{U[1]}$ of Proposition 8.2.1, though this is an elementary case of that proposition since the map $U \to \mathrm{spec}(A)$ is smooth. By Corollary 8.1.5 what we now need to verify is that for every smooth A-scheme X and every integer n the resulting sequence

$$\mathrm{GW}^{\mathrm{s}}(Z, \mathcal{O}_Z[n-1]) \to \mathrm{GW}^{\mathrm{s}}(X, \mathcal{O}_X[n]) \to \mathrm{GW}^{\mathrm{s}}(V, \mathcal{O}_X[n])$$

is exact, where $Z := X \times_{\mathrm{spec}(A)} \mathrm{spec}(A/a)$ and $V := X \times_{\mathrm{spec}(A)} \mathrm{spec}(A[a^{-1}])$. Indeed, this is Devissage for symmetric GW, see Theorem 5.2.1. □

A. Derived categories of schemes

In this appendix, we summarize the ∞-categorical approach to derived categories of schemes. The principal exports which are use in the body of the text are the results of §A.5 concerning perfect complexes, and in particular Corollary A.5.9, which says that the formation of perfect derived categories sends Nisnevich squares to Karoubi squares. The results of §A.6 also play an important part in §5.1.

A.1. **Grothendieck abelian categories and their derived counterparts.** Recall that a Grothendieck abelian category is an abelian category which is presentable and in which the class of monomorphisms is closed under filtered colimits. Any Grothendieck abelian category $\mathcal{A}$ has enough injectives, and one may endow the category $\mathrm{Ch}(\mathcal{A})$ of unbounded (homologically graded) complexes in $\mathcal{A}$ with a model structure in which the weak equivalences are quasi-isomorphisms of complexes, the cofibrations are the levelwise monomorphisms, and every fibrant object is levelwise injective (though in general not every levelwise injective complex is fibrant). We write $\mathcal{D}(\mathcal{A})$ for the underlying ∞-category of this model structure, so that $\mathcal{D}(\mathcal{A})$ is by definition the localisation of $\mathrm{Ch}(\mathcal{A})$ by the collection of quasi-isomorphisms. It can also be realized more explicitly as the dg-nerve of the full subcategory of fibrant objects, see [Lur17a, §1.3.5]. It is always a stable presentable ∞-category, and admits a t-structure which is compatible with filtered colimits, see [Lur17a, Proposition 1.3.5.21].

The injective model structure is convenient for many purposes, but its functorial dependence on $\mathcal{A}$ is rather weak: an adjunction $f^* \dashv f_*$ of Grothendieck abelian categories induces a Quillen adjunction on complexes equipped with injective model structures if and only if the left adjoint f^* is exact, which is a rather strong condition. In the context of the present appendix, the Grothendieck abelian categories we are interested in here are the categories $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-module sheaves over a scheme X. Furthermore, for any morphism $f: X \to Y$ of schemes, we would like to consider the associated pullback-push-forward adjunction $f^* \dashv f_*$, and so we would like that the induced adjunction on the level complexes be a Quillen adjunction. This will not work in general for the injective model structure, since f^* is usually not exact. To surmount this issue one can work with the associated *flat* model structure. Recall that if $\mathcal{A}$ is a symmetric monoidal Grothendieck abelian category then an object $x \in \mathcal{A}$ is called flat if the functor $(-) \otimes x$ is exact. Let us then say that $\mathcal{A}$ is model-flat if $\mathrm{Ch}(\mathcal{A})$ admits a combinatorial model structure whose weak equivalences are the quasi-isomorphisms and the cofibrant objects are those which consist of flat objects. We note that if such a model structure exists then it is unique, since model structures are determined by the weak equivalences and cofibrant objects. In addition, the symmetric monoidal structure on $\mathcal{A}$ induces a symmetric monoidal structure on $\mathrm{Ch}(\mathcal{A})$, and this symmetric monoidal structure is always compatible (essentially by design) with the flat model structure. Given a symmetric monoidal left adjoint functor $f: \mathcal{A} \to \mathcal{B}$ between model-flat Grothendieck abelian categories, if f sends monomorphisms with flat cokernel to monomorphisms with flat cokernel, then the induced functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ is a symmetric monoidal left Quillen functor with respect to the flat model structure on both sides. We may call such f model-flat left functors. Finally, the model category $\mathrm{Ch}(\mathcal{A})$ is stable and comes equipped with a built-in t-structure. This t-structure is multiplicative, that is, the collection of connective objects contains the unit and is closed under tensor products, and for any model-flat left adjoint functor $f: \mathcal{A} \to \mathcal{B}$ the induced left Quillen functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ preserves connective objects (that is, it is right t-exact). The formation of flat model structures then assembles to form a functor

$$\mathrm{SMGro}^\flat \to \mathrm{SMModCat}_{\mathrm{st,t}} \qquad \mathcal{A} \mapsto (\mathrm{Ch}(\mathcal{A}), \mathcal{C}\mathrm{of}^\flat, \mathcal{F}\mathrm{ib}^\flat, \mathrm{QIso}) \tag{44}$$

where $\mathrm{SMGro}^\flat$ is the category whose objects are model-flat symmetric monoidal Grothendieck abelian categories and model-flat symmetric monoidal left functors between them and $\mathrm{SMModCat}_{\mathrm{st,t}}$ is the category of symmetric monoidal stable combinatorial model categories equipped with multiplicative t-structures, and whose morphisms are right t-exact symmetric monoidal left Quillen functors.

Unwinding the definitions, the category $\mathrm{SMModCat}_{\mathrm{st,t}}$ can be identified with the category commutative algebra objects in the colored operad $\mathrm{ModCat}^{\otimes}_{\mathrm{st,t}}$ whose multi-maps $(\mathcal{M}_1, \dots, \mathcal{M}_n) \to \mathrm{N}$ are the left Quillen multi-functors $f : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathrm{N}$ such that $f(x_1, \dots, x_n)$ is connective whenever $x_i \in \mathcal{M}_i$ is connective for every $i = 1, \dots, n$. Here, the null operations $() \to \mathrm{N}$ are by definition the cofibrant connective objects.

On the other hand, recall that the underlying ∞-category of any combinatorial model category is presentable, and that Quillen multi-functors induce multi-functors which preserve colimits in each variable separately. In addition, the underlying ∞-category of a stable model category is stable, and the notion of a t-structure on a stable model category or on its associated ∞-category is the same, since it can be defined purely in terms of the associated (triangulated) homotopy category. In particular, the functor which inverts weak equivalences induces a map of ∞-operads

$$\mathrm{ModCat}^{\otimes}_{\mathrm{st,t}} \to (\mathrm{Pr}^{L}_{\mathrm{st,t}})^{\otimes}, \tag{45}$$

where, on the right hand side, we have the ∞-operad whose objects are the stable presentable ∞-categories with t-structures $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and whose multi-maps

$$((\mathcal{C}_1, (\mathcal{C}_1)_{\geq 0}, (\mathcal{C}_1)_{\leq 0}), \dots, (\mathcal{C}_n, (\mathcal{C}_n)_{\geq 0}, (\mathcal{C}_n)_{\leq 0})) \to (\mathcal{D}, \mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$$

are the functors $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ which preserve colimits in each variable separately and that send $(\mathcal{C}_1)_{\geq 0} \times \cdots \times (\mathcal{C}_n)_{\geq 0}$ to $\mathcal{D}_{\geq 0}$. We note that without the t-structure part, this is exactly the underlying ∞-operad of the symmetric monoidal category of stable presentable ∞-categories, equipped with the (restriction to stable ∞-categories) of the Lurie tensor product of presentable ∞-categories. In fact, it is not hard to verify that the ∞-operad $(\mathrm{Pr}^{L}_{\mathrm{st,t}})^{\otimes}$ is also a symmetric monoidal ∞-category, where the tensor product of $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and $(\mathcal{D}, \mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ is given by $\mathcal{C} \otimes \mathcal{D}$, equipped with the unique t-structure for which the connective objects form the smallest full subcategory containing the image of $\mathcal{C}_{\geq 0} \otimes \mathcal{D}_{\geq 0}$ and closed under colimits and extensions. We also note that commutative algebra objects in $(\mathrm{Pr}^{L}_{\mathrm{st,t}})^{\otimes}$ correspond to stable presentably symmetric monoidal ∞-categories $\mathcal{C}$ equipped with a multiplicative t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, that is, a t-structure such that $\mathcal{C}_{\geq 0}$ contains the unit and is closed under tensor products.

Composing (44) and the functor on commutative algebra objects induced from (45) we now obtain a functor

$$\mathrm{SMGro}^{\flat} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{st,t}}) \qquad \mathcal{A} \mapsto \mathrm{Ch}(\mathcal{A})[\mathrm{QIso}^{-1}] \tag{46}$$

encoding the formation of derived ∞-categories of model-flat symmetric monoidal Grothendieck abelian categories.

A.2. **Derived categories of $\mathcal{O}_X$-modules.** The relevance of the above discussion to the present context is that if X is any scheme then the symmetric monoidal Grothendieck abelian category $\mathrm{Mod}(X)$ of $\mathcal{O}$-modules is model-flat, and for any morphism of schemes $f : X \to Y$, the pullback functor f^* is symmetric monoidal and model-flat [Rec19]. Associating to X the Grothendieck abelian category $\mathrm{Mod}(X)$ and post-composing with the formation of derived ∞-categories (46) we obtain a functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st,t}}) \qquad X \mapsto (\mathcal{D}(X), \mathcal{D}(X)_{\geq 0}, \mathcal{D}(X)_{\leq 0}) \tag{47}$$

with $\mathcal{D}(X) = \mathrm{Ch}(\mathrm{Mod}(X))[\mathrm{QIso}^{-1}]$ with associated t-structure as above.

Remark A.2.1. By [Lur17b, Corollary 2.1.2.3] the ∞-category $\mathcal{D}(X)$ is naturally equivalent to the full subcategory of $\mathrm{Mod}_{\mathcal{O}_S}(\mathrm{Sh}(X_{\text{ét}}, \mathcal{S}\mathrm{p}))$ spanned by those of $\mathcal{O}_X$-module étale spectral sheaves whose underline sheaf is hypercomplete. It seems likely that the whole ∞-category of $\mathcal{O}_X$-module spectral sheaves coincides with the *unseparated derived category* of $\mathrm{Mod}(X)$, which can be modeled as the dg-nerve of all complexes of injectives (as opposed to only fibrant complexes), see [Lur17b, § C.5.0]. This subtlety however becomes irrelevant below when passing to quasi-coherent sheaves, since these are automatically hypercomplete, see [Lur17b, Proposition 2.2.6.1, Corollary 2.2.6.2].

Proposition A.2.2 (Open base change)**.** *If*

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{j} & Y \end{array}$$

is a cartesian square of schemes whose horizontal arrows are open embeddings then the Beck-Chevalley transformation $j^* g_* \Rightarrow i^* f_*$ is an equivalence of functors $\mathcal{D}(X) \to \mathcal{D}(V)$.

Proof. For an open embedding $i : U \hookrightarrow X$ the pullback functor $i^* : \mathrm{Mod}(X) \to \mathrm{Mod}(U)$ has an exact left adjoint, and hence i^* preserves fibrant complexes of module sheaves (with respect to the injective model structure). Using this, the claim reduces to showing to the 1-categorical statement about the corresponding Beck-Chevalley transformation on the level of module sheaves, which is straightforward. □

Proposition A.2.3 (Zariski descent)**.** *Let X be a scheme equipped with a finite open covering $X = \cup_{i=1}^n U_i$ by opens. For every non-empty subsets $S \subseteq \{1, \dots, n\}$ let us denote by $U_S := \cap_{i \in S} U_i$ and $j_S : U_S \hookrightarrow X$ the associated inclusion. Then the functor*

(48) $$\mathcal{D}(X) \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} \mathcal{D}(U_S) \qquad M \mapsto \{j_S^* M\}_S$$

is an equivalence of ∞-categories.

Proof. The collection of push-forward functors j_{S*} assemble to give a right adjoint

$$\lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} \mathcal{D}(U_S) \to \mathcal{D}(X) \qquad \{M_S\}_S \mapsto \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S*} M_S.$$

Since the functors j_S^* are jointly conservative, to show that the adjunction we get is an equivalence it will suffice to show that its counit is an equivalence. For a given family $\{M_S\}_S$, the component of the counit at a given $\emptyset \neq T \subseteq \{1, .., n\}$ is the map

$$j_T^* \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S*} M_S \to M_T$$

adjoint to the projection $\lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S*} M_S \to j_{T*} U_T$. Now since j_T^* preserves finite limits we can rewrite this map as the composite

$$\lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_T^* j_{S*} M_S \to j_T^* j_{T*} M_T \to M_T.$$

By open base change (Proposition A.2.2) the second map is an equivalence and the first is the projection

$$\lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{T,S*} M_{S \cup T} = \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{T,S*} j_{T,S}^* M_S \to M_T$$

where $j_{S,T} : U_{S \cup T} \to U_T$ is the open inclusion of the intersection $U_{S \cup T} = U_S \cap U_T$. Now the claim that this last projection is an equivalence amounts to saying that the punctured cube

(49) $$S \mapsto j_{S,T*} M_{S \cup T} \qquad \emptyset \neq S \subseteq \{1, \dots, n\}$$

in $\mathcal{D}(U_T)$ can be extended to a cartesian cube whose edge relating the limit corner $\emptyset$ to T is an equivalence. Indeed, the formula (49) simply makes sense for any $S \subseteq \{1, \dots, n\}$, including the empty one, and the resulting n-cube has the property that it sends any $S \subseteq S'$ such that $S \cup T = S' \cup T$ to an equivalence. Such an n-cube is always cartesian (see, e.g., [Lur17a, Proposition 6.1.1.13]). □

Remark A.2.4. As explained in the proof of Proposition A.2.3, the functor (A.2.3) is part of an adjunction, and since (A.2.3) is an equivalence the associated unit map is an equivalence. Explicitly, this unit is the map

$$M \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S*} j_S^* M,$$

and so get in particular, that every $M \in \mathcal{D}(X)$ admits a canonical presentation as a finite limit of complexes pushed forward from the various U_S's.

Given a scheme X and a qcqs open subset $U \hookrightarrow X$ with complement $Z = X \setminus U$ we denote by $\mathcal{D}_Z(X) \subseteq \mathcal{D}(X)$ the full subcategory spanned by those $M \in \mathcal{D}_U(X)$ such that $M|_U = 0$.

Remark A.2.5. One often encounters situations where several open subsets are considered simultaneously. For this, it is convenient to point out that if X is a scheme and $U, V \subseteq X$ two qcqs open subsets with closed complements $Z, Y \subseteq X$, respectively, then the restriction functor

$$j^* : \mathcal{D}(X) \to \mathcal{D}(V)$$

along the inclusion $j: V \hookrightarrow X$ sends $\mathcal{D}_Z(X)$ to $\mathcal{D}_{V\cap Z}(V)$, and by open base change (Proposition A.2.2) the push-forward functor

$$j_*: \mathcal{D}(V) \to \mathcal{D}(X)$$

sends $\mathcal{D}_{V\cap Z}(V)$ to $\mathcal{D}_Z(X)$. We conclude that $j^* \dashv j_*$ restricts to an adjunction

$$j^*: \mathcal{D}_Z(X) \rightleftarrows \mathcal{D}_{V\cap Z}(V) : j_*$$

Corollary A.2.6 (Excision)**.** *Let X be a scheme, $U \subseteq X$ an open subscheme with closed complement $Z = X \setminus U$ and V an open subscheme which contains Z. Then the functor*

$$\mathcal{D}_Z(X) \to \mathcal{D}_Z(V),$$

induced by restriction, is an equivalence.

Proof. This is the induced functor on the horizontal fibres of the square

$$\begin{array}{ccc} \mathcal{D}(X) & \longrightarrow & \mathcal{D}(U) \\ \downarrow & & \downarrow \\ \mathcal{D}(V) & \longrightarrow & \mathcal{D}(V\cap U)\,, \end{array}$$

which is cartesian by Proposition A.2.3. □

Remark A.2.7. Suppose given a scheme X, a closed subscheme $Z \subseteq X$, with complement $U = X \setminus Z$ and a finite open covering $X = \cup_{i=1}^n X_i$. Write $U_i = U \cap X_i$ and $Z_i = Z \cap X_i$. We may then apply Zariski descent to $\mathcal{D}^{\mathrm{qc}}_Z(X)$ by placing it in the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_Z(X) & \longrightarrow & \mathcal{D}(X) & \longrightarrow & \mathcal{D}(U) \\ \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ \lim_{\emptyset\neq S\subseteq\{1,\dots,n\}} \mathcal{D}_{Z_i}(X_i) & \longrightarrow & \lim_{\emptyset\neq S\subseteq\{1,\dots,n\}} \mathcal{D}(X_i) & \longrightarrow & \lim_{\emptyset\neq S\subseteq\{1,\dots,n\}} \mathcal{D}(U_i)\,. \end{array}$$

where the rows are fibre sequences and the middle and right most vertical maps are equivalences by Proposition A.2.3. We may then conclude that the left most vertical map is an equivalence as well. In other words, ∞-categories of $\mathcal{O}_X$-module complexes with support also satisfy Zariski descent.

A.3. **Quasi-coherent complexes.** For a scheme X, and a sheaf M on X we write $\Gamma_X(M) = M(X)$ for the global sections of M. In particular $\Gamma(X) = \Gamma_X(\mathcal{O}_X)$ is the ring of global regular functions on X, and if M is an $\mathcal{O}_X$-module sheaf then $\Gamma_X(M)$ is a $\Gamma(X)$-module. We may consequently view Γ_X as a functor

$$\Gamma_X: \mathrm{Mod}(X) \to \mathrm{Mod}(\Gamma(X)).$$

As such, it is compatible with push-forwards, in the sense that for a map $f: X \to Y$ then the square

$$\begin{array}{ccc} \mathrm{Mod}(X) & \xrightarrow{\Gamma_X} & \mathrm{Mod}(\Gamma(X)) \\ \downarrow & & \downarrow \\ \mathrm{Mod}(Y) & \xrightarrow{\Gamma_Y} & \mathrm{Mod}(\Gamma(Y))\,, \end{array}$$

in which the left vertical map is given by push-forward and the right vertical map by the forgetful functor (which simply restricts the action along $\Gamma(Y) \to \Gamma(X)$), is commutative. The functor Γ_X admits a left adjoint

$$\Lambda_X: \mathrm{Mod}(\Gamma(X)) \to \mathrm{Mod}(X)$$

where for an A-module M the sheaf $\Lambda_X(M)$ is such that for any $a \in A$, the sections of $\Lambda_X(M)$ over $\mathrm{spec}(A[f^{-1}])$ are given by $M[f^{-1}] = M \otimes_A A[f^{-1}]$. We note that the fact that Γ_X is compatible with push-forward implies that Λ_X is compatible with pullback. When $X = \mathrm{spec}(A)$ is affine the left adjoint

$$\Lambda_{\mathrm{spec}(A)}: \mathrm{Mod}(A) \to \mathrm{Mod}(\mathrm{spec}(A))$$

is fully-faithful. An $\mathcal{O}_{\mathrm{spec}(A)}$-module is said to be *quasi-coherent* if it is in the essential image of $\Lambda_{\mathrm{spec}(A)}$. For a general scheme X, we say that an $\mathcal{O}_X$-module is quasi-coherent if its restriction along every map of the form $\mathrm{spec}(A) \to X$ is quasi-coherent in the above sense. The compatibility of Λ_X with pullbacks implies that this definition indeed extends the definition for affine schemes. We say that a complex $M \in \mathcal{D}(X)$ is *quasi-coherent* if all its cohomology sheaves are quasi-coherent, and we write $\mathcal{D}^{qc}(X) \subseteq \mathcal{D}(X)$ for the full subcategory spanned by the quasi-coherent complexes. This is a stable subcategory which is closed under colimits.

Remark A.3.1. Being quasi-coherent is a local property. In particular, a complex $M \in \mathcal{D}(X)$ is quasi-coherent if and only if its restriction to each open subset $U \subseteq X$ is quasi-coherent, and if and only if there exists some open cover by $X = \cup_i U_i$ such that each $M|_{U_i}$ is quasi-coherent.

Remark A.3.2. The adjunction $\Lambda_X \dashv \Gamma_X$ of Grothendieck abelian categories induces a Quillen adjunction

$$\mathrm{Ch}(\mathrm{Mod}(\Gamma(X))) \underset{\perp}{\rightleftarrows} \mathrm{Ch}(\mathrm{Mod}(X)),$$

on the model categorical level (where we can either endow both with the flat model structure or with the injective one), and hence an adjunction

$$\mathcal{D}(\Gamma(X)) \underset{\perp}{\rightleftarrows} \mathcal{D}(X)$$

on the level of derived ∞-categories. The essential image of the left adjoint is then $\mathcal{D}(\mathrm{Mod}^{qc}(A)) \simeq \mathcal{D}^{qc}(\mathrm{spec}(A))$ (see Remark A.3.3((5)) below).

Remark A.3.3. The ∞-subcategory $\mathcal{D}^{qc}(X)$ inherits from $\mathcal{D}(X)$ a t-structure such that the inclusion $\mathcal{D}^{qc}(X) \subseteq \mathcal{D}(X)$ is t-exact. This t-structure has the following properties:
(1) The heart of $\mathcal{D}^{qc}(X)$ is the ordinary category $\mathrm{Mod}^{qc}(X)$ of quasi-coherent $\mathcal{O}_X$-modules.
(2) Pullbacks along flat maps (and in particular open inclusions) are t-exact.
(3) When $X = \mathrm{spec}(A)$ is affine the t-structure on $\mathcal{D}^{qc}(X) = \mathcal{D}(A)$ coincides with the usual t-structure on A-complexes.
(4) If X is qcqs then the t-structure on $\mathcal{D}^{qc}(X)$ is left complete (this follows by Zariski descent for quasi-coherent complexes and the fact that the t-structure on $\mathcal{D}^{qc}(\mathrm{spec}(A)) = \mathcal{D}(A)$ is left complete for any ring A).
(5) When X is quasi-compact and separated the canonical functor $\mathcal{D}(\mathrm{Mod}^{qc}(X)) \to \mathcal{D}^{qc}(X)$ is an equivalence (this is the Bökstedt-Neeman theorem).

For any map of schemes $f : X \to Y$ the pullback functor

$$f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$$

sends quasi-coherent complexes to quasi-coherent complexes. Indeed, by Remark A.3.1 it suffices to prove this in the case where both X and Y are affine, in which case it is a consequence of Serre's affine theorem (see Remark A.3.2). See also [LH09, Proposition 3.9.1] for a more homological algebra proof.

We shall now restrict our attention from general to quasi-compact quasi-coherent schemes (qcqs for short). In this setting, the push-forward functor also preserves quasi-coherent complexes, see, e.g., [LH09, Proposition 3.9.2]). For the convenience of the reader we provide a proof below in the present language.

Proposition A.3.4. *Let $f : X \to Y$ be a qcqs morphism of schemes (e.g., f is any map between qcqs schemes). Then*

$$f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$$

sends quasi-coherent complexes to quasi-coherent complexes.

In particular, for f qcqs the pullback-push-forward adjunction restricts to an adjunction

$$f^* : \mathcal{D}^{qc}(Y) \underset{\perp}{\rightleftarrows} \mathcal{D}^{qc}(X) : f_*.$$

Since the full subcategory $\mathcal{D}^{qc}(X) \subseteq \mathcal{D}(X)$ of quasi-coherent sheaves is closed under tensor products ([SP, Tag 01CE]) it inherits from $\mathcal{D}(X)$ its symmetric monoidal structure such that all pullback functors are symmetric monoidal. In light of Remark A.3.3 and the discussion preceding it, we get that the functor (47) refines to a functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st,t}}) \qquad X \mapsto (\mathcal{D}^{qc}(X), \mathcal{D}^{qc}(X)_{\geq 0}, \mathcal{D}^{qc}(X)_{\leq 0}). \tag{50}$$

Corollary A.3.5. *When all schemes involved are qcqs, the properties of open base change (Proposition A.2.2), Zariski descent (Proposition A.2.3) and Excision (Corollary A.2.6) hold verbatim if one replaces everywhere $\mathcal{D}(-)$ by $\mathcal{D}^{qc}(-)$.*

The proof of Proposition A.3.4 uses the following type of argument, which we later use again several times:

Lemma A.3.6. *Suppose that P is a property of schemes which is local in the following sense: a scheme X which admits a finite covering $X = \cup_{i=1}^n X_i$ has property P if $X_S = \cap_{i\in S} X_i$ has property P for every $\emptyset \neq S \subseteq \{1,\ldots,n\}$. Then every qcqs scheme has property P if every affine scheme has property P.*

Proof. Any qcqs scheme X admits a finite open covering $X = \cup_i X_i$ such that each X_i is affine and each intersection X_S is quasi-compact. If X is furthermore *separated*, then the open affines in X are closed under intersection and hence each X_S is actually affine. We conclude that each quasi-compact separated scheme has property P. Now if X is quasi-compact and quasi-separated and $X = \cup_i X_i$ is a finite affine open covering then each X_i is in particular separated and hence each X_S is also separated, since being separated is inherited to open subsets. Hence each X_S is quasi-compact and separated, so has property P by the above. We conclude that X has property P as well. □

Proof of Proposition A.3.4. Being a quasi-coherent complex on Y is a property that we can test by restricting to all affine open subschemes of Y. By open base change we may thus assume without loss of generality that Y is affine. Let us say that a scheme X is *good* if the statement of the proposition holds for X for any qcqs map $f : X \to Y$ with affine target. We note that if X admits a qcqs map to an affine scheme then X itself is qcqs, and on the other hand, if X is qcqs then every map to an affine scheme is qcqs. Our goal is now to show that all qcqs schemes are good. Since the collection of quasi-coherent sheaves on Y is closed under finite limits, Remark A.2.4 implies the following: if X admits a finite open covering $X = \cup_{i=1}^n X_i$ such that for every $\emptyset \neq S \subseteq \{1,\ldots,n\}$ the scheme $X_S = \cap_{i\in S} X_i$ is good then X itself is good. Invoking Lemma A.3.6 it will hence suffice to show that any affine scheme is good. For this, we will show that for any map of commutative rings $A \to B$ (associated to a map of affine schemes $f : \mathrm{spec}(B) \to \mathrm{spec}(A)$), and every B-module complex $M \in \mathcal{D}(B)$ the Beck-Chevalley map

$$\Lambda_{\mathrm{spec}(A)}(M) \to f_*\Lambda_{\mathrm{spec}(B)}(M) \tag{51}$$

is an equivalence in $\mathrm{Mod}(\mathrm{spec}(A))$ (where $\Lambda_{\mathrm{spec}(A)}$ and $\Lambda_{\mathrm{spec}(B)}$ are the (derived) left adjoints of $\Gamma_{\mathrm{spec}(A)}$ and $\Gamma_{\mathrm{spec}(B)}$, respectively, see discussion above). Now $\Lambda_{\mathrm{spec}(A)}, \Lambda_{\mathrm{spec}(B)}$ and the forgetful functor $\mathcal{D}(B) \to \mathcal{D}(A)$ are induced on derived categories by exact functors of abelian categories, while the push-forward forward functor $\mathrm{Mod}(\mathrm{spec}(A)) \to \mathrm{Mod}(\mathrm{spec}(B))$ is exact on quasi-coherent sheaves by Serre's vanishing theorem. Passing to homotopy sheaves, it will hence suffice to show that the above Beck-Chevalley map is an isomorphism on the level of sheaves. For this, we can check that the map induces an equivalence on sections over opens of the form $\mathrm{spec}(A[g^{-1}])$ for $g \in A$. Then $\mathrm{spec}(B) \times_{\mathrm{spec}(A)} \mathrm{spec}(A[g^{-1}]) = \mathrm{spec}(B \otimes_A A[g^{-1}])$ and the map induced by (51) on sections over $\mathrm{spec}(A[g^{-1}])$ is just the isomorphism

$$M \otimes_A A[g^{-1}] \xrightarrow{\cong} M \otimes_B B \otimes A[g^{-1}].$$ □

In the setting of quasi-coherent sheaves, the open base change property of Proposition A.2.2 can be extended to *flat base change*:

Proposition A.3.7 (Flat base change)**.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{q} & Y \end{array}$$

be a cartesian square of qcqs schemes such that either f or q is flat. Then the Beck-Chevalley transformation $q^*g_* \Rightarrow p^*f_*$ is an equivalence of functors $\mathcal{D}(X) \to \mathcal{D}(W)$.

Proof. We reduce to the case where all schemes in the above square are affine as follows. First, we reduce to the case where both W and Y are affine. For this, let us pick an open affine covering $Y = \cup_{i=1}^n Y_i$. Let us write $X_i = X \times_Y Y_i$, $W_i = W \times_Y Y_i$ and $Z_i = Z \times_Y Y_i$. For each $i = 1,\ldots,n$ let us pick an open affine covering $W_i = \cup_{j=1}^m W_{i,j}$ and write $Z_{i,j} = W_{i,j} \times_{Y_i} X_i$. For a given $M \in \mathcal{D}^{\mathrm{qc}}(Y)$, the Beck-Chevalley map $\eta : q^*g_*(M) \Rightarrow p^*f_*(M)$ is an equivalence in $\mathcal{D}^{\mathrm{qc}}(W)$ if and only if $\eta|_{W_{i,j}}$ is an equivalence $\mathcal{D}^{\mathrm{qc}}(W_{i,j})$

for every $i = 1, \dots, n$. But by open base change we have that $\eta|_{W_{i,j}}$ is the Beck-Chevalley transformation $p_i^* f_{i*}(M|_{X_i}) \to (p_i^*) f_{i*}(M|_{X_i})$ for the cartesian square

$$\begin{array}{ccc} Z_{i,j} & \xrightarrow{p_{i,j}} & X_i \\ {\scriptstyle f_{i,j}}\downarrow & & \downarrow{\scriptstyle g_i} \\ W_{i,j} & \xrightarrow{q_{i,j}} & Y_i \, . \end{array}$$

In addition, if the map $q\colon W \to Y$ was flat then the map $q_{i,j}\colon W_{i,j} \to Y_i$ is flat as well, being the composite of the open embedding $W_{i,j} \to W_i$ and the base change $W_i \to Y_i$ of q along $Y_i \subseteq Y$. We may hence assume without loss of generality that W and Y are affine. Let us therefore write $Y = \operatorname{spec}(A)$ and $W = \operatorname{spec}(B)$, so that we have a square of the form

$$\begin{array}{ccc} X_B & \xrightarrow{p} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{spec}(B) & \xrightarrow{q} & \operatorname{spec}(A) \end{array}$$

Let us now say that X is *good* if the flat base change statement holds for any square as above whenever either $\operatorname{spec}(B) \to \operatorname{spec}(A)$ or $X \to \operatorname{spec}(A)$ are flat. We claim that if X admits a finite open covering $X = \cup_i X_i$ such that for any $\emptyset \neq S \subseteq \{1, \dots, n\}$ the open subscheme $X_S = \cap_{i \in S} X_i$ is good then X itself is good. To see this, let $M \in \mathcal{D}^{\mathrm{qc}}(X)$ be an quasi-coherent complex and write $M_S = M|_{X_S}$. By Remark A.2.4 the map

$$M \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S*} M_S$$

is an equivalence, where $j_S\colon X_S \hookrightarrow X$ is the inclusion. Since the functors $q^* g_*$ both commutes with finite limits we consequently have

$$q^* g_* M = \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} q^* g_* j_{S*} M_S.$$

Let $X_{S,B} = X_S \times_X X_B$ and $p_{S,B}\colon X_{S,B} \to X_S$ and $j_{S,B}\colon X_{S,B} \to X_B$ the two associated maps. Applying Remark A.2.4 to X_B we get that the map

$$p^* M \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S,B*} M_{S,B} = \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} j_{S,B*} p_{S,B}^* M_S$$

is also an equivalence, where $M_{S,B} = (p^* M)_{X_{S,B}}$, and hence

$$f_* p^* M = \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} f_* j_{S,B*} p_{S,B}^* M_S.$$

The Beck-Chevalley transformation for $q^* g_* M \to f_* p^* M$ can then be identified with the map

$$\lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} q^* g_* j_{S*} M_S \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} f_* j_{S,B*} p_{S,B}^* M_S$$

induced on limits by the Beck-Chevalley transformation of $M_S \in \mathcal{D}^{\mathrm{qc}}(X_S)$ associated to the cartesian square

$$\begin{array}{ccc} X_{S,B} & \xrightarrow{p} & X_S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{spec}(B) & \xrightarrow{q} & \operatorname{spec}(A) \, . \end{array}$$

We also note that if $X \to \operatorname{spec}(A)$ is flat then $X_S \to \operatorname{spec}(A)$ is flat as well. We conclude that if each X_S is good then X is good. By Lemma A.3.6 it will hence suffice to show that any affine scheme is good. In particular, we need to show that the Beck-Chevalley transformation is an equivalence for squares of the form

$$\begin{array}{ccc} \operatorname{spec}(B \otimes_A C) & \xrightarrow{p} & \operatorname{spec}(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{spec}(B) & \xrightarrow{q} & \operatorname{spec}(A) \, . \end{array}$$

where either B or C are flat over A. Explicitly, we need to show that in this case for every C-module M, the map

$$q^*g_*M = B \otimes^L_A M \to (B \otimes_A C) \otimes^L_C M = f_*p^*M$$

is an equivalence. Since both sides preserves colimits it is enough to check this for $M = C$, where it becomes the map

$$B \otimes^L_A C \to B \otimes_A C$$

form the derived to the underived tensor product over A. This map is indeed an equivalence whenever either B or C are flat.

□

A.4. **Flat excision.** Our goal in the present subsection is to prove the following proposition, which we will use to establish Corollary A.5.9 below.

Proposition A.4.1 (Flat excision)**.** *Let X be a qcqs scheme, $f : V \to X$ a flat qcqs map and $Z \subseteq X$ a finitely presented closed subscheme such the induced map $V \times_X Z \to Z$ is an isomorphism. Set $U = X \setminus Z$ and $W = V \times_X W$. Then the square*

$$\begin{array}{ccc} \mathcal{D}^{qc}(X) & \longrightarrow & \mathcal{D}^{qc}(V) \\ \downarrow & & \downarrow \\ \mathcal{D}^{qc}(U) & \longrightarrow & \mathcal{D}^{qc}(W) \end{array}$$

is cartesian.

Remark A.4.2.

(1) In Proposition A.4.1, the condition that Z is finitely presented implies that U (and hence W) are quasi-compact, and hence qcqs.
(2) On the other hand, if $U \subseteq X$ is any qcqs open subscheme then we can always choose a finitely presented subscheme structure on the closed complement Z of U. The assumption that $f : V \to X$ is flat insures that the condition that $V \times_X Z \to Z$ is an isomorphism does not depend on the choice of a finitely presented subscheme structure on Z.

The proof of Proposition A.4.1 will require a few preliminaries. We first establish the following more general observation:

Lemma A.4.3. *Let X be a qcqs scheme $f : V \to X$ a flat qcqs map and $Z \hookrightarrow X$ a closed subscheme with qcqs complement $j : U \hookrightarrow X$. Set $W = f^{-1}(U)$ and $Y = f^{-1}(Z)$. Assume that the map $U \coprod V \to X$ is surjective. Then the following statements are equivalent:*

(1) The square

$$\begin{array}{ccc} \mathcal{D}^{qc}(X) & \xrightarrow{f^*} & \mathcal{D}^{qc}(V) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ \mathcal{D}^{qc}(U) & \xrightarrow{f^*} & \mathcal{D}^{qc}(W) \end{array}$$

is cartesian, where, by abuse of notation, we denote the base change of f again by f and the base change of j again by j.

(2) The functor $\mathcal{D}^{qc}_Z(X) \to \mathcal{D}^{qc}_Y(V)$ induced by f^* is an equivalence.

In addition, of f is affine then these conditions are also equivalent to:

(3) The functor $\mathcal{D}^{qc}_Z(X) \to \mathcal{D}^{qc}_Y(V)$ induced by f^* is fully-faithful.

Proof. We note that by definition, the functor appearing in (2) is just the functor induced on vertical fibres by the square in (1). Hence (1) ⇒ (2) and the implication in the other direction follows from [CDH+II, Lemma 1.5.3] since j^* is a (right split) Verdier projection and the square is right adjointable by flat base change. On the other hand, if f is affine then the functor $\mathcal{D}^{qc}_Y(V) \to \mathcal{D}^{qc}_Z(X)$ induced by f_* is conservative (since f_* is), and hence the adjunction $\mathcal{D}^{qc}_Z(X) \rightleftarrows \mathcal{D}^{qc}_Y(V)$ is an equivalence if and only if its left adjoint (induced by f^*) is fully-faithful. □

Lemma A.4.4. *Let X be a scheme and $i : Z \hookrightarrow X$ a finitely presented closed subscheme with open complement $U \subseteq X$. If $M \in \mathcal{D}^{\mathrm{qc}}(X)$ is such that M restricts to 0 on both U and Z then $M = 0$.*

Proof. Testing the triviality of M can be done locally, so we may as well assume that $X = \mathrm{spec}(A)$ for A a ring. In this case Z is affine as well, and corresponds to $\mathrm{spec}(A/I)$ for some finitely generated ideal $I = \langle f_1, \dots, f_n\rangle \subseteq A$. If $I = 0$ then $Z = X$ and the claim is trivially true, so we may assume $n \geq 1$ with each $f_j \neq 0$.

Let $\mathcal{E} \subseteq \mathcal{D}(A)$ be the full (stable) subcategory spanned by those A-complexes N such that $N \otimes_A M = 0$. For $i = 1, \dots, n$ let $E_j = \mathrm{cof}[f_j : A \to A] \in \mathcal{D}(A)$ and $E = E_1 \otimes_A \cdots \otimes_A E_n$. We then claim that $\mathcal{E}$ contains E. To see this, we note that $\mathcal{E}$ contains i_*N for every $N \in \mathcal{D}(A/I)$, since $M \otimes i_*N = i_*(i^*M \otimes N) = 0$. Now since E is connective and n-truncated we may prove by induction on m that $\tau_{\leq m}E$ is in $\mathcal{E}$. At each point in the induction (including the base), it will suffice to show that $\pi_m E[m]$ is in $\mathcal{E}$. Unwinding the definitions, we see that each of the A-modules $\pi_m E[m]$ admits a filtration whose successive quotients are obtained from A by iteratively taking either the kernel or cokernel of f_i for each i; these successive quotients are in particular A/I-modules, and so we conclude that $\mathcal{E}$ contains E.

We now prove by descending induction on j that $\mathcal{E}$ contains $E_1 \otimes_A \cdots \otimes_A E_j$ for every $j = 0, \dots, n$ (so that, for $j = 0$ we get that $\mathcal{E}$ contains A itself and hence $M = 0$). Indeed, let $M' = M \otimes_A E_1 \otimes_A \cdots \otimes_A E_{j-1}$ and suppose that $M' \otimes_A E_j = \mathrm{cof}[f_j : M' \to M'] = 0$. Then f_j acts invertibly on M', and so M' is pushed forward from $U_j = \mathrm{spec}(A[f_j^{-1}]) \subseteq \mathrm{spec}(A)$. But U_j is contained in U, and so $M'|_{U_j} = 0$, so that $M' = 0$. The proof is hence finished. □

Proof of Proposition A.4.1. Since $f : V \to X$ is flat and both X and V are qcqs we may find open coverings $X = \cup_{i=1}^n X_i$ and $V = \cup_{i=1}^n V_i$ such that for each $i = 1, \dots, n$ the schemes X_i and V_i are affine, $f(V_i) \subseteq X_i$ and the induced map $V_i \to X_i$ is flat. Write $Z_i = V_i \times_X Z$, which we can view as an open subset of Z_i (since $V \times_X Z \to Z$ is an isomorphism by assumption). Set $Y_i = (Z \cap X_i) \setminus Z_i$, so that Y_i can be considered as a closed subscheme of Z. We note that Z_i is affine, being a closed subset of the affine scheme V_i. In particular, Z_i is quasi-compact. We may consequently find a finite collection of affine opens $X_{i,1}, \dots, X_{1,m} \subseteq X_i \setminus Y_i$ such that $Z_i \subseteq \cup_j X_{1,j}$. Set $V_{i,j} = V_i \times_{X_j} X_{i,j}$. Then $V_{i,j} = V_i \times_{X_j} X_{i,j}$ is a fibre product of affine schemes, and is hence affine. Furthermore, the map $V_{i,j} \to X_{i,j}$ is a base change of the map $V_i \to X_i$, and is hence flat. Finally, by construction we have that $V_{i,j} \times_X Z \to X_{i,j} \times_X Z$ is an isomorphism. Let $X^\circ = \cup_{i,j} X_{i,j}$ and $V^\circ = \cup_{i,j} V_{i,j}$. We note that X° contains Z and $V^\circ = V \times_X X^\circ$ contains $V \times_X Z \cong Z$. Let $\mathcal{I} = \{1, \dots, n\} \times \{1, \dots, m\}$ be the set of all pairs of indices (i, j) with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. For each $\emptyset \neq S \subseteq \mathcal{I}$ let us denote by $X_S = \cap_{(i,j)\in S} X_{i,j}$ and $V_S = \cap_{(i,j)\in S} V_{i,j}$. We may then consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{qc}}_Z(X) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}_{V\times_X Z}(V) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{D}^{\mathrm{qc}}_Z(X^\circ) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}_{V\times_X Z}(V^\circ) \\
\downarrow \simeq & & \downarrow \simeq \\
\lim_{\emptyset\neq S\subseteq\mathcal{I}} \mathcal{D}^{\mathrm{qc}}_{X_S\times_X Z}(X_S) & \longrightarrow & \lim_{\emptyset\neq S\subseteq\mathcal{I}} \mathcal{D}^{\mathrm{qc}}_{V_S\times_X Z}(V_S)
\end{array}$$

where the vertical arrows are equivalences by Zariski descent (see Remark A.2.7) and excision (Corollary A.2.6). To prove Proposition A.4.1 for X, V and Z it will hence suffice to so for each of $X_{i,j}, V_{i,j}$ and $X_{i,j} \times_X Z$. Since each $X_{i,j}$ is affine we have that each W_S is separated, and hence it will suffice to prove Proposition A.4.1 under the addition assumption that X and V are separated. On the other hand, if X and V are separated then each of the $X_{i,j}$ and $V_{i,j}$ are affine, and hence it will suffice to prove Proposition A.4.1 under the additional assumption that X and V are affine.

Let us then write $X = \mathrm{spec}(A)$ and $V = \mathrm{spec}(B)$ where A is a noetherian ring and B is a flat A-algebra. Let $I \subseteq A$ be the ideal corresponding to the closed set $Z = \mathrm{spec}(A) \setminus U$. Then condition that $\mathrm{spec}(B) \setminus f^{-1}(U) \to Z$ is an isomorphism can then be rephrased as saying that the map $A/I \to B \otimes_A A/I$ is an isomorphism. This means in particular that $\mathrm{cof}[A \to B] \otimes_A A/I = 0$, so that $\mathrm{cof}[A \to B]$ restricts to zero on $Z = \mathrm{spec}(A/I)$. Using criterion (3) of Lemma A.4.3, what we need to show in this case is that the functor

$$\mathcal{D}^{\mathrm{qc}}_Z(\mathrm{spec}(A)) \to \mathcal{D}^{\mathrm{qc}}_Z(\mathrm{spec}(B))$$

is fully-faithful. Concretely, this means that if $M \in \mathcal{D}^{\mathrm{qc}}(\mathrm{spec}(A)) = \mathcal{D}(A)$ is such that $j^*M = 0$ then the map $M \to M \otimes_A B$ is an equivalence. Equivalently, the object $C = \mathrm{cof}[M \to M \otimes_A B] = M \otimes_A \mathrm{cof}[A \to B]$ is zero. Indeed, C is a tensor product of an object M which restricts to 0 on U and an object $\mathrm{cof}[A \to B]$ which restricts to zero on Z. Since restriction functors are symmetric monoidal we conclude that C restricts to 0 on both U and Z, and hence $C = 0$ by Lemma A.4.4. □

A.5. **Perfect complexes.** We now proceed to discuss compact objects in $\mathcal{D}^{\mathrm{qc}}(X)$, which are our primary objects of interest. We begin with the following useful statement, which follows relatively directly form Zariski descent:

Proposition A.5.1. *Let $f : X \to Y$ be a map of qcqs schemes. Then the push-forward functor $f_* : \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(Y)$ preserves filtered colimits (and hence all colimits).*

Proof. A right adjoint whose target is a compactly generated category preserves filtered colimits if and only if its left adjoint preserves compact objects. Since equivalences of sheaves can be tested locally and pullback functors preserve colimit we may use open base change to reduce to the case where Y is affine. In this case, $\mathcal{O}_Y$ is a compact generator of $\mathcal{D}^{\mathrm{qc}}(Y)$, hence it will suffice to show that $f^*\mathcal{O}_Y = \mathcal{O}_X$ is a compact object of $\mathcal{D}^{\mathrm{qc}}(X)$. Let us say that a qcqs scheme X is *good* if $\mathcal{O}_X$ is compact in $\mathcal{D}^{\mathrm{qc}}(X)$. Now by Zariski descent (see Corollary A.3.5) we get that if we can find a finite covering $X = \cup_{i=1}^n U_i$ by qcqs schemes such that for each $\emptyset \neq S \subseteq \{1, \dots, n\}$ the intersection $U_S = \cap_{i \in S} U_i$ is good then X itself is good. On the other hand, we know that affine schemes are good. The desired result hence follows from Lemma A.3.6. □

Corollary A.5.2. *Let $f : X \to Y$ be a map of qcqs schemes. Then the pullback functor $f^* : \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(Y)$ preserves compact objects.*

Lemma A.5.3. *Let X be a qcqs scheme and $M \in \mathcal{D}^{\mathrm{qc}}(X)$ be a quasi-coherent complex. Then the following conditions are equivalent:*

(1) M is a compact object of $\mathcal{D}^{\mathrm{qc}}(X)$.
(2) For every affine $\mathrm{spec}(A) \subseteq X$ the restricted complex $M|_{\mathrm{spec}(A)}$ is compact in $\mathcal{D}^{\mathrm{qc}}(\mathrm{spec}(A))$.
(3) There exists a finite covering $X = \cup_{i=1}^n \mathrm{spec}(A_i)$ by affines such that for every $i = 1, \dots, n$ the restricted complex $M|_{\mathrm{spec}(A_i)}$ is compact in $\mathcal{D}^{\mathrm{qc}}(\mathrm{spec}(A_i))$ for every $i = 1, \dots, n$.
(4) There exists a finite covering $X = \cup_{i=1}^n \mathrm{spec}(A_i)$ by affines such that for every $i = 1, \dots, n$ the A_i-module complex $M(\mathrm{spec}(A)) \in \mathcal{D}(A_i)$ is quasi-isomorphic to a bounded complex of projective A_i-modules.

Proof. The implication (1) ⇒ (2) follows from pullback functors preserving compact objects (Corollary A.5.2), and the implication (2) ⇒ (3) is because any qcqs X admits a finite covering by affines. In addition, (3) and (4) are equivalent by the affine Serre theorem (Remark A.3.2) and the standard description of compact objects in module categories. We now show that (3) ⇒ (1). For this, let us fix a finite covering $X = \cup_{i=1}^n U_i$ by affine schemes and an $M \in \mathcal{D}^{\mathrm{qc}}(X)$ whose restriction to each U_i is compact. For every $\emptyset \neq S \subseteq \{1, \dots, n\}$ let $U_S = \cap_{i \in S} U_i$. Since $M|_{U_i}$ is compact in $\mathcal{D}(U_i)$ for every i we get from Corollary A.5.2 that $M|_{U_S}$ is compact in $\mathcal{D}^{\mathrm{qc}}(U_S)$ for every $\emptyset \neq S \subseteq \{1, \dots, n\}$. But by Zariski descent we that

$$\mathcal{D}^{\mathrm{qc}}(X) = \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} \mathcal{D}^{\mathrm{qc}}(U_S)$$

and so M is compact in $\mathcal{D}^{\mathrm{qc}}(X)$. □

Definition A.5.4. A quasi-coherent complex $M \in \mathcal{D}^{\mathrm{qc}}(X)$ is *perfect* if it satisfies the equivalent conditions of Lemma A.5.3. We denote by $\mathcal{D}^{\mathrm{p}}(X) \subseteq \mathcal{D}^{\mathrm{qc}}(X)$ the full subcategory spanned by the perfect complexes. For any map $f : X \to Y$ between qcqs schemes the pullback functor restricts to a functor $f^* : \mathcal{D}^{\mathrm{p}}(Y) \to \mathcal{D}^{\mathrm{p}}(X)$, which we denote identically.

By the fourth characterization in Lemma A.5.3 we see that $\mathcal{D}^{\mathrm{p}}(X)$ is closed in $\mathcal{D}^{\mathrm{qc}}(X)$ under tensor products and so $\mathcal{D}^{\mathrm{p}}(X)$ inherits from $\mathcal{D}(X)$ its symmetric monoidal structure, and the functor (50) then refines to a functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}) \qquad X \mapsto (\mathcal{D}^{\mathrm{p}}(X), \mathcal{D}^{\mathrm{qc}}(X)_{\geq 0}, \mathcal{D}^{\mathrm{qc}}(X)_{\leq 0}) \tag{52}$$

where $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}}$ and its monoidal enhancement $(\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{t}})^{\otimes}$ come from Definition 3.2.1.

Remark A.5.5. Observing the equivalent conditions of Lemma A.5.3 that the property of being perfect is local in the following sense: for any open covering $X = \cup_i U_i$ we have that a given $M \in \mathcal{D}^{\mathrm{qc}}(X)$ is perfect if and only if $M|_{U_i} \in \mathcal{D}^{\mathrm{qc}}(U_i)$ is perfect for every i.

Remark A.5.6. If X is qcqs then any $M \in \mathcal{D}^{\mathrm{qc}}(X)_{\geq 0}$ can be written as a filtered colimits of objects in $\mathcal{D}^{\mathrm{p}}(X)_{\geq 0} = \mathcal{D}^{\mathrm{p}}(X) \cap \mathcal{D}^{\mathrm{qc}}(X)_{\geq 0}$, see [Lur17b, 9.6.1.2]. It then follows that $\mathcal{D}^{\mathrm{qc}}(X)_{\geq 0} = \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}(X)_{\geq 0})$ in this case.

Now let X be a qcqs scheme and $U \subseteq X$ a qcqs open subset. We will say that $M \in \mathcal{D}^{\mathrm{qc}}_Z(X)$ is perfect if it is perfect when considered as an object of $\mathcal{D}^{\mathrm{qc}}(X)$. We then similarly denote by $\mathcal{D}^{\mathrm{p}}_Z(X) \subseteq \mathcal{D}^{\mathrm{qc}}_Z(X)$ the full subcategory spanned by the perfect complexes. In particular, we have a fibre sequence

$$\mathcal{D}^{\mathrm{p}}_Z(X) \to \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U) \tag{53}$$

of stable ∞-categories. More generally, if $V \subseteq X$ is another qcqs open subset with closed complement $Y = X \setminus V$, then we have a fibre sequence

$$\mathcal{D}^{\mathrm{p}}_{Z\cap Y}(X) \to \mathcal{D}^{\mathrm{p}}_Y(X) \to \mathcal{D}^{\mathrm{p}}_{Y\cap U}(U) \tag{54}$$

Remark A.5.7. Since being perfect is a local property the Zariski descent and excision properties (see Corollary A.3.5) are inherited by perfect complexes. In particular, if $X = V \cup U$ with V, U qcqs and $Z := X \setminus U \subseteq V$ then the square

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{p}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(U) \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{p}}(V) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(U \cap V) \end{array}$$

is cartesian and the functor $\mathcal{D}^{\mathrm{p}}_Z(X) \to \mathcal{D}^{\mathrm{p}}_Z(V)$ is an equivalence.

Theorem A.5.8. *Let X be a qcqs scheme. Then the following holds:*

(1) For every closed subscheme $Z \subseteq X$ with qcqs complement U we have that $\mathcal{D}^{\mathrm{qc}}_Z(X)$ is generated under filtered colimits by $\mathcal{D}^{\mathrm{p}}_Z(X)$.

(2) For every pair of closed subschemes $Z, Y \subseteq X$ with qcqs complements U, V respectively the sequence (54) is a Karoubi sequence.

Corollary A.5.9. *Let X be a qcqs scheme, $f : V \to X$ a qcqs étale map and $U \subseteq X$ a qcqs open subset such that the induced map $V \setminus f^{-1}(U) \to X \setminus U$ is an isomorphism. Then*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{p}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(V) \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{p}}(U) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(f^{-1}(U)) \end{array}$$

is a Karoubi square of stable ∞-categories.

Proof. By Theorem A.5.8 the vertical arrows are Karoubi projections, and hence it will suffice to show that the square is cartesian. Since the analogous square for quasi-coherent complexes is cartesian by Proposition A.4.1, it will suffice to show that a quasi-coherent complex $M \in \mathcal{D}^{\mathrm{qc}}(X)$ is perfect if and only if its restriction to U, V and $f^{-1}(U)$ is perfect. Indeed, the only if direction is because pullback functors preserve perfect complexes, and on the other hand, if the restrictions of M to U, V and $f^{-1}(U)$ are all perfect and the image of M under the equivalence

$$\mathcal{D}^{\mathrm{qc}}(X) \xrightarrow{\simeq} \mathcal{D}^{\mathrm{qc}}(U) \times_{\mathcal{D}^{\mathrm{qc}}(f^{-1}(U))} \mathcal{D}^{\mathrm{qc}}(V)$$

is compact, and hence M itself is compact. □

We prove Theorem A.5.8 in several steps.

Lemma A.5.10. *For a fixed qcqs X, Claim (1) of Theorem A.5.8 implies Claim (2).*

Proof. Assume (1) holds for X and let us prove that (2) holds as well. Since (54) is a fibre sequence it will suffice to show that its second map is a Karoubi projection. By excision for perfect complexes (see Remark A.5.7) we may identify the last term in (54) as $\mathcal{D}^{\mathrm{p}}_{U\cap Y}(U) = \mathcal{D}^{\mathrm{p}}_{U\cap Y}(U \cup V) = \mathcal{D}^{\mathrm{p}}_{(U\cup V)\cap Y}(U \cup V)$. Replacing U with $U \cup V$ we may hence assume without loss of generality that $V \subseteq U$ and $Z \subseteq Y$. Now in the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}^{\mathrm{p}}_{Y}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}_{Y\cap U}(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{p}}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(U) & \longrightarrow & \mathcal{D}^{\mathrm{p}}(V) \end{array}$$

the right square and external rectangle are cartesian, and hence the left square is cartesian as well. We conclude that the functor $\mathcal{D}^{\mathrm{p}}_{Y}(X) \to \mathcal{D}^{\mathrm{p}}_{Y\cap U}(U)$ is base changed from $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$. Since Karoubi projections are closed under base change ([CDH[+]II, Lemma A.3.10]) it will suffice to show that $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$ is a Karoubi projection. Since we know (1) holds for X we know that $\mathcal{D}^{\mathrm{qc}}(X)$ and $\mathcal{D}^{\mathrm{qc}}_{Z}(X)$ are generated under filtered colimits by $\mathcal{D}^{\mathrm{p}}(X)$ and $\mathcal{D}^{\mathrm{p}}_{Z}(X)$. This also implies that $\mathcal{D}^{\mathrm{qc}}(U)$ is generated under filtered colimits by $\mathcal{D}^{\mathrm{p}}(U)$: indeed, given $M \in \mathcal{D}^{\mathrm{qc}}(U)$ we can write j_*M as a filtered colimits of perfect complexes over X, and thus obtain a similar presentation over U by applying the colimit preserving functor j^*. In particular, applying the functor Ind to the sequence

$$\mathcal{D}^{\mathrm{p}}_{Z}(X) \to \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U)$$

yields the sequence

$$\mathcal{D}^{\mathrm{qc}}_{Z}(X) \to \mathcal{D}^{\mathrm{qc}}(X) \xrightarrow{j^*} \mathcal{D}^{\mathrm{qc}}(U).$$

The last sequence is a right split Verdier sequence (of large stable ∞-categories) since the functor j^* has a fully-faithful right adjoint j_*. It then follows that the former sequence is a Karoubi sequence by [CDH[+]II, Theorem A.3.11]. □

Lemma A.5.11. *Theorem A.5.8 holds whenever $X = \mathrm{spec}(A)$ is affine.*

Proof. By Lemma A.5.10 it will suffice to show (1) in the case where $X = \mathrm{spec}(A)$ is affine. If U is empty then $\mathcal{D}^{\mathrm{qc}}_{Z}(X) = \mathcal{D}^{\mathrm{qc}}(X) = \mathcal{D}(A)$ and we know that $\mathcal{D}(A)$ is generated under filtered colimits by $\mathcal{D}^{\mathrm{p}}(A)$. We may hence assume that U is not empty. For brevity let us write $\mathcal{D}_Z(A)$ and $\mathcal{D}^{\mathrm{p}}_{Z}(A)$ for $\mathcal{D}^{\mathrm{qc}}_{Z}(\mathrm{spec}(A))$ and $\mathcal{D}^{\mathrm{p}}_{Z}(\mathrm{spec}(A))$, respectively. Let $I \subseteq A$ be the ideal of elements vanishing on Z. Then U can be covered by the opens $\mathrm{spec}(A[f^{-1}])$ for $f \in I$, and since U is quasi-compact we may choose a finite subset $\{f_1, \dots, f_n\} \in I$ such that $\mathrm{spec}(A[f_1^{-1}]), \dots, \mathrm{spec}(A[f^{-1}])$ cover U. Let $j_i : \mathrm{spec}(A[f_i]) \hookrightarrow \mathrm{spec}(A)$ be the inclusion. For each $i = 1, \dots, n$ let

$$E_i = \mathrm{cof}[A \xrightarrow{f_i} A] \in \mathcal{D}^{\mathrm{p}}(A)$$

and let $E = \otimes_i E_i$. We note that E is a perfect complex. In addition, we have that $j_i^* E_i = 0$ for every i and hence $j^*E = 0$. We conclude that $E \in \mathcal{D}^{\mathrm{p}}_{Z}(A)$.

Let us write $\mathrm{Hom}_A(-,-)$ for the mapping complex between two A-module complexes. We claim that for an A-module M, if $\mathrm{Hom}_A(E, M) = 0$ then $M = 0$. In other words, we claim that the functor

$$\mathrm{Hom}_A(E, -) \colon \mathcal{D}_Z(A) \to \mathcal{D}_Z(A)$$

has trivial kernel. This implies that via standard arguments that $\mathcal{D}_Z(A)$ is generated under filtered colimits by its smallest stable subcategory containing K. By definition, $\mathrm{Hom}_A(E, -)$ is the composite of the functors

$$\mathrm{Hom}_A(E_i, -) \colon \mathcal{D}_Z(A) \to \mathcal{D}_Z(A)$$

for $i = 1, \dots, n$ (to avoid confusion, note that the E_i's do not belong to $\mathcal{D}_Z(A)$, but $\mathcal{D}_Z(A)$ is closed in $\mathcal{D}(A)$ under exponentiation with any perfect complex). It will hence suffice to show that each of the functors $\mathrm{Hom}_A(E_i, -)$ has trivial kernel in $\mathcal{D}_Z(A)$. Indeed, if $\mathrm{Hom}_A(E_i, M) = 0$ then f_i acts invertibly on M, and hence

$$M = M[f_i^{-1}] = (j_i)_* j_i^* M.$$

But $j_i^* M = 0$ (because $j_M^* = 0$), and so we conclude that $M = 0$. □

Lemma A.5.12. *Suppose that the qcqs scheme X can be covered by two qcqs open subsets $X = X' \cup X''$. If Theorem A.5.8 holds for X' and for X'' then it holds for X.*

Proof. By Lemma A.5.10 it will suffice to prove that (1) of Theorem A.5.8 holds for X. Let $Z \subseteq X$ be a closed subset of qcqs complements U. We need to show that $\mathcal{D}^{\mathrm{qc}}_Z(X)$ is generated under filtered colimits by $\mathcal{D}^{\mathrm{p}}_Z(X)$. Write $Z' = Z \cap X'$, $Z'' = Z \cap X''$, $Y' = X \setminus X'$ and $Y'' = X \setminus X''$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}^{\mathrm{p}}_{Z\cap Y''}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}_{Z}(X) & \longrightarrow & \mathcal{D}^{\mathrm{p}}_{Z''}(X'') \\ \downarrow{\scriptstyle\simeq} & & \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{p}}_{Z'\cap Y''}(X') & \longrightarrow & \mathcal{D}^{\mathrm{p}}_{Z'}(X') & \longrightarrow & \mathcal{D}^{\mathrm{p}}_{Z'\cap Z''}(X'\cap X'') \end{array}$$

Here, the rows are fibre sequences and the right square is a pullback square by Zariski descent (see Remark A.2.7). Now since Theorem A.5.8 is assumed known for X' the bottom row is a Karoubi sequence and hence the top row is a Karoubi sequence by [CDH+II, Lemma A.3.10]. Now consider the diagram

$$\begin{array}{ccccc} \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}_{Z\cap Y''}(X)) & \longrightarrow & \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}_{Z}(X)) & \longrightarrow & \mathrm{Ind}(\mathcal{D}^{\mathrm{p}}_{Z''}(X'')) \\ \downarrow{\scriptstyle\simeq} & & \downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathcal{D}^{\mathrm{qc}}_{Z\cap Y''}(X) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}_{Z}(X) & \longrightarrow & \mathcal{D}^{\mathrm{qc}}_{Z''}(X'') \end{array}$$

where the right vertical map is an equivalence since we assume Theorem A.5.8 for X'' and the left vertical map is an equivalence since $\mathcal{D}^{\mathrm{qc}}_{Z\cap Y''}(X) = \mathcal{D}^{\mathrm{qc}}_{Z'\cap Y''}(X')$ by excision and we assume Theorem A.5.8 for X'. Here the bottom row is a right split Verdier sequence since the pullback functor $\mathcal{D}^{\mathrm{qc}}_Z(X) \to \mathcal{D}^{\mathrm{qc}}_{Z''}(X'')$ has a fully-faithful right adjoint given by push-forward (see Remark A.2.5), and the top row is a right split Verdier sequence since it is obtained by applying Ind to a Karoubi sequence, see [NS18, Proposition I.3.5]. Now the middle vertical map is at least fully-faithful, since every perfect complex is compact in $\mathcal{D}^{\mathrm{qc}}_Z(X)$ (since it is compact in $\mathcal{D}^{\mathrm{qc}}(X)$). This implies that the right adjoint of the bottom Verdier inclusion gives a right adjoint to the top Verdier inclusion when restricted to $\mathrm{Ind}(\mathcal{D}^{\mathrm{p}}_Z(X))$. In other words, the left square is right adjointable, in the sense that it remains commutative when replacing the two horizontal Verdier inclusions by their right adjoints. Since both rows are right split Verdier sequences, the counit of the Verdier inclusion and the unit of the Verdier projection form a fibre sequence. It follows that the right square is right adjointable as well, in the sense that it remains commutative when replacing the two horizontal Verdier projections by their right adjoints. It then follows that the essential image of the middle vertical functor contain the essential image of $\mathcal{D}^{\mathrm{qc}}_{Z''}(X'')$ via push-forward. Since it also contains $\mathcal{D}^{\mathrm{qc}}_{Z\cap Y''}(X)$ it must contain all of $\mathcal{D}^{\mathrm{qc}}_Z(X)$, and so the proof is complete. ☐

Proof of Theorem A.5.8. Since X is qcqs it admits a finite covering $X = \cup_{i=1}^n U_i$ such that each U_i is affine. We then argue by induction on the minimal number n of affine opens needed to cover X. If $n = 1$ then X is affine and the claim is covered by Lemma A.5.11. Otherwise, X can be covered by two open subsets each of which can be covered by less than n affines, and hence the claim follows from the induction hypothesis via Lemma A.5.12. ☐

Corollary A.5.13. *Let X be a qcqs scheme and $p \in X$ a (not necessarily closed) point. Then the restriction functor*

$$\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathcal{O}_{X,p})$$

is a Karoubi projection.

The proof of Corollary A.5.13 will require the following lemma:

Lemma A.5.14. *Let $\mathcal{I}$ be a filtered poset and $\{U_\alpha\}_{\alpha\in\mathcal{I}^{\mathrm{op}}}$ an $\mathcal{I}^{\mathrm{op}}$-indexed diagram of qcqs schemes such that each of the maps $p_{\alpha,\beta} \colon U_\beta \to U_\alpha$ in the diagram is affine. Let $U = \lim_{\alpha\in\mathcal{I}} U_\alpha$ and $p_\alpha \colon U \to U_\alpha$ the canonical maps. Then the functor $\mathrm{colim}_{\alpha\in\mathcal{I}} \mathcal{D}^{\mathrm{p}}(U_\alpha) \to \mathcal{D}^{\mathrm{p}}(U)$ induced by $(M, \alpha) \mapsto p_\alpha^* M$ is an equivalence.*

Proof. The condition that each of the maps in the diagram $\{U_\alpha\}$ are affine implies that each of the maps $p_\alpha : U \to U_\alpha$ is affine. In particular, U is qcqs. Without loss of generality we may assume that $\mathcal{I}$ has an terminal element $\alpha_0 \in \mathcal{I}$ (otherwise, pick any $\alpha \in \mathcal{I}$ and replace $\mathcal{I}$ by the cofinal subposet $\{\beta \in \mathcal{I} | \beta \geq \alpha\}$. Since U_{α_0} is qcqs we may cover it by a finite collection of open affine subschemes. Since each of the $p_{\alpha,\beta} : U_\beta \to U_\alpha$ is affine this covering determines an open affine covering of each U_β, and similarly an open affine covering of U. Applying Zariski descent for perfect complexes (Remark A.5.7) and using the commutation of filtered colimits with finite limits we may reduce to the case where all the U_α's are affine. Transforming the problem to the setting of commutative rings, we find a filtered diagram $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ of rings, and we need to show that the functor

$$\operatorname*{colim}_\alpha \mathcal{D}^{\mathrm{p}}(A_\alpha) \to \mathcal{D}^{\mathrm{p}}(A) \qquad (M, \alpha) \mapsto M \otimes_{A_\alpha} A$$

is an equivalence, where $A = \operatorname{colim}_\alpha A_\alpha$. We first show that this functor is fully-faithful. For this, it will suffice to show that for any $\alpha \in \mathcal{I}$ and any $M, M' \in \mathcal{D}^{\mathrm{p}}(A_\alpha)$ the map

$$\operatorname*{colim}_{\beta\geq\alpha} \hom_{A_\beta}(A_\beta \otimes_{A_\alpha} M, A_\beta \otimes_{A_\alpha} M) \to \hom_A(A \otimes_{A_\alpha} M, A \otimes_{A_\alpha} M)$$

is an equivalence. By adjunction, we can identify this map with the map

$$\operatorname*{colim}_{\beta\geq\alpha} \hom_A(M, A_\beta \otimes_{A_\alpha} M) \to \hom_{A_\alpha}(M, A \otimes_{A_\alpha} M),$$

and since M is perfect as an A_α-module we can also take $\hom_A(M, -)$ out of the colimit, so that it will suffice to show that the map

$$\operatorname*{colim}_{\beta\geq\alpha} A_\beta \otimes_A M \to A \otimes_A M$$

is an equivalence. Indeed, this is clear since $A = \operatorname{colim}_\beta A_\beta = \operatorname{colim}_{\beta\geq\alpha} A_\beta$ and $(-) \otimes_A M$ preserves colimits.

We now show that this functor is essentially surjective. Since we already have that the functor in question is fully-faithful it will suffice to show that its image generates all of $\mathcal{D}^{\mathrm{p}}(A)$ under finite colimits and desuspensions. In particular, it will suffice to show that every finitely generated projective A-module N is of the form $N = M \otimes_{A_\alpha} A$ for some finitely generated projective A_α-module. Indeed, N is a retract of A^n for some n, and the coefficients of the associated idempotent $(n\times n)$-matrix must be in the image of $A_\alpha \to A$ for some sufficiently large α. □

Proof of Corollary A.5.13. We may write $\operatorname{spec}(\mathcal{O}_{x,p}) = \lim_\alpha U_\alpha$, where the U_α ranges over all affine open neighborhoods of p. Since each of the functor $\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(U_\alpha)$ is a Karoubi projection by Theorem A.5.8 and the collection of Karoubi projections is closed under colimits, we get from Lemma A.5.14 that

$$\mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(\mathcal{O}_{X,p}) = \operatorname*{colim}_\alpha \mathcal{D}^{\mathrm{p}}(U_\alpha)$$

is a Karoubi projection as well. □

A.6. **Quasi-perfect maps.** Recall from Proposition A.5.1 that for any map $f : X \to Y$ of qcqs schemes, the push-forward functor $f_* : \mathcal{D}^{\mathrm{qc}}(X) \to \mathcal{D}^{\mathrm{qc}}(Y)$ preserves colimits. Since $\mathcal{D}^{\mathrm{qc}}(X)$ is presentable this means that f_* admits a right adjoint $f^! : \mathcal{D}^{\mathrm{qc}}(Y) \to \mathcal{D}^{\mathrm{qc}}(X)$.

Remark A.6.1 (Flat base change for pushforward and upper shriek). Given a cartesian square

$$\begin{array}{ccc} V & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{j} & Y \end{array} \tag{55}$$

of qcqs schemes with j and i flat, we have by flat base change (Proposition A.3.7) an equivalence $j^*g_* \xrightarrow{\simeq} f_*i^*$. Replacing all functors by their right adjoints we get an equivalence

$$g^!j_* \xrightarrow{\simeq} i_*f^!$$

Otherwise put, since the square of push-forward functors associated to (56) is horizontally left adjointable by flat base change and all functors have right adjoints, it follows formally that it is also vertically right adjointable, and we obtain a commutative square

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{qc}}(V) & \xrightarrow{i_*} & \mathcal{D}^{\mathrm{qc}}(X) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle g^!} \\ \mathcal{D}^{\mathrm{qc}}(U) & \xrightarrow{j_*} & \mathcal{D}^{\mathrm{qc}}(Y) \end{array} \tag{56}$$

Let $f : X \to Y$ be a map of qcqs schemes. Then $f^* : \mathcal{D}^{\mathrm{qc}}(Y) \to \mathcal{D}^{\mathrm{qc}}(X)$ is symmetric monoidal, and hence can in particular be considered as a $\mathcal{D}^{\mathrm{qc}}(Y)$-linear functor (where $\mathcal{D}^{\mathrm{qc}}(Y)$ acts on $\mathcal{D}^{\mathrm{qc}}(X)$ by restricting $\mathcal{D}^{\mathrm{qc}}(X)$'s self action via f^*). This $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure induces a lax $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure on the right adjoint f_*, which the projection formula asserts is actually an honest $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure. This last $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure then induces a lax $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure on its further right adjoint $f^! : \mathcal{D}^{\mathrm{qc}}(Y) \to \mathcal{D}^{\mathrm{qc}}(X)$. Concretely, given a map $f : X \to Y$ of qcqs schemes and $M, N \in \mathcal{D}^{\mathrm{qc}}(Y)$, the map

$$f_*(f^*(M) \otimes f^!(N)) \simeq M \otimes f_* f^!(N) \to M \otimes N$$

obtained using the projection formula and the counit of $f_* \dashv f^!$ determines a natural map

$$\tau : f^*(M) \otimes f^!(N) \to f^!(M \otimes N),$$

encoding a lax $\mathcal{D}^{\mathrm{qc}}(Y)$-linear structure on $f^! : \mathcal{D}^{\mathrm{qc}}(Y) \to \mathcal{D}^{\mathrm{qc}}(X)$.

Lemma A.6.2. *The map τ is an equivalence if M is perfect. If f is quasi-perfect then τ is an equivalence for any M.*

Proof. The first claim follows from Lemma 4.1.6, by considering the underlying lax $\mathcal{D}^{\mathrm{p}}(Y)$-linear structure on $f^!$. If f is quasi-perfect then $f^!$ preserves colimits and so the domain and target of τ both depend on M in a colimit preserving manner. Since $\mathcal{D}^{\mathrm{p}}(Y)$ generates $\mathcal{D}^{\mathrm{qc}}(Y)$ under colimits the second part follows from the first part. □

Definition A.6.3. A map $f : X \to Y$ is said to be *quasi-perfect* if f_* preserves perfect complexes, or, equivalently, if $f^!$ preserves colimits.

Clearly quasi-perfect maps are closed under composition. They are also closed under base change against open embeddings:

Lemma A.6.4. *Consider a cartesian square of qcqs schemes as in (55), and suppose in addition that i and j are open embeddings. If g is quasi-perfect then f is quasi-perfect and the square (56) is left adjointable, that is, the associated mate transformation*

$$\eta : i^* g^! \Rightarrow f^! j^*$$

is an equivalence.

Proof. To see that f is quasi-perfect let $M \in \mathcal{D}^{\mathrm{qc}}(U)$ be a perfect complex. Since i is an open embedding we have that every perfect complex $M \in \mathcal{D}^{\mathrm{p}}(U)$ is a retract of one of the form i^*N for $N \in \mathcal{D}^{\mathrm{p}}(X)$. Now since g is quasi-perfect we have that $f_* i^* \simeq j^* g_* : \mathcal{D}^{\mathrm{p}}(X) \to \mathcal{D}^{\mathrm{p}}(V)$ preserves perfect complexes and hence f is quasi-perfect as well.

We now prove that η is an equivalence. For this, note that any $M \in \mathcal{D}^{\mathrm{qc}}(Y)$ sits in a fibre sequence of the form

$$N \to M \to j_* j^* M$$

with N supported on the complement of U. It will hence suffice to show that η is an equivalence on objects which are either in the image of j_* or in the kernel of j^*. If M is in the kernel of j^* then $f^! j^* M = 0$ and

$$i^* g^! M = i^*(g^* M \otimes g^! \mathcal{O}_Y) = i^* g^* M \otimes i^* g^! \mathcal{O}_Y = j^* f^* M \otimes i^* g^! \mathcal{O}_Y = 0$$

by Lemma A.6.2. Now assume that M is in the image of j_*. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}^{qc}(V) & \xleftarrow{i^*} & \mathcal{D}^{qc}(X) & \xleftarrow{i_*} & \mathcal{D}^{qc}(V) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{D}^{qc}(U) & \xleftarrow{j^*} & \mathcal{D}^{qc}(Y) & \xleftarrow{j_*} & \mathcal{D}^{qc}(U) \end{array}$$

obtained by gluing the square encoding the functoriality of push-forward for the square (55) with its horizontal mate (which is itself a commutative square by open base change). Now in the external rectangle the horizontal arrows are equivalences (since i_* and j_* are fully-faithful) and the vertical arrows have right adjoints and hence this rectangle is automatically vertically right adjointable. In addition, the right square is vertically right adjointable by Remark A.6.1. It then follows that the vertical right mate transformation of the left square is an equivalence on objects in the image of j_*. But this vertical mate is the same as the horizontal left mate of the corresponding square of right adjoints (56), which is η.

□

We deduce that being quasi-perfect is a local property on the codomain:

Corollary A.6.5. *Let $f: X \to Y$ be a map of qcqs schemes, $Y = \cup_i V_i$ an open covering of Y. Then f is quasi-perfect if and only if its base change $f_i: U_i = X \times_Y V_i \to V_i$ to V_i is quasi-perfect for every i. In addition, when these equivalent conditions hold we have that $(f^! M|)_{U_i} \simeq f_i^!(M|_{V_i})$ for every i.*

Proof. If f is quasi-perfect then each f_i is quasi-perfect by the first part of Lemma A.6.4. On the other hand, if each f_i is quasi-perfect and $M \in \mathcal{D}^p(X)$ is a perfect complex then $(f_*M)|_{V_i} \simeq (f_i)_*(M|_{U_i})$ is perfect for every i and so f_*M is perfect (see Remark A.5.5). The last part now follows from the second part of Lemma A.6.4. □

B. The Zariski and Nisnevich topologies

Throughout this section, we fix a presentable ∞-category $\mathcal{A}$.

B.1. **Extensive ∞-categories and coverages.** Recall that an ∞-category $\mathcal{C}$ is said to be *extensive* if it admits coproducts and the functor

$$\mathcal{C}_{/x} \times \mathcal{C}_{/y} \to \mathcal{C}_{/x \coprod y} \qquad ([a \to x], [b \to y]) \mapsto \left[a \coprod b \to x \coprod y\right]$$

is an equivalence for every $x, y \in \mathcal{C}$. On extensive ∞-categories one can construct Grothendieck topologies from a certain data known as a *coverage*, which is a collection of morphisms which behave like covering morphisms. More precisely, a collection of morphisms in an extensive ∞-category $\mathcal{C}$ is called a coverage if it is closed under composition, base change and coproducts (that is, if $x \to y$ and $z \to w$ are in the collection, then so is $x \coprod z \to y \coprod w$). One then refers to the morphisms in the coverage as *epimorphisms*. To a coverage one may associate a Grothendieck topology by declaring that a sieve on y is covering if and only if it contains a finite collection $\{x_i \to y\}_{i=1,\dots,n}$ such that $\coprod_i x_i \to y$ is an epimorphism, see [Lur17b, Proposition 3.2.1]. This Grothendieck topology is finitary by construction. In addition, as proven in [Lur17b, Proposition 3.3.1], a presheaf $\mathcal{F}: \mathcal{C}^{op} \to \mathcal{A}$ is a sheaf with respect to this Grothendieck topology if and only if it satisfies the following two conditions:

(1) $\mathcal{F}$ sends finite coproducts in $\mathcal{C}$ to products in $\mathcal{A}$.
(2) For every epimorphism $p: x \to y$ in the coverage, $\mathcal{F}$ satisfies Čech descent for p, that is, the induced map

$$\mathcal{F} \to \mathrm{Tot}\mathcal{F}(U_\bullet(p))$$

is an equivalence. Here $U_\bullet(f)$ is the Čech resolution of p, that is, the simplicial object whose n-simplices consist of the $(n+1)$-fold fibre product $x \times_y \cdots \times_y x$.

We record the following result:

Lemma B.1.1. *Let $\mathcal{C}$ be an extensive ∞-category equipped with a coverage and let*

$$\begin{array}{ccc} x & \xrightarrow{p'} & y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ z & \xrightarrow{p} & w \end{array} \tag{57}$$

be a commutative square all of whose arrows are epimorphisms. Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be a presheaf which satisfies Čech descent for the morphisms $U_n(p') \to U_n(p)$ and $U_m(q') \to U_m(q)$ for every $n, m \geq 0$. Then $\mathcal{F}$ satisfies Čech descent for q if and only if it satisfies Čech descent for p.

Proof. Let $\Delta_+ = \Delta \cup \{\emptyset\} = \Delta^{\triangleleft}$ be the extended simplex category, so functors $\Delta_+^{\mathrm{op}} \to \mathcal{C}$ correspond to augmented simplicial object and the formation of Čech resolutions (including their augmentation) can be identified with right Kan extension along the functor $\iota\colon \Delta^1 \to \Delta_+^{\mathrm{op}}$ picking the arrow $[0] \leftarrow \emptyset$ of Δ_+^{op}. Considering the square (57) as a functor $\rho\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$, we may right Kan extend it along $\iota \times \iota\colon \Delta^1 \times \Delta^1 \to \Delta_+^{\mathrm{op}} \times \Delta_+^{\mathrm{op}}$ to obtain a functor

$$W\colon \Delta_+^{\mathrm{op}} \times \Delta_+^{\mathrm{op}} \to \mathcal{C}.$$

We then have that $W_{\bullet,\emptyset}$ is the Čech resolution of p, $W_{\emptyset,\bullet}$ is the Čech resolution of q, $W_{\bullet,0}$ is the Čech resolution of p' and $W_{0,\bullet}$ is the Čech resolution of q'. Similarly, $W_{\bullet,m}$ is the Čech construction of $U_m(q') \to U_m(q)$ and $W_{n,\bullet}$ is the Čech resolution of $U_n(p') \to U_n(p)$. Our assumptions hence imply that in the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{F}(w) & & \\ & \swarrow & & \searrow & \\ \lim_{n\in\Delta} \mathcal{F}(U_n(p)) & & & & \lim_{m\in\Delta} \mathcal{F}(U_m(q)) \\ & \searrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\swarrow & \\ & & \lim_{(n,m)\in\Delta\times\Delta} \mathcal{F}(W_{n,m}) & & \end{array}$$

the two bottom arrows are equivalences. We then conclude that the top left arrow is an equivalence if and only if the top right arrow is one, as desired. □

Corollary B.1.2. *Let $\mathcal{C}$ be an extensive ∞-category equipped with a coverage and let $z \xrightarrow{g} y \xrightarrow{f} w$ be a composable sequence of maps such that f and $f \circ g$ are epis. Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be a presheaf. If $\mathcal{F}$ satisfies Čech descent with respect to $f \circ g$ and all of its base changes, then $\mathcal{F}$ also satisfies Čech descent with respect to f.*

Proof. Apply Lemma B.1.1 to the square

$$\begin{array}{ccc} z \times_w y & \longrightarrow & y \\ \downarrow & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle f} \\ z & \xrightarrow{f\circ g} & w \end{array}$$

and use the fact that any presheaf satisfies Čech descent with respect to split epis (that is, those that admit a section), because the Čech resolution extends in this case to a split simplicial object. □

B.2. **The Zariski and Nisnevich coverages.** Let S be a qcqs scheme. The category $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ of qcqs schemes over S is extensive: indeed, this follows from the fact that the category of all schemes is extensive and the collection of qcqs schemes is closed under fibre products and finite coproducts. We may hence use the formalism of coverages to define Grothendieck topologies on $\mathrm{Sch}^{\mathrm{qq}}$. As we will see below, both the Zariski and Nisnevich topologies can be obtained in this manner. Our main motivation for taking up this point of view is that it allows for a simple argument establishing Proposition B.3.1 below.

Definition B.2.1. We will say that a map $p\colon V \to X$ of qcqs S-schemes is

(1) A *Zariski epi* if p is étale and there exists a finite open cover $Y = \cup_i U_i$ such that for every $i = 1, \dots, n$ there exists a dotted lift

$$\begin{array}{ccc} & & V \\ & \nearrow & \downarrow p \\ U_i & \longrightarrow & Y \end{array}$$

as indicated.

(2) A *Nisnevich epi* if p is étale and there exists a finite sequence of open subsets $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ of X such that for every $i = 1, \dots, n$ there exists a dotted lift

$$\begin{array}{ccc} & & V \\ & \nearrow & \downarrow p \\ U_i \setminus U_{i-1} & \longrightarrow & X \end{array}$$

as indicated.

Example B.2.2. If $X = \cup_i U_i$ is a finite open cover then $\coprod_i U_i \to X$ is a Zariski epi.

Example B.2.3. Any Zariski epi is also a Nisnevich epi.

Lemma B.2.4. *Both the collection of Zariski epis and the collection of Nisnevich epis satisfy the axioms of a coverage, that is, they are closed under composition, base change and coproducts.*

Proof. We begin with the collection of Zariski epis. Here, closure under base change and disjoint union is straightforward. For composition, suppose that $V \to Y \to X$ is a composable sequence of Zariski epis. Then by definition we have a finite open cover $X = \cup_i U_i$ such that the base change $Y \times_X U_i \to U_i$ admits a section $s_i : U_i \to Y \times_X U_i$ for every i and a finite open cover $Y = \cup W_j$ such that $V \times_Y W_j \to W_j$ admits a section for every j. Setting $Z_{i,j} := s_i^{-1}(W_j \times_X U_i)$ we obtain a finite open cover $X = \cup_{i,j} Z_{i,j}$ such that each of the base changes $V \times_X Z_{i,j} \to Z_{i,j}$ admits a section, as desired.

We now consider the collection of Nisnevich epis. Closure under base change remains straightforward. Now consider two Nisnevich epis $V \to X$ and $W \to Y$. We want to show that $V \coprod W \to X \coprod Y$ is a Nisnevich epi. □

Remark B.2.5. If k is a field then the only non-empty open subset of spec(k) is spec(k) itself, and hence any Nisnevich epi $V \to \mathrm{spec}(k)$ admits a section. It then follows that for any scheme X if $p : V \to X$ is a Nisnevich epi then the induced map $p(k) : V(k) \to X(k)$ on k-points is surjective for any field k. If X is Noetherian then the inverse implication holds as well. In other words, Nisnevich epis over Noetherian schemes can be characterized as those morphisms which are surjective on k-points for any field k.

Our next goal is to show that the Zariski and Nisnevich Grothendieck topologies on qcqs schemes coincides, respectively, with the Grothendieck topologies associated the Zariski and Nisnevich coverages, as above. We begin with the Zariski topology.

Proposition B.2.6. *Let $\mathcal{E} \subseteq \mathrm{Sch}_{/X}$ be a sieve on X. Then the following are equivalent:*

(1) $\mathcal{E}$ is a covering sieve for the Zariski topology.

(2) $\mathcal{E}$ contains a finite collection of open embeddings $\{U_i \to X\}_{i=1,\dots,n}$ such that the induced map $\coprod_i U_i \to X$ is a Zariski epi in the sense of Definition B.2.1.

(3) $\mathcal{E}$ contains a finite collection of maps $\{Z_i \to X\}_{i=1,\dots,n}$ such that the induced map $\coprod_i Z_i \to X$ is a Zariski epi in the sense of Definition B.2.1.

Proof. If $\mathcal{E}$ is a Zariski covering sieve then by definition it contains a finite collection of open embeddings $\{U_i \to X\}_{i=1,\dots,n}$ which form an open covering of X. In this case clearly the induced map $\coprod_i U_i \to X$ is Zariski epi, so that (1) ⇒ (2). Clearly also (2) ⇒ (3). We now prove that (3) ⇒ (1).

Suppose there exists a finite $\{Z_i \to X\}_{i=1,\dots,n}$ such that the induced map $Z = \coprod_i Z_i \to X$ is a Zariski epi. Then there exists finite collection of open subschemes $U_1, \dots, U_m \subseteq X$ such that $Z \times U_j \to U_j$ admits a section $s : U_j \to Z \times_X U_j$. Now each Z_i is an open and closed subscheme of Z, and so we the section s determines a decomposition $U_j = \coprod_i U_{i,j}$ where $U_{i,j} = s^{-1}(Z_i \times_X U_j)$. Then each of the open embeddings

$U_{i,j} \hookrightarrow X$ factors through $Z \to X$ and is hence contained in $\mathcal{E}$. We conclude that the sieve $\mathcal{E}$ contains an open covering of X and is hence a covering sieve with respect to the Zariski topology. □

Corollary B.2.7. *Let $\mathcal{F}\colon (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}} \to \mathcal{A}$ be a presheaf. Then the following are equivalent:*

(1) $\mathcal{F}$ is a sheaf with respect to the Zariski topology.
(2) $\mathcal{F}$ sends coproducts in $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ to products in $\mathcal{A}$ and satisfies Čech descent with respect to any Zariski epi $p\colon X \to Y$.
(3) $\mathcal{F}$ sends coproducts in $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ to products in $\mathcal{A}$ and satisfies Čech descent with respect to any Zariski epi of the form $\coprod_{i=1}^{n} U_i \to X$, where $U_1, ..., U_n$ are an open cover of X.
(4) $\mathcal{F}(\emptyset)$ is terminal in $\mathcal{A}$ and for every finite open covering $X = U_1 \cup \cdots \cup U_n$ the induced map

$$\mathcal{F}(X) \to \lim_{\emptyset \neq S \subseteq \{1,\dots,n\}} \mathcal{F}(\cap_{i\in S} U_i)$$

is an equivalence.
(5) $\mathcal{F}(\emptyset)$ is terminal in $\mathcal{A}$ and for every Zariski covering of the form $X = U_1 \cup U_2$ of qcqs S-schemes, the square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_2) & \longrightarrow & \mathcal{F}(U_1 \cap U_2) \end{array}$$

is cartesian.

Proof. The equivalence of (1) and (2) follows from Proposition B.2.6 and the characterization of [Lur17b, Proposition A.3.3.1]. The equivalence of (2) and (3) follows from Corollary B.1.2, since any Zariski epi $Z \to X$ refines to a Zariski epi of the form $\coprod_{i=1}^{n} U_i \to X$. The equivalence of (3) and (4) is standard and follows from the fact the forgetful functor from the simplex category of $\mathrm{cosk}_0(\{1, ..., n\})$ to the poset of non-empty subsets in $\{1, ..., n\}$ is coinitial. Finally, the equivalence of (4) and (5) follows in a standard manner by induction. □

As for the Nisnevich topology, it is often considered in the literature exclusively for Noetherian schemes. Since we do not want to make this assumption at this point, we will work with the following definition in the general qcqs case (see [Lur17b]):

Definition B.2.8. Let X be a qcqs scheme. A collection of maps $\{V_\alpha \to X\}_\alpha$ is said to be a Nisnevich covering if the following holds:

(1) Each $V_\alpha \to X$ is étale.
(2) There exists a finite sequence of open subsets $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ of X, such that for every $i = 1, \dots, n$ there exists an index α_i such that the dotted lift

$$\begin{array}{ccc} & & V_{\alpha_i} \\ & \nearrow & \downarrow \\ U_i \setminus U_{i-1} & \longrightarrow & X \end{array}$$

exists.

The *Nisnevich topology* on $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ is then by definition the Grothendieck topology generated by Nisnevich coverings. In other words, a sieve on $X \in \mathrm{Sch}^{\mathrm{qq}}_{/S}$ is a covering sieve if and only if it contains a collection of morphisms $V_\alpha \to X$ which form a Nisnevich covering. We note that every Zariski covering of a qcqs scheme is also a Nisnevich covering, and so the Nisnevich topology on $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ is finer than the Zariski topology.

Remark B.2.9. In Definition B.2.8, if $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ is a finite sequence of opens satisfying (2) of that definition, then any refinement of this sequence (that is, a sequence obtained by factoring each inclusion $U_i \subseteq U_{i+1}$ into a finite sequence $U_i = U_i^0 \subseteq U_i^1 \subseteq \cdots \subseteq U_i^m = U_{i+1}$) will again satisfy (2). It hence follows that if $\{V_\alpha \to X\}$ is a Nisnevich covering such that $X = \mathrm{spec}(A)$ is affine then we may assume (by passing to a refinement) that the open subsets $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ are of the form

$U_i = \mathrm{spec}(A[f_i^{-1}])$ for $f_i \in A$. In particular, a Nisnevich covering among affine schemes in this sense coincides with the notion appearing in [Lur17b, Definition B.4.1.1].

The Nisnevich topology is again finitary: indeed, if $\{V_\alpha \to X\}_\alpha$ is a Nisnevich covering then $\{V_{\alpha_i} \to X\}_{i=1,\dots,n}$ is again a Nisnevich covering, where the α_i's are determined by the finite sequence $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ as in (2) of Definition B.2.8.

Proposition B.2.10. *Let $\mathcal{E} \subseteq \mathrm{Sch}_{/X}$ be a sieve on X. Then $\mathcal{E}$ is covering with respect to the Nisnevich topology if and only if it contains a finite collection of étale maps $\{U_i \to X\}_{i=1,\dots,n}$ such that the induced map $\coprod_i U_i \to X$ is a Nisnevich epi in the sense of Definition B.2.1. In particular, the Nisnevich topology on $\mathrm{Sch}_{/X}$ coincides with the one induced by the coverage of Nisnevich epis.*

Proof. □

By [Lur17b, Proposition A.3.3.1], we then conclude:

Corollary B.2.11. *A presheaf $\mathcal{F}\colon (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}} \to \mathcal{A}$ is a Nisnevich sheaf if and only if it sends coproducts in $\mathrm{Sch}^{\mathrm{qq}}_{/S}$ to products in $\mathcal{A}$ and satisfies Čech descent for Nisnevich epis.*

B.3. **Characterization of Nisnevich sheaves.** Let $\mathrm{Sch}^{\mathrm{qcs}}_{/S} \subseteq \mathrm{Sch}^{\mathrm{qq}}_{/S}$ be the full subcategory spanned by the morphisms $X \to S$ such that X is separated, and let $\mathrm{Aff}_{/S} \subseteq \mathrm{Sch}^{\mathrm{qcs}}_{/S}$ be the full subcategory therein spanned by those $X \to S$ such that X is affine. Both of these full subcategories are closed under direct sums and fibre products are hence themselves extensive. We then define the Nisnevich topologies on $\mathrm{Sch}^{\mathrm{qcs}}_{/S}$ and $\mathrm{Aff}_{/S}$ to be those associated to the coverage of Nisnevich epis among separated and affine schemes, respectively. By the same argument as in the proof of Proposition B.2.10 we have that the resulting Grothendieck topologies on $\mathrm{Sch}^{\mathrm{qcs}}_{/S}$ and $\mathrm{Aff}_{/S}$ coincide with the ones generated by the Nisnevich coverings among separated and affine schemes.

Proposition B.3.1. *Let S be a qcqs scheme and let $\mathcal{F}\colon (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}} \to \mathcal{A}$ be a functor valued in some presentable ∞-category $\mathcal{A}$. Then the following are equivalent:*

(1) $\mathcal{F}$ is a Nisnevich sheaf.
(2) $\mathcal{F}$ is a Zariski sheaf and the restriction of $\mathcal{F}$ to $(\mathrm{Sch}^{\mathrm{qcs}}_{/S})^{\mathrm{op}} \subseteq (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}}$ is a Nisnevich sheaf.
(3) $\mathcal{F}$ is a Zariski sheaf and the restriction of $\mathcal{F}$ to $(\mathrm{Aff}_{/S})^{\mathrm{op}} \subseteq (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}}$ is a Nisnevich sheaf.

The proof of Proposition B.3.1 will require the following lemma:

Lemma B.3.2. *Let*

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a pullback square of qcqs schemes all of whose legs are Nisnevich epis, and let $p\colon W' \to W$ be a map of qcqs schemes. Then p is a Nisnevich epi if and only if the composites $W' \to W \to Z$ and $W' \to W \to Y$ are Nisnevich epis.

Proof. The only if direction follows from the fact that Nisnevich epis are closed under composition. To prove the if the direction, let $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = Z$ and $\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = Y$ be ascending sequences of open subschemes such that each of the inclusions $U_i \setminus U_{i-1} \hookrightarrow Z$ and $V_j \setminus V_{j-1} \hookrightarrow Y$ lifts to W'. We construct a sequence of open subschemes

$$W_{1,1} \subseteq W_{1,2} \subseteq \cdots \subseteq W_{1,m} \subseteq W_{2,1} \subseteq \cdots \subseteq W_{1,m} \subseteq \cdots \subseteq W_{n,m} = W$$

inductively as follows. First we set $W_{1,1} = U_1 \times_X V_1$. Let $1 \le i \le n$ and $1 \le j \le m$ be such that $W_{i',j'}$ has been defined for every (i', j') such that either $i' < i$ or $i' = i$ and $j' < j$, that is, all (i', j') lexicographically smaller than (i, j). Let (i', j') be the largest element smaller than (i, j) in the lexicographical order. Then we define $W_{i,j}$ by

$$W_{i,j} = W_{i',j'} \cup (U_i \times_X V_j).$$

We now observe that by construction, both $U_{i-1} \times_X V_j$ and $U_i \times V_{j-1}$ are contained in $W_{i',j'}$, and hence $W_{i,j} \setminus W_{i',j'}$ is contained in $(U_i \setminus U_{i-1}) \times_X (V_j \setminus V_{j-1})$ for every lexicographically consecutive pair $(i', j') < (i, j)$ in $\{1, \dots, n\} \times \{1, \dots, m\}$ (this is also true if i or j are 1, since $U_0 = V_0 = \emptyset$) It then follows that the inclusion $W_{1,1} \hookrightarrow W$, as well as each of the inclusions $W_{i,j} \setminus W_{i',j'}$ for every lexicographically consecutive pair $(i', j') < (i, j)$ in $\{1, \dots, n\} \times \{1, \dots, m\}$, lifts to W', and so $W' \to W$ is a Nisnevich epi, as desired. □

Proof of Proposition B.3.1. Assume first that $\mathcal{F} : (\mathrm{Sch}^{\mathrm{qq}}_{/S})^{\mathrm{op}} \to \mathcal{A}$ is a Nisnevich sheaf. By Corollary B.2.11 $\mathcal{F}$ satisfies čech descent for Nisnevich epis. Since any Zariski epi is a Nisnevich epi $\mathcal{F}$ is also a Zariski sheaf. In addition, the restriction of $\mathcal{F}$ to $\mathrm{Sch}^{\mathrm{qcs}}_{/S}$ satisfies Čech descent for Nisnevich epis among separated and affine schemes and so (1) ⇒ (2), and similarly (2) ⇒ (3) for the same reason.

We now prove that (3) ⇒ (2) ⇒ (1). Suppose that $\mathcal{F}$ is a Zariski sheaf whose restriction to $(\mathrm{Aff}_{/S})^{\mathrm{op}}$ is a Nisnevich sheaf. We prove that the restriction of $\mathcal{F}$ to $\mathrm{Sch}^{\mathrm{qcs}}_{/S}$ is also a Nisnevich sheaf. Now since $\mathcal{F}$ is a Zariski sheaf it takes coproducts of schemes to products in $\mathcal{A}$ by Corollary B.2.7. It will hence suffice to show that it satisfies Čech descent with respect to Nisnevich epis among separated schemes. Let $q : V \to X$ be a Nisnevich epi of quasi-compact S-schemes such that X and V are separated. Choose a finite open covering $X = \cup_{i=1,\dots,n} U_i$ by affines and write $U = \coprod_i U_i$, so that $p : U \to X$ is a Zariski epi and U is affine (and in particular quasi-compact and separated). Let $Y = U \times_X V$. Being a fibre product of quasi-compact and separated schemes, Y is also quasi-compact separated. Let us now choose again a finite open covering $Y = \cup_{j=1,\dots,m} W_i$, and set $W = \coprod_j W_j$, so that $W \to Y$ is a Zariski epi and W is affine. Since Nisnevich epis are closed under base change and composition, we have that the composed map $W \to Y \to U$ is a Nisnevich epi, and similarly, the composed map $W \to Y \to V$ is a Zariski epi. Consider the commutative square

$$\begin{array}{ccc} W & \xrightarrow{p'} & V \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow{p} & X \end{array} \tag{58}$$

whose vertical arrows are Nisnevich epis and whose horizontal arrows are Nisnevich epis. We claim that this square satisfies the assumptions of Lemma B.1.1. For this, first note that since Nisnevich epis are closed under base change and composition we have that all the maps in Čech resolutions of p and p' are Zariski epis. Combined with Lemma B.3.2, this implies that each of the maps $U_n(p') \to U_n(p)$ is a Nisnevich epi. At the same time, $U_n(p)$ decomposes as a finite disjoint union of the form

$$U_n(p) = \coprod_{i_1,\dots,i_{n+1}} [U_{i_1} \cap \dots \cap U_{i_{n+1}}]$$

and each $U_{i_1} \cap \dots \cap U_{i_{n+1}}$ is affine, being the intersection of affine open subschemes inside a separated scheme. By assumption we thus have that $\mathcal{F}$ satisfies Čech descent with respect to each $U_n(p') \to U_n(p)$. We now need to show that $\mathcal{F}$ also satisfies Čech descent with respect to each $U_m(q') \to U_m(q)$. Now we claim that each of these maps is a Zariski epi. To see this, let us factor the above square as the pasting of two squares as follows

$$\begin{array}{ccccc} W & \xrightarrow{p''} & Y & \xrightarrow{p'} & V \\ {\scriptstyle q'}\downarrow & & {\scriptstyle q''}\downarrow & & \downarrow{\scriptstyle q} \\ U & = & U & \xrightarrow{p} & X \end{array}$$

We claim that the maps $U_m(q') \to U_m(q'')$ and $U_m(q'') \to U_m(q)$ are both Zariski epis (and hence the composite is such as well). For the latter, by construction this is the map

$$W \times_U \dots \times_U W = \coprod_{i_1,\dots,i_{m+1}} [W_{i_1} \times_U \dots \times_U W_{i_{m+1}}] \to Y \times_U \dots \times_U Y$$

which is a Zariski epi since each of the maps

$$W_{i_1} \times_U \dots \times_U W_{i_{m+1}} \to Y \times_U \dots \times_U Y$$

is an open embedding.

We have thus shown that the square (58) satisfies the assumption of Lemma 58, and hence we may conclude that $\mathcal{F}$ satisfies Čech descent with respect to p if and only if it satisfies Čech descent with respect to q. Since p is a Zariski epi $\mathcal{F}$ satisfies descent with respect to p and hence we conclude that $\mathcal{F}$ satisfies Čech descent with respect to q.

This shows that (3) ⇒ (2). The proof that (2) ⇒ (1) proceeds in exactly the same manner, where we note that the only place we used the assumption that X is separated was in order to conclude that each $U_{i_1} \cap \cdots \cap U_{i_{n+1}}$ is affine, being the intersection of affine open subschemes inside a separated scheme. In order to prove (2) ⇒ (1) we no longer need each $U_{i_1} \cap \cdots \cap U_{i_{n+1}}$ to be affine, just separated. But this automatic from the fact that these are open subschemes of the affine (and in particular separated) scheme U_{i_1}, and being separated is a property which is inherited by open subschemes. □

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