

# PUSH-PULL OPERATORS ON THE FORMAL AFFINE DEMAZURE ALGEBRA AND ITS DUAL

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## 1. INTRODUCTION

In the present work we pursue the ‘algebraization approach’ for oriented cohomology theories started in [CPZ] and continued in [HMSZ], [CZZ] and [CZZ2]; the general idea is to match (equivariant) oriented cohomology rings of flag varieties and elements of classical interest in them – such as classes of Bott-Samelson resolutions, Schubert varieties and their duals – with explicit combinatorial objects that can be introduced purely algebraically, in the spirit of Demazure [Dem73], Kostant-Kumar [KK86, KK90], Arabia [Ar86, Ar89], Bressler-Evens [BE90], Brion [Br97] and others. We mainly focus on algebraic constructions pertaining to  $T$ -equivariant oriented cohomology rings and associated duality pairings, where  $T$  is a maximal

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torus in a split semisimple linear algebraic group  $G$ . Proofs and details of how these constructions match cohomology groups and geometry can be found in [CZZ2].<sup>1</sup>

We now give a more precise overview of the approach.

In a series of papers [KK86, KK90], Kostant and Kumar introduced and successfully applied the techniques of nil (resp. 0-) Hecke algebras to study equivariant singular cohomology (resp.  $K$ -theory) of flag varieties. In particular, they showed that the dual of the nil Hecke algebra serves as an algebraic model for  $T$ -equivariant singular cohomology of the variety of Borel subgroups  $G/B$ . The algebraic part of this formalism was generalized in [HMSZ] and [CZZ]: given a formal group law  $F$  and a root datum (i.e. a root system sitting in a lattice), one constructs a *formal affine Demazure algebra*  $\mathbf{D}_F$ , with a coproduct. Its presentation in terms of generators and relations resembles the one of an affine Hecke algebra. Indeed, when  $F$  is the additive (resp. multiplicative) formal group law, it coincides with the completion of the nil (resp. 0-) affine Hecke algebra (see [HMSZ]). Moreover, in section 9, we show that for some degenerate elliptic formal group law  $F$  and a root system of type  $A$  the non-affine part of  $\mathbf{D}_F$  is isomorphic to the classical Iwahori-Hecke algebra (this result was recently generalized to arbitrary Kac-Moody root systems in [Le15]).

The link with geometry goes as follows. We start with an equivariant version of an algebraic oriented cohomology theory  $\mathfrak{h}$  in the sense of Levine-Morel [LM07]; algebraic cobordism  $\Omega$  is a universal example of such a theory. Given a root datum associated to  $(G, T)$  and a formal group law  $F$  associated to  $\mathfrak{h}$ , one can prove (see [CZZ2, Thm. 8.2]) that the dual  $\mathbf{D}_F^*$  of the coalgebra  $\mathbf{D}_F$  is isomorphic to the  $T$ -equivariant oriented cohomology ring  $\mathfrak{h}_T(G/B)$ . Specializing to the additive (resp. the multiplicative) formal group laws, one then recovers the results of Kostant-Kumar for singular cohomology (resp. for  $K$ -theory).

When dealing with an arbitrary formal group law (thus an arbitrary  $\mathfrak{h}$ ), one needs to take into account the following observation. Demazure [Dem73, Dem74] proved that for Chow groups and  $K$ -theory, the class of a Bott-Samelson desingularization corresponding to a reduced decomposition of an element  $w$  of the Weyl group actually only depends on  $w$  and not on the chosen reduced decomposition. It actually is the class of the – possibly singular – Schubert variety corresponding to  $w$  in Chow groups or of its structural sheaf in  $K$ -theory. This independence plays a crucial role in the arguments used in the literature when dealing with Chow groups or  $K$ -theory: see Th. 1, Prop. 3, Cor. 1, 2, 3, and 4 in [Dem73], Prop. 4.2 and 4.3 in [KK86], [Ar86], etc. However, Bressler and Evens [BE90, Th. 3.7] proved that if the formal group law of  $\mathfrak{h}$  is anything more complicated than multiplicative, this independence result fails. And indeed, for an arbitrary oriented cohomology theory, for example for  $\Omega$ , different Bott-Samelson desingularizations of the same Schubert variety give different classes. This means that one needs to come up with genuinely new arguments to understand the combinatorics of the general case, beyond Chow groups and  $K$ -theory.

The following key tools are extensively used in the study of equivariant cohomology and  $K$ -theory:

<sup>1</sup>In that reference, the algebro-geometric setting is used, but the same algebraic formalism also describes complex oriented cohomology theories applied to projective homogeneous varieties under complex Lie groups.

- I. A moment map (to  $T$ -fixed points),
- II. Push-forward operators to parabolic quotients  $G/P$ , or push-pull operators when subsequently pulling back to  $G/B$ ,
- III. The Poincaré duality pairing and the basis dual to Bott-Samelson classes for this pairing.

In the present paper we generalize all of them (i.e. give algebraic descriptions using the algebra  $\mathbf{D}_F$  and associated constructions) for an arbitrary oriented equivariant cohomology theory  $\mathfrak{h}$  over a base field. In turn, these algebraic descriptions imply geometric results (mostly stated in [CZZ2]).

For convenience, let us nevertheless give a walk-through of the geometric interpretation of our algebraic results. Our notation for the spectrum of the base field is  $\text{pt}$  and we'll use  $R = \mathfrak{h}(\text{pt})$  and  $S = \mathfrak{h}_T(\text{pt})$  as coefficient rings.<sup>2</sup>

I. The  $T$ -fixed points of  $G/B$  are naturally in bijection with the Weyl group  $W$  [CZZ2, §6]. This gives a pull-back to the fixed locus map (the so-called moment map)

$$\mathfrak{h}_T(G/B) \longrightarrow \mathfrak{h}_T(W) \simeq \bigoplus_{w \in W} \mathfrak{h}_T(\text{pt}).$$

The algebraic version of the moment map is precisely the embedding of Definition 10.1

$$\mathbf{D}_F^* \longrightarrow S_W^* \simeq \bigoplus_{w \in W} S,$$

hence, proving its injectivity; we don't know any direct geometric proof of the injectivity of the moment map in this generality.

II. Then, we look at push-pull operators of the form

$$\mathfrak{h}_T(G/P_{\Xi'}) \xrightarrow{p_*} \mathfrak{h}_T(G/P_{\Xi}) \xrightarrow{p^*} \mathfrak{h}_T(G/P_{\Xi'})$$

given by the push-forward followed by the pull-back along the natural quotient map  $p: G/P_{\Xi'} \rightarrow G/P_{\Xi}$ , where  $P_{\Xi'} \subseteq P_{\Xi}$  are two parabolic subgroups of  $G$ . Again  $p^*$  happens to be injective, and it identifies  $\mathfrak{h}_T(G/P_{\Xi})$  with a subring of  $\mathfrak{h}_T(G/P_{\Xi'})$ , namely the subring of invariants under the action of the corresponding parabolic subgroup  $W_{\Xi}$  of the Weyl group  $W$  (cf. [CZZ2, Thm. 9.1]). This does not seem to be straightforward from the geometry either, and it once more follows from our algebraic description (see Lemma 15.1): for subsets  $\Xi' \subseteq \Xi$  of a given set of simple roots  $\Pi$  (giving rise to the parabolic subgroups  $P_{\Xi'}$  and  $P_{\Xi}$ ), the push-pull operator is the bottom row of the diagram

$$\begin{array}{ccccc} (\mathbf{D}_F^*)^{W_{\Xi'}} & \xrightarrow{A_{\Xi/\Xi'}} & (\mathbf{D}_F^*)^{W_{\Xi}} & \subseteq & (\mathbf{D}_F^*)^{W_{\Xi'}} \\ p_{\Xi'}^* \uparrow \wr & & p_{\Xi}^* \uparrow \wr & & p_{\Xi'}^* \uparrow \wr \\ \mathbf{D}_{F,\Xi'}^* & \xrightarrow{A_{\Xi/\Xi'}} & \mathbf{D}_{F,\Xi}^* & \xrightarrow{p_{\Xi/\Xi'}^*} & \mathbf{D}_{F,\Xi'}^* \end{array}$$

while the top is obtained as the action of an element  $Y_{\Xi/\Xi'}$  (see 5.3), and the vertical isomorphisms correspond to pull-backs to (their image in)  $\mathfrak{h}_T(G/B)$ . The formula for the element  $Y_{\Xi/\Xi'}$  with  $\Xi' = \emptyset$  had already appeared in related contexts, namely,

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<sup>2</sup>We will require that  $S$  be 'complete' in some precise sense, but this is a technical point, that we prefer to hide here for simplicity. See [CZZ2, Definition 2.1]

in discussions around the Becker-Gottlieb transfer for topological complex-oriented theories (see [BE90, (2.1)] and [GR12, §4.1]).

III. We also provide the algebraic counterpart of the Poincaré pairing (cf [CZZ2, Thm. 9.3])

$$\mathfrak{h}_T(G/P_\Xi) \otimes \mathfrak{h}_T(G/P_\Xi) \rightarrow \mathfrak{h}_T(\text{pt})$$

obtained by multiplication and push-forward to the point. It is a pairing

$$\mathbf{D}_{F,\Xi}^* \otimes \mathbf{D}_{F,\Xi}^* \rightarrow S.$$

We show in Theorem 15.6 that it is non-degenerate, and that algebraic classes corresponding to chosen Bott-Samelson desingularizations of Schubert varieties (by [CZZ2, Lem. 8.8]) form a basis  $\{\mathcal{A}_\Xi(A_{I_w^{\text{rev}}}(x_\Pi f_e))\}_{w \in W^\Xi}$  of  $\mathbf{D}_{F,\Xi}^*$ ; we describe explicitly the dual basis with respect to this pairing, its elements being algebraic duals  $\{(Y_{I_w^\Xi}^\Xi)^*\}_{w \in W^\Xi}$  of certain natural elements  $\{Y_{I_w^\Xi}^\Xi\}_{w \in W^\Xi}$  in  $\mathbf{D}_{F,\Xi}$ . This generalizes (to parabolic subgroups and to arbitrary equivariant oriented cohomology groups) and unifies several results in the literature: [Dem74, Prop. 1 p. 69], [Ar89, Thm. 4.2.1], [KK86, KK90], [CPZ, §14]. We also construct another pair of dual bases

$$\{\mathcal{A}_\Xi(B_{I_w^{\text{rev}}}(x_\Pi f_e))\}_{w \in W^\Xi} \text{ and } \{(X_{I_w^\Xi}^\Xi)^*\}_{w \in W^\Xi}.$$

In the case of Chow groups this pair is the same as the previous one up to sign, but for K-theory they are already quite different (see [LZ, §5]).

A final technical remark: as in [KK90], it is convenient to invert certain elements in  $S$  because most formulas have a nicer expression using denominators. But contrary to [KK90], in which  $S$  is formal power series over  $R = \mathbb{Z}$  and the localization used is simply the fraction field of  $S$ , in our case  $R$  can be arbitrary, and we must be very careful about localizing as little as possible. Thus, we localize  $S$  at a multiplicative subset generated by Chern classes of line bundles corresponding canonically to roots, which gives injections  $S \subseteq Q$  (the localization),  $S_W \subseteq Q_W$  and  $S_W^* \subseteq Q_W^*$ . Although we do not know good geometric interpretations of  $Q$ ,  $Q_W$  or  $Q_W^*$ , all the formulas and operators we are interested in are easily defined at that localized level. The main technical difficulties then lie in proving that these operators actually restrict to  $S$ ,  $S_W^*$ ,  $\mathbf{D}_F^*$  etc., or so to speak, that the denominators cancel out.

The paper is organized as follows. In sections 2 and 3, we recall definitions and basic properties from [CPZ, §2, §3], [HMSZ, §6] and [CZZ, §4, §5]: the formal group algebra  $S$ , the Demazure and push-pull operators  $\Delta_\alpha$  and  $C_\alpha$  for every root  $\alpha$ , the formal twisted group algebra  $Q_W$  and its Demazure and push-pull elements  $X_\alpha$  and  $Y_\alpha$ . In section 4, we introduce a left  $Q_W$ -action ‘ $\bullet$ ’ on the dual  $Q_W^*$ . It induces both an action of the Weyl group  $W$  on  $Q_W^*$  (the Weyl-action) and an action of  $X_\alpha$  and  $Y_\alpha$  on  $Q_W^*$  (the Hecke-action). In sections 5 and 6, we introduce and study more general push-pull elements in  $Q_W$  and operators on  $Q_W^*$  with respect to given coset representatives of parabolic quotients of the Weyl group. In section 7 we study relationships between some technical coefficients. In section 8, we construct a basis of the subring of invariants of  $Q_W^*$ , which generalizes [KK90, Lemma 2.27]. In section 9, we recall the definition and basic properties of the formal (affine) Demazure algebra  $\mathbf{D}_F$  following [HMSZ, §6], [CZZ, §5] and [Zh13]. We show that for a certain elliptic formal group law (Example 2.2), the formal Demazure algebra can be identified with the classical Iwahori-Hecke algebra. In section 10, we define

the algebraic restriction to the fixed locus map which is used in section 12 to restrict all our push-pull operators and elements to  $\mathbf{D}_F$  and its dual  $\mathbf{D}_F^*$  as well as to restrict the non-degenerate pairing on  $\mathbf{D}_F^*$ . In section 11, we define the algebraic restriction to the fixed locus map on  $G/P$  for a parabolic subgroup  $P$ . In section 13, we define an involution on  $\mathbf{D}_F^*$  which is used to relate the equivariant characteristic map with the push-pull operators. In section 14, we define and discuss the non-degenerate pairing on the subring of invariants of  $\mathbf{D}_F^*$  under a parabolic subgroup of the Weyl group. At last, in section 15, in the parabolic case, we identify the Weyl group invariant subring  $(\mathbf{D}_F^*)^{W_\Xi}$  with  $\mathbf{D}_{F,\Xi}^*$ , the dual of a quotient of  $\mathbf{D}_F$ , which matches more naturally to  $\mathfrak{h}_T(G/P)$ .

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## 2. FORMAL PUSH-PULL AND DEMAZURE OPERATORS

In this section we recall definitions of the formal group algebra and of the formal Demazure and push-pull operators, following [CPZ, §2, §3] and [CZZ].

Let  $R$  be a commutative ring with unit, and let  $F$  be a one-dimensional commutative *formal group law* (FGL) over  $R$ , i.e.  $F(x, y) \in R[[x, y]]$  satisfies

$$F(x, 0) = 0, \quad F(x, y) = F(y, x) \quad \text{and} \quad F(x, F(y, z)) = F(F(x, y), z).$$

**Example 2.1.** The *additive* FGL is defined by  $F_a(x, y) = x + y$ , and a *multiplicative* FGL is defined by  $F_m(x, y) = x + y - \beta xy$  with  $\beta \in R$  (not necessarily a unit). The coefficient ring of the *universal* FGL  $F_u(x, y) = x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j$  is generated by the coefficients  $a_{i,j}$  modulo relations induced by the commutativity and associativity of the formal group law and is called the *Lazard ring*.

**Example 2.2.** Consider an elliptic curve given in Tate coordinates by

$$(1 - \mu_1 t - \mu_2 t^2)s = t^3.$$

The corresponding FGL over the coefficient ring  $R = \mathbb{Z}[\mu_1, \mu_2]$  is given by [BB10, Cor. 2.8]

$$F(x, y) := \frac{x+y-\mu_1 xy}{1+\mu_2 xy}.$$

Its genus is the 2-parameter generalized Todd genus introduced and studied by Hirzebruch in [Hi66]. Its exponent is given by the rational function  $\frac{e^{\epsilon_1 x} + e^{\epsilon_2 x}}{\epsilon_1 e^{\epsilon_1 x} + \epsilon_2 e^{\epsilon_2 x}}$  in  $e^x$ , where  $\mu_1 = \epsilon_1 + \epsilon_2$  and  $\mu_2 = -\epsilon_1 \epsilon_2$  which suggests to call  $F$  a *hyperbolic* FGL and to denote it by  $F_h$ .

By definition we have

$$F_h(x, y) = x + y - xy(\mu_1 + \mu_2 F_h(x, y))$$

and, thus, that the formal inverse of  $F_h$  is identical to the one of  $F_m$  (i.e.  $\frac{x}{\mu_1 x - 1}$ ) and  $F_h(x, x) = \frac{2x - \mu_1 x^2}{1 + \mu_2 x^2}$ .

Let  $\Lambda$  be an Abelian group and let  $R[[x_\Lambda]]$  be the ring of formal power series with variables  $x_\lambda$  for all  $\lambda \in \Lambda$ . Define the *formal group algebra*  $S := R[[\Lambda]]_F$  to be the quotient of  $R[[x_\Lambda]]$  by the closure of the ideal generated by elements  $x_0$  and  $x_{\lambda_1 + \lambda_2} - F(x_{\lambda_1}, x_{\lambda_2})$  for any  $\lambda_1, \lambda_2 \in \Lambda$  [CPZ, Def.2.4]. Here 0 is the identity element in  $\Lambda$ . Let  $\mathcal{I}_F$  denote the kernel of the augmentation map  $\epsilon: S \rightarrow R, x_\alpha \mapsto 0$ .

Let  $\Lambda$  be a free Abelian group of finite rank and let  $\Sigma$  be a finite subset of  $\Lambda$ . A *root datum* is an embedding  $\Sigma \hookrightarrow \Lambda^\vee$ ,  $\alpha \mapsto \alpha^\vee$  into the dual of  $\Lambda$  satisfying certain conditions [SGA, Exp. XXI, Def. 1.1.1]. The *rank* of the root datum is the  $\mathbb{Q}$ -rank of  $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ . The *root lattice*  $\Lambda_r$  is the subgroup of  $\Lambda$  generated by  $\Sigma$ , and the *weight lattice*  $\Lambda_w$  is the Abelian group defined by

$$\Lambda_w := \{\omega \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid \alpha^\vee(\omega) \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\}.$$

We always assume that the root datum is reduced and *semisimple* (the  $\mathbb{Q}$ -ranks of  $\Lambda_r$ ,  $\Lambda_w$  and  $\Lambda$  are the same and no root is twice another one). We say that a root datum is *simply connected* (resp. *adjoint*) if  $\Lambda = \Lambda_w$  (resp.  $\Lambda = \Lambda_r$ ), and then use the notation  $\mathcal{D}_n^{sc}$  (resp.  $\mathcal{D}_n^{ad}$ ) for irreducible root data where  $\mathcal{D} = A, B, C, D, E, F, G$  is one of the Dynkin types and  $n$  is the rank.

The *Weyl group*  $W$  of a root datum  $(\Lambda, \Sigma)$  is the subgroup of  $\text{Aut}_{\mathbb{Z}}(\Lambda)$  generated by simple reflections  $s_\alpha$  for all  $\alpha \in \Sigma$  defined by

$$s_\alpha(\lambda) := \lambda - \alpha^\vee(\lambda)\alpha, \quad \lambda \in \Lambda.$$

We fix a set of *simple roots*  $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Sigma$ , i.e. a basis of the root datum: each element of  $\Sigma$  is an integral linear combination of simple roots with either all positive or all negative coefficients. This partitions  $\Sigma$  into the subsets  $\Sigma^+$  and  $\Sigma^-$  of *positive* and *negative* roots. Let  $\ell$  denote the *length function* on  $W$  with respect to the set of simple roots  $\Pi$ . Let  $w_0$  be the *longest element* of  $W$  with respect to  $\ell$  and let  $N := \ell(w_0)$ .

Following [CZZ, Def. 4.4] we say that the formal group algebra  $S$  is  $\Sigma$ -*regular* if  $x_\alpha$  is not a zero divisor in  $S$  for all roots  $\alpha \in \Sigma$ . We will always assume that:

*The formal group algebra  $S$  is  $\Sigma$ -regular.*

By [CZZ, Lemma 2.2] this holds if  $x +_F x$  is not a zero divisor in  $R[[x]]$ , in particular if 2 is not a zero divisor in  $R$ , or if the root datum does not contain any symplectic datum  $C^{sc}$  as an irreducible component.

Following [CPZ, Definitions 3.5 and 3.12] for each  $\alpha \in \Sigma$  we define two  $R$ -linear operators  $\Delta_\alpha$  and  $C_\alpha$  on  $S$  as follows:

$$(2.1) \quad \Delta_\alpha(y) := \frac{y - s_\alpha(y)}{x_\alpha}, \quad C_\alpha(y) := \kappa_\alpha y - \Delta_\alpha(y) = \frac{y}{x - \alpha} + \frac{s_\alpha(y)}{x_\alpha}, \quad y \in S,$$

where  $\kappa_\alpha := \frac{1}{x_\alpha} + \frac{1}{x - \alpha}$  (note that  $\kappa_\alpha \in S$ ). The operator  $\Delta_\alpha$  is called the *Demazure operator* and the operator  $C_\alpha$  is called the *push-pull operator* or the *BGG operator*.

**Example 2.3.** For the hyperbolic formal group law  $F_h$  we have  $\kappa_\alpha = \mu_1 + \mu_2 F_h(x_{-\alpha}, x_\alpha) = \mu_1$  for each  $\alpha \in \Sigma$ . If the root datum is of type  $A_1^{sc}$ , we have  $\Sigma = \{\pm\alpha\}$ ,  $\Lambda = \langle \omega \rangle$  with simple root  $\alpha = 2\omega$  and

$$C_\alpha(x_\alpha) = \frac{x_\alpha}{x - \alpha} + \frac{x - \alpha}{x_\alpha} = \mu_1 x_\alpha - 1 + \frac{1}{\mu_1 x_\alpha - 1}, \quad C_\alpha(x_\omega) = \frac{x_\omega}{x - \alpha} + \frac{x - \omega}{x_\alpha} = \mu_1 x_\omega - \frac{1 + \mu_2 x_\omega^2}{1 - \mu_1 x_\omega}.$$

If it is of type  $A_2^{sc}$  we have  $\Sigma = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$ ,  $\Lambda = \langle \omega_1, \omega_2 \rangle$  with simple roots  $\alpha_1 = 2\omega_1 - \omega_2$ ,  $\alpha_2 = 2\omega_2 - \omega_1$  and  $x_{\alpha_1} = \frac{2x_1 - \mu_1 x_1^2 - x_2 - \mu_2 x_1^2 x_2}{1 + \mu_2 x_1^2 - \mu_1 x_2 - 2\mu_2 x_1 x_2}$ ,

$$C_{\alpha_2}(x_1) = \mu_1 x_1, \quad C_{\alpha_1}(x_1) = \mu_1 x_1 - \frac{1 + \mu_2 x_1^2 - \mu_1 x_2 - 2\mu_2 x_1 x_2}{1 - \mu_1 x_1 - \mu_2 x_1 x_2},$$

where  $x_1 := x_{\omega_1}$  and  $x_2 := x_{\omega_2}$ .

According to [CPZ, §3] the operators  $\Delta_\alpha$  satisfy the twisted Leibniz rule

$$(2.2) \quad \Delta_\alpha(xy) = \Delta_\alpha(x)y + s_\alpha(x)\Delta_\alpha(y), \quad x, y \in S,$$

i.e.  $\Delta_\alpha$  is a twisted derivation. Moreover, they are  $S^{W_\alpha}$ -linear, where  $W_\alpha = \{e, s_\alpha\}$ , and

$$(2.3) \quad s_\alpha(x) = x \text{ if and only if } \Delta_\alpha(x) = 0.$$

**Remark 2.4.** Properties (2.2) and (2.3) suggest that the Demazure operators can be effectively studied using the theory of twisted derivations and the invariant theory of  $W$ . On the other hand, push-pull operators do not satisfy properties (2.2) and (2.3) but according to [CPZ, Theorem 12.4] they correspond to the push-pull maps between flag varieties and, hence, are of geometric origin.

For the  $i$ -th simple root  $\alpha_i$ , let  $\Delta_i := \Delta_{\alpha_i}$  and  $s_i := s_{\alpha_i}$ . Given a non-empty sequence  $I = (i_1, \dots, i_m)$  with  $i_j \in \{1, \dots, n\}$  define

$$\Delta_I := \Delta_{i_1} \circ \dots \circ \Delta_{i_m} \text{ and } C_I := C_{i_1} \circ \dots \circ C_{i_m}.$$

We say that a sequence  $I$  is *reduced* in  $W$  if  $s_{i_1}s_{i_2}\dots s_{i_m}$  is a reduced expression of the element  $w = s_{i_1}s_{i_2}\dots s_{i_m}$  in  $W$ , i.e. it is of minimal length among such decompositions of  $w$ . In this case we also say that  $I$  is a *reduced sequence* for  $w$  of length  $\ell(w)$ . For the neutral element  $e$  of  $W$ , we set  $I_e = \emptyset$  and  $\Delta_\emptyset = C_\emptyset = \text{id}_S$ .

**Remark 2.5.** It is well-known that for a nontrivial root datum the composites  $\Delta_{I_w}$  and  $C_{I_w}$  are independent of the choice of a reduced sequence  $I_w$  of  $w \in W$  if and only if  $F$  is of the form  $F(x, y) = x + y + \beta xy$ ,  $\beta \in R$ . The “if” part of the statement is due to Demazure [Dem73, Th. 1] and the “only if” part is due to Bressler-Evens [BE90, Theorem 3.7]. So for such  $F$  we can define  $\Delta_w := \Delta_{I_w}$  and  $C_w := C_{I_w}$  for each  $w \in W$ .

The operators  $\Delta_w$  and  $C_w$  play a crucial role in the Schubert calculus and computations of the singular cohomology ( $F = F_a$ ) and the  $K$ -theory ( $F = F_m$ ) rings of flag varieties.

For a general  $F$  (e.g. for  $F = F_h$ ) the situation becomes much more intricate as we have to rely on choices of reduced decomposition  $I_w$ .

Let us now prove a Euclid type lemma for later use.

**Lemma 2.6.** *If  $f \in xR[[x]]$  is regular in  $R[[x]]$  and  $g \in yR[[y]]$ , then  $f(x) +_F g(y)$  is regular in  $R[[x, y]]$ .*

*Proof.* Consider  $f +_F g$  in  $R[[x, y]] = (R[[x]])[[y]]$  and note that its degree 0 coefficient (in  $R[[x]]$ ) is  $f$  and is regular by assumption, so  $f +_F g$  is regular by [CZZ, Lemma 12.3.(a)].  $\square$

**Lemma 2.7.** *For each irreducible component of the root datum, assume that the corresponding integers or formal integers listed in Table 1 are regular in  $R$  or  $R[[x]]$  (and that 2 is invertible for  $C_1^{sc}$ ). In particular,  $S$  is  $\Sigma$ -regular. Then  $x_\alpha | x_\beta x'$  implies that  $x_\alpha | x'$  for any two positive roots  $\alpha \neq \beta$  and for any  $x' \in S$ .*

(For example, in adjoint type  $E_7$  we require that either  $2 \cdot_F x$  or  $3 \cdot_F x$  is regular in  $R[[x]]$ , and in simply connected type  $E_7$ , we require that 2 is regular in  $R$ .)

Type	$A_l$ ( $l \geq 2$ )	$B_l$ ( $l \geq 3$ )	$C_l$ ( $l \geq 2$ )	$D_l$ ( $l \geq 4$ )	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
adjoint	$\emptyset$	$2 \cdot_F$	$2 \cdot_F$	$\emptyset$	$2 \cdot_F$ and $3 \cdot_F$	$2 \cdot_F$	$\emptyset$	$2 \cdot_F$ or $3 \cdot_F$	$2 \cdot_F$ or $3 \cdot_F$
non adjoint	$ \Lambda/\Lambda_r $	2	$2 \in R^\times$	2	-	-	3	2	-

TABLE 1. Integers and formal integers assumed to be regular in  $R$  or  $R[[x]]$  in Lemma 2.7. In the simply connected  $C_2$  case, we require 2 invertible in  $R$ .

*Proof of Lemma 2.7.* It is equivalent to show that  $x_\beta$  is regular in  $S/(x_\alpha)$ .

If  $\alpha$  and  $\beta$  belong to different irreducible components, we can complete  $\alpha$  and  $\beta$  into bases of the lattices of their respective components by [CZZ, Lemma 2.1], and then complete the union of the two sets into a basis of  $\Lambda$ . By [CPZ, Cor. 2.13], this gives an isomorphism  $S \simeq R[[x_1, \dots, x_l]]$  sending  $x_\alpha$  to  $x_1$  and  $x_\beta$  to  $x_2$ , so the conclusion is obvious in this case.

If  $\alpha$  and  $\beta$  belong to the same irreducible component, we can assume that the root datum is irreducible.

*Adjoint case.* Complete  $\alpha$  to a basis  $(\alpha_i)_{1 \leq i \leq l}$  of simple roots of  $\Sigma$  and express  $\beta = \sum_i n_i \alpha_i$ . Still by [CPZ, Cor. 2.13], this yields an isomorphism  $S \simeq R[[x_1, \dots, x_l]]$ , sending  $x_\alpha$  to  $x_1$  and  $x_\beta$  to  $(n_1 \cdot_F x_1) +_F \dots +_F (n_l \cdot_F x_l)$ . A repeated application of Lemma 2.6 shows that  $x_\beta$  is regular provided  $n_i \cdot_F x$  is regular in  $R[[x]]$  for at least one  $i \neq 1$ . Using Planche I to IX in [Bo68] giving coefficients of positive roots decomposed in terms of simple ones, one checks for every type that it is always the case under the assumptions. For example, in the  $E_6$  case, there are always two 1's in any decomposition (except if the root is simple), hence the absence of any requirement. In the  $E_7$  case, the same is true except for the longest root, in which there is a 1, a 2 and a 3, hence the requirement that  $2 \cdot_F x$  or  $3 \cdot_F x$  is regular in  $R[[x]]$ . All other cases are as easy and left to the reader.

*Non adjoint case.* By [CZZ, Lemma 1.2], the natural morphism  $R[[\Lambda_r]]_F \rightarrow R[[\Lambda]]_F$  induced by the inclusion of the root lattice  $\Lambda_r \subset \Lambda$  is injective. Furthermore, it becomes an isomorphism if  $q = |\Lambda/\Lambda_r|$  is invertible in  $R$ .

Since  $\alpha$  can be completed to a basis of  $\Lambda$  or to a basis of  $\Lambda_r$ , both  $R[[\Lambda_r]]_F/x_\alpha$  and  $R[[\Lambda]]_F/x_\alpha$  are isomorphic to power series ring (in one less variable) and therefore respectively inject in  $R[\frac{1}{q}][[\Lambda_r]]_F/x_\alpha$  and  $R[\frac{1}{q}][[\Lambda]]_F/x_\alpha$ , which are isomorphic. By the adjoint case,  $x_\beta$  is regular in the latter, and thus in its subring  $S/x_\alpha = R[[\Lambda]]_F/x_\alpha$ .  $\square$

**Remark 2.8.** Since  $n \cdot_F x$  is regular in  $R[[x]]$  if  $n$  is regular in  $R$ , the conclusion of Lemma 2.7 holds when formal integers are replaced by usual integers in  $R$  in the adjoint case.

Moreover, if the formal group law is the multiplicative one  $x + y - xy$ , then  $2 \cdot_F x$  is regular in  $R[[x]]$  for any noetherian ring  $R$  (to show this consider the ideal generated by the coefficients of a power series annihilating  $2 \cdot_F x$ ). In particular, Lemma 2.7 still holds for all adjoint types in the case  $R = \mathbb{Z}[a, b]/(2a, 3b)$  (in which neither 2 nor 3 are regular).



## 3. TWO BASES OF THE FORMAL TWISTED GROUP ALGEBRA

We now recall definitions and basic properties of the formal twisted group algebra  $Q_W$ , Demazure elements  $X_\alpha$  and push-pull elements  $Y_\alpha$ , following [HMSZ] and [CZZ]. For a chosen set of reduced sequences  $\{I_w\}_{w \in W}$  we introduce two  $Q$ -bases  $\{X_{I_w}\}_{w \in W}$  and  $\{Y_{I_w}\}_{w \in W}$  of  $Q_W$  and describe transformation matrices  $(a_{v,w}^X)$  and  $(a_{v,w}^Y)$  with respect to the canonical basis  $\{\delta_w\}_{w \in W}$  of  $Q_W$ .

Let  $S_W$  be the *twisted group algebra* of  $S$  and the group ring  $R[W]$ , i.e.  $S_W = S \otimes_R R[W]$  as an  $R$ -module and the multiplication is defined by

$$(3.1) \quad (x \otimes \delta_w)(x' \otimes \delta_{w'}) = xw(x') \otimes \delta_{ww'}, \quad x, x' \in S, \quad w, w' \in W,$$

where  $\delta_w$  is the canonical element corresponding to  $w$  in  $R[W]$ . The algebra  $S_W$  is a free  $S$ -module with basis  $\{1 \otimes \delta_w\}_{w \in W}$ . Note that  $S_W$  is not an  $S$ -algebra since the embedding  $S \hookrightarrow S_W$ ,  $x \mapsto x \otimes \delta_e$  is not central.

Since the formal group algebra  $S$  is  $\Sigma$ -regular, it embeds into the localization  $Q = S[\frac{1}{x_\alpha} \mid \alpha \in \Sigma]$ . Let  $Q_W$  be the  $Q$ -module obtained by localizing the  $S$ -module  $S_W$ , i.e.  $Q_W = Q \otimes_S S_W$ . The product on  $S_W$  extends to  $Q_W$  using the same formula (3.1) on basis elements ( $x$  and  $x'$  are now in  $Q$ ).

Inside  $Q_W$ , we use the notation  $q := q \otimes \delta_e$  and  $\delta_w := 1 \otimes \delta_w$ ,  $1 := \delta_e$  and  $\delta_\alpha := \delta_{s_\alpha}$  for a root  $\alpha \in \Sigma$ . Thus  $q\delta_w = q \otimes \delta_w$  and  $\delta_w q = w(q) \otimes \delta_w$ . By definition,  $\{\delta_w\}_{w \in W}$  is a basis of  $Q_W$  as a left  $Q$ -module, and  $S_W$  injects into  $Q_W$  via  $\delta_w \mapsto \delta_w$ .

Similarly to (2.1) for each  $\alpha \in \Sigma$  we define the following elements of  $Q_W$  (corresponding to the operators  $\Delta_\alpha$  and  $C_\alpha$ , respectively, by the action of (4.3)):

$$X_\alpha := \frac{1}{x_\alpha} - \frac{1}{x_\alpha} \delta_\alpha, \quad Y_\alpha := \kappa_\alpha - X_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha} \delta_\alpha$$

called the *Demazure elements* and the *push-pull elements*, respectively.

Direct computations show that for each  $\alpha \in \Sigma$  we have

$$(3.2) \quad \begin{aligned} X_\alpha^2 &= \kappa_\alpha X_\alpha = X_\alpha \kappa_\alpha \quad \text{and} \quad Y_\alpha^2 = \kappa_\alpha Y_\alpha = Y_\alpha \kappa_\alpha, \\ X_\alpha q &= s_\alpha(q) X_\alpha + \Delta_\alpha(q) \quad \text{and} \quad Y_\alpha q = s_\alpha(q) Y_\alpha + \Delta_{-\alpha}(q), \quad q \in Q, \\ X_\alpha Y_\alpha &= Y_\alpha X_\alpha = 0. \end{aligned}$$

We set  $\delta_i := \delta_{s_i}$ ,  $X_i := X_{\alpha_i}$  and  $Y_i := Y_{\alpha_i}$  for the  $i$ -th simple root  $\alpha_i$ . Given a sequence  $I = (i_1, i_2, \dots, i_m)$  with  $i_j \in \{1, \dots, n\}$ , the product  $X_{i_1} X_{i_2} \dots X_{i_m}$  is denoted by  $X_I$  and the product  $Y_{i_1} Y_{i_2} \dots Y_{i_m}$  by  $Y_I$ . We set  $X_\emptyset = Y_\emptyset = 1$ .

By [Bo68, Ch. VI, §1, No 6, Cor. 2] if  $v \in W$  has a reduced decomposition  $v = s_{i_1} s_{i_2} \dots s_{i_m}$ , then

$$(3.3) \quad v\Sigma^- \cap \Sigma^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} s_{i_2} \dots s_{i_{m-1}}(\alpha_{i_m})\}.$$

We define

$$x_v := \prod_{\alpha \in v\Sigma^- \cap \Sigma^+} x_\alpha.$$

In particular,  $x_{w_0} = \prod_{\alpha \in \Sigma^+} x_\alpha$  if  $w_0$  is the longest element of  $W$ .

**Lemma 3.1.** *We have*

- (a)  $s_\alpha \Sigma^- \cap \Sigma^+ = \{\alpha\}$  and  $x_{s_\alpha} = x_\alpha$ ;
- (b) if  $\ell(vs_i) = \ell(v) + 1$ , then

$$vs_i \Sigma^- \cap \Sigma^+ = (v\Sigma^- \cap \Sigma^+) \sqcup \{v(\alpha_i)\} \quad \text{and} \quad x_{vs_i} = x_v x_{v(\alpha_i)};$$

(c) if  $\ell(s_i v) = \ell(v) + 1$ , then

$$s_i v \Sigma^- \cap \Sigma^+ = s_i (v \Sigma^- \cap \Sigma^+) \sqcup \{\alpha_i\} \text{ and } x_{s_i v} = s_i(x_v) x_{\alpha_i};$$

(d) if  $w = uv$  and  $\ell(w) = \ell(u) + \ell(v)$ , then

$$w \Sigma^- \cap \Sigma^+ = (u \Sigma^- \cap \Sigma^+) \sqcup u(v \Sigma^- \cap \Sigma^+) \text{ and } x_w = x_u u(x_v);$$

(e) for any  $v \in W$ ,  $\frac{v(x_{w_0})}{x_{w_0}}$  is invertible in  $S$ .

*Proof.* Items (a)-(d) follow immediately from the definition. As for (e) we have

$$v \Sigma^+ = (v \Sigma^+ \cap \Sigma^-) \sqcup (v \Sigma^+ \cap \Sigma^+) = (-(v \Sigma^- \cap \Sigma^+)) \sqcup (v \Sigma^+ \cap \Sigma^+) \text{ and} \\ \Sigma^+ = v \Sigma \cap \Sigma^+ = (v \Sigma^- \cap \Sigma^+) \sqcup (v \Sigma^+ \cap \Sigma^+), \text{ therefore,}$$

$$\frac{v(x_{w_0})}{x_{w_0}} = \frac{\prod_{\alpha \in v \Sigma^+} x_\alpha}{\prod_{\alpha \in \Sigma^+} x_\alpha} = \prod_{\alpha \in v \Sigma^- \cap \Sigma^+} \frac{x_{-\alpha}}{x_\alpha},$$

which is invertible in  $S$  since so is  $\frac{x_{-\alpha}}{x_\alpha}$ .  $\square$

**Lemma 3.2.** *Let  $I_v$  be a reduced sequence for an element  $v \in W$ .*

*Then  $X_{I_v} = \sum_{w \leq v} a_{v,w}^X \delta_w$  for some  $a_{v,w}^X \in Q$ , where the sum is taken over all elements of  $W$  less or equal to  $v$  with respect to the Bruhat order and  $a_{v,v}^X = (-1)^{\ell(v)} \frac{1}{x_v}$ . Moreover, we have  $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w}$  for some  $b_{v,w}^X \in S$  such that  $b_{v,e}^X = 1$  and  $b_{v,v}^X = (-1)^{\ell(v)} x_v$ .*

*Proof.* It follows from [CZZ, Lemma 5.4, Corollary 5.6] and the fact that  $\delta_\alpha = 1 - x_\alpha X_\alpha$ .  $\square$

Similarly, for  $Y$ 's we have

**Lemma 3.3.** *Let  $I_v$  be a reduced sequence for an element  $v \in W$ .*

*Then  $Y_{I_v} = \sum_{w \leq v} a_{v,w}^Y \delta_w$  for some  $a_{v,w}^Y \in Q$  and  $a_{v,v}^Y = \frac{1}{x_v}$ . Moreover, we have  $\delta_v = \sum_{w \leq v} b_{v,w}^Y Y_{I_w}$  for some  $b_{v,w}^Y \in S$  and  $b_{v,v}^Y = x_v$ .*

*Proof.* We follow the proof of [CZZ, Lemma 5.4] replacing  $X$  by  $Y$ . By induction we have

$$Y_{I_v} = \left( \frac{1}{x_{-\beta}} + \frac{1}{x_\beta} \delta_\beta \right) \sum_{w \leq v'} a_{v',w}^Y \delta_w = \frac{1}{x_\beta} s_\beta (a_{v',v'}^Y) \delta_v + \sum_{w < v} a_{v,w}^Y \delta_w,$$

where  $I_v = (i_1, \dots, i_m)$  is a reduced sequence of  $v$ ,  $\beta = \alpha_{i_1}$  and  $v' = s_\beta v$ . This implies the formulas for  $Y_{I_v}$  and for  $a_{v,v}^Y$ . The remaining statements involving  $b_{v,w}^Y$  follow by the same arguments as in the proof of [CZZ, Corollary 5.6] using the fact that  $\delta_\alpha = x_\alpha Y_\alpha - \frac{x_\alpha}{x_{-\alpha}}$  and  $\frac{x_\alpha}{x_{-\alpha}} \in S^\times$ .  $\square$

As in the proof of [CZZ, Corollary 5.6], Lemmas 3.2 and 3.3 immediately imply:

**Corollary 3.4.** *The family  $\{X_{I_v}\}_{v \in W}$  (resp.  $\{Y_{I_v}\}_{v \in W}$ ) is a basis of  $Q_W$  as a left or as a right  $Q$ -module.*

**Example 3.5.** For the root data  $A_1^{ad}$  or  $A_1^{sc}$  and the formal group law  $F_h$  we have  $x_\Pi = x_{-\alpha}$  and

$$(a_{v,w}^Y)_{v,w \in W} = \begin{pmatrix} 1 & 0 \\ \mu_1 - \frac{1}{x_\alpha} & \frac{1}{x_\alpha} \end{pmatrix},$$

where the first row and column correspond to  $e \in W$  and the second to  $s_\alpha \in W$ .

Multiplying the base-change matrices of Lemma 3.2 and 3.3 we obtain

**Corollary 3.6.** *For each  $v \in W$  we have*

$$Y_{I_v} = (-1)^{\ell(v)} X_{I_v} + \sum_{w < v} c_{v,w} X_{I_w}, \quad c_{v,w} \in Q.$$

#### 4. THE WEYL AND THE HECKE ACTIONS

In the present section we recall several basic facts concerning the  $Q$ -linear dual  $Q_W^*$  following [HMSZ] and [CZZ]. We introduce a left  $Q_W$ -action ‘ $\bullet$ ’ on  $Q_W^*$  which induces an action of the Weyl group  $W$  on  $Q_W^*$  (the *Weyl-action*) and the action by means of  $X_\alpha$  and  $Y_\alpha$  on  $Q_W^*$  (the *Hecke-action*). These two actions will play an important role in the sequel.

Let  $Q_W^* := \text{Hom}_Q(Q_W, Q)$  denote the  $Q$ -linear dual of the left  $Q$ -module  $Q_W$ . By definition,  $Q_W^*$  is a left  $Q$ -module via  $(qf)(z) := qf(z)$  for any  $z \in Q_W$ ,  $f \in Q_W^*$  and  $q \in Q$ . Moreover, there is a  $Q$ -basis  $\{f_w\}_{w \in W}$  of  $Q_W^*$  dual to the canonical basis  $\{\delta_w\}_{w \in W}$  defined by  $f_w(\delta_v) := \delta_{w,v}^{\text{Kr}}$  (the Kronecker symbol) for  $w, v \in W$ .

**Definition 4.1.** We define a left action of  $Q_W$  on  $Q_W^*$  as follows:

$$(z \bullet f)(z') := f(z'z), \quad z, z' \in Q_W, f \in Q_W^*.$$

By definition, this action is left  $Q$ -linear, i.e.  $z \bullet (qf) = q(z \bullet f)$  and it induces a different left  $Q$ -module structure on  $Q_W^*$  via the embedding  $q \mapsto q\delta_e$ , i.e.

$$(q \bullet f)(z) := f(zq).$$

It also induces a  $Q$ -linear action of  $W$  on  $Q_W^*$  via  $w(f) := \delta_w \bullet f$ .

**Lemma 4.2.** *We have  $q \bullet f_w = w(q)f_w$  and  $w(f_v) = f_{vw^{-1}}$  for any  $q \in Q$  and  $w, v \in W$ .*

*Proof.* We have  $(q \bullet f_w)(\delta_v) = f_w(v(q)\delta_v) = v(q)\delta_{w,v}^{\text{Kr}}$  which shows that  $q \bullet f_v = v(q)f_v$ . For the second equality, we have  $[w(f_v)](\delta_u) = f_v(\delta_u\delta_w) = \delta_{v,uw}^{\text{Kr}}$ , so  $w(f_v) = f_{vw^{-1}}$ .  $\square$

There is a coproduct on the twisted group algebra  $S_W$  that extends to  $Q_W$  defined by [CZZ, Def. 8.9]:

$$\Delta : Q_W \rightarrow Q_W \otimes_Q Q_W, \quad q\delta_w \mapsto q\delta_w \otimes \delta_w.$$

Here  $\otimes_Q$  is the tensor product of left  $Q$ -modules. The coproduct is cocommutative with co-unit  $\varepsilon : Q_W \rightarrow Q$ ,  $q\delta_w \mapsto q$  [CZZ, Prop. 8.10]. The coproduct structure on  $Q_W$  induces a product structure on  $Q_W^*$ , which is  $Q$ -bilinear for the natural action of  $Q$  on  $Q_W^*$  (not the one using  $\bullet$ ). In terms of the basis  $\{f_w\}_{w \in W}$  this product is given by component-wise multiplication:

$$(4.1) \quad \left( \sum_{v \in W} q_v f_v \right) \left( \sum_{w \in W} q'_w f_w \right) = \sum_{w \in W} q_w q'_w f_w, \quad q_w, q'_w \in Q.$$

In other words, if we identify the dual  $Q_W^*$  with the  $Q$ -module of maps  $\text{Hom}(W, Q)$  via

$$Q_W^* \rightarrow \text{Hom}(W, Q), \quad f \mapsto f', \quad f'(w) := f(\delta_w),$$

then the product is the classical multiplication of ring-valued functions.

The multiplicative identity  $\mathbf{1}$  of this product corresponds to the counit  $\varepsilon$  and equals  $\mathbf{1} = \sum_{w \in W} f_w$ . We also have

$$(4.2) \quad q \bullet (ff') = (q \bullet f)f' = f(q \bullet f') \quad \text{for } q \in Q \text{ and } f, f' \in Q_W^*.$$

**Lemma 4.3.** *For any  $\alpha \in \Sigma$  and  $f, f' \in Q_W^*$  we have  $s_\alpha(ff') = s_\alpha(f)s_\alpha(f')$ , i.e. the Weyl group  $W$  acts on the algebra  $Q_W^*$  by  $Q$ -linear automorphisms.*

*Proof.* By  $Q$ -linearity of the action of  $W$  and of the product, it suffices to check the formula on basis elements  $f = f_w$  and  $f' = f_v$ , for which it is straightforward.  $\square$

Observe that the ring  $Q$  can be viewed as a left  $Q_W$ -module via the following action:

$$(4.3) \quad (q\delta_w) \cdot q' := qw(q'), \quad q, q' \in Q, \quad w \in W.$$

Then by definition we have

$$(4.4) \quad (q \bullet \mathbf{1})(z) = z \cdot q, \quad z \in Q_W.$$

**Definition 4.4.** For  $\alpha \in \Sigma$  we define two  $Q$ -linear operators on  $Q_W^*$  by

$$A_\alpha(f) := Y_\alpha \bullet f \quad \text{and} \quad B_\alpha(f) := X_\alpha \bullet f, \quad f \in Q_W^*.$$

An action by means of  $A_\alpha$  or  $B_\alpha$  will be called a *Hecke-action* on  $Q_W^*$ .

**Remark 4.5.** If  $F = F_m$  (resp.  $F = F_a$ ) one obtains actions introduced by Kostant–Kumar in [KK90, I<sub>18</sub>] (resp. in [KK86, I<sub>51</sub>]).

As in (2.2) and (2.3) we have

$$(4.5) \quad B_\alpha(ff') = B_\alpha(f)f' + s_\alpha(f)B_\alpha(f') \quad \text{and} \quad B_\alpha \circ s_\alpha = -B_\alpha, \quad \text{for } f, f' \in Q_W^*,$$

$$(4.6) \quad B_\alpha(f) = 0 \quad \text{if and only if } f \in (Q_W^*)^{W_\alpha}.$$

Indeed, using (4.2) and Lemma 4.3 we obtain

$$\begin{aligned} B_\alpha(f)f' + s_\alpha(f)B_\alpha(f') &= \left[\frac{1}{x_\alpha}(1 - \delta_\alpha) \bullet f\right]f' + s_\alpha(f)\left[\frac{1}{x_\alpha}(1 - \delta_\alpha) \bullet f'\right] \\ &= \left[\frac{1}{x_\alpha} \bullet (f - s_\alpha(f))\right]f' + s_\alpha(f)\left[\frac{1}{x_\alpha} \bullet (f' - s_\alpha(f'))\right] \\ &= \frac{1}{x_\alpha} \bullet (ff' - s_\alpha(f)s_\alpha(f')) = B_\alpha(ff') \end{aligned}$$

and  $B_\alpha(s_\alpha(f)) = \frac{1}{x_\alpha}(1 - \delta_\alpha) \bullet s_\alpha(f) = \frac{1}{x_\alpha} \bullet (s_\alpha(f) - f) = -B_\alpha(f)$ . As for (4.6) we have  $0 = B_\alpha(f) = X_\alpha \bullet f = \frac{1}{x_\alpha} \bullet [(1 - \delta_\alpha) \bullet f]$  which is equivalent to  $f = s_\alpha(f)$ .

And as in (3.2), we obtain

$$(4.7) \quad A_\alpha^{\circ 2}(f) = \kappa_\alpha \bullet A_\alpha(f) = A_\alpha(\kappa_\alpha \bullet f), \quad B_\alpha^{\circ 2}(f) = \kappa_\alpha \bullet B_\alpha(f) = B_\alpha(\kappa_\alpha \bullet f),$$

$$A_\alpha \circ B_\alpha = B_\alpha \circ A_\alpha = 0.$$

We set  $A_i = A_{\alpha_i}$  and  $B_i := B_{\alpha_i}$  for the  $i$ -th simple root  $\alpha_i$ . We set  $A_I = A_{i_1} \circ \dots \circ A_{i_m}$  and  $B_I = B_{i_1} \circ \dots \circ B_{i_m}$  for a non-empty sequence  $I = (i_1, \dots, i_m)$  with  $i_j \in \{1, \dots, n\}$  and  $A_\emptyset = B_\emptyset = \text{id}$ . The operators  $A_I$  and  $B_I$  are key ingredients in the proof that the natural pairing of Theorem 12.4 on the dual of the formal affine Demazure algebra is non-degenerate.

## 5. PUSH-PULL ELEMENTS AND OPERATORS

Let us now introduce and study a key notion of the present paper, the notion of push-pull operators (resp. elements) on  $Q$  (resp. in  $Q_W$ ) with respect to given coset representatives in parabolic quotients of the Weyl group.

Let  $(\Sigma, \Lambda)$  be a root datum with a chosen set of simple roots  $\Pi$ . Let  $\Xi \subseteq \Pi$  and let  $W_\Xi$  denote the subgroup of the Weyl group  $W$  of the root datum generated by simple reflections  $s_\alpha$ ,  $\alpha \in \Xi$ . We thus have  $W_\emptyset = \{e\}$  and  $W_\Pi = W$ . Let  $\Sigma_\Xi := \{\alpha \in \Sigma \mid s_\alpha \in W_\Xi\}$  and let  $\Sigma_\Xi^\pm := \Sigma_\Xi \cap \Sigma^\pm$ ,  $\Sigma_\Xi^- := \Sigma_\Xi \cap \Sigma^-$  be subsets of positive and negative roots respectively.

Given subsets  $\Xi' \subseteq \Xi$  of  $\Pi$ , let  $\Sigma_{\Xi/\Xi'}^+ := \Sigma_\Xi^+ \setminus \Sigma_{\Xi'}^+$  and  $\Sigma_{\Xi/\Xi'}^- := \Sigma_\Xi^- \setminus \Sigma_{\Xi'}^-$ . We define

$$x_{\Xi/\Xi'} := \prod_{\alpha \in \Sigma_{\Xi/\Xi'}^-} x_\alpha \quad \text{and set } x_\Xi := x_{\Xi/\emptyset}.$$

In particular,  $x_\Pi = \prod_{\alpha \in \Sigma^-} x_\alpha = w_0(x_{w_0})$ .

**Lemma 5.1.** *Given subsets  $\Xi' \subseteq \Xi$  of  $\Pi$  we have*

$$v(\Sigma_{\Xi/\Xi'}^-) = \Sigma_{\Xi/\Xi'}^-, \text{ and } v(\Sigma_{\Xi/\Xi'}^+) = \Sigma_{\Xi/\Xi'}^+, \text{ for any } v \in W_{\Xi'}.$$

*Proof.* We prove the first statement only, the second one can be proven similarly. Since  $v$  acts faithfully on  $\Sigma_\Xi$ , it suffices to show that for any  $\alpha \in \Sigma_{\Xi/\Xi'}^-$ , the root  $\beta := v(\alpha) \notin \Sigma_{\Xi'}$  and is negative. Indeed, if  $\beta \in \Sigma_{\Xi'}$ , then so is  $\alpha = v^{-1}(\beta)$  (as  $v^{-1} \in W_{\Xi'}$ ), which is impossible. On the other hand, if  $\beta$  is positive, then

$$\beta = v(\alpha) \in v\Sigma_\Xi^- \cap \Sigma_\Xi^+ = v\Sigma_{\Xi'}^- \cap \Sigma_\Xi^+,$$

where the latter equality follows from (3.3) and the fact that  $v \in W_{\Xi'}$ . So  $\alpha = v^{-1}(\beta) \in \Sigma_{\Xi'}$ , a contradiction.  $\square$

**Corollary 5.2.** *For any  $v \in W_{\Xi'}$ , we have  $v(x_{\Xi/\Xi'}) = x_{\Xi/\Xi'}$ .*

**Definition 5.3.** Given a set of left coset representatives  $W_{\Xi/\Xi'}$  of  $W_\Xi/W_{\Xi'}$  we define a *push-pull operator* on  $Q$  with respect to  $W_{\Xi/\Xi'}$  by

$$C_{\Xi/\Xi'}(q) := \sum_{w \in W_{\Xi/\Xi'}} w\left(\frac{q}{x_{\Xi/\Xi'}}\right), \quad q \in Q,$$

and a *push-pull element* with respect to  $W_{\Xi/\Xi'}$  by

$$Y_{\Xi/\Xi'} := \left( \sum_{w \in W_{\Xi/\Xi'}} \delta_w \right) \frac{1}{x_{\Xi/\Xi'}}.$$

We set  $C_\Xi := C_{\Xi/\emptyset}$  and  $Y_\Xi := Y_{\Xi/\emptyset}$  (so they do not depend on the choice of  $W_{\Xi/\emptyset} = W_\Xi$  in these two special cases).

By definition, we have  $C_{\Xi/\Xi'}(q) = Y_{\Xi/\Xi'} \cdot q$ , where  $Y_{\Xi/\Xi'}$  acts on  $q \in Q$  by (4.3). Also in the trivial case where  $\Xi = \Xi'$ , we have  $x_{\Xi/\Xi} = 1$ , while  $C_{\Xi/\Xi} = \text{id}_Q$  and  $Y_{\Xi/\Xi} = 1$  if we choose  $e$  as representative of the only coset. Observe that for  $\Xi = \{\alpha_i\}$  we have  $W_\Xi = \{e, s_i\}$  and  $C_\Xi = C_i$  (resp.  $Y_\Xi = Y_i$ ) is the push-pull operator (resp. element) introduced before and preserves  $S$ .

**Example 5.4.** For the formal group law  $F_h$  and the root datum  $A_2$ , we have  $x_\Pi = x_{-\alpha_1}x_{-\alpha_2}x_{-\alpha_1-\alpha_2}$  and

$$C_\Pi(1) = \sum_{w \in W} w\left(\frac{1}{x_\Pi}\right) = \mu_1\left(\frac{1}{x_{-\alpha_2}x_{-\alpha_1-\alpha_2}} + \frac{1}{x_{-\alpha_1}x_{\alpha_2}} + \frac{1}{x_{\alpha_1}x_{\alpha_1+\alpha_2}}\right) = \mu_1^3 + \mu_1\mu_2.$$

**Lemma 5.5.** *The operator  $C_{\Xi/\Xi'}$  restricted to  $Q^{W_{\Xi'}}$  is independent of the choices of representatives  $W_{\Xi/\Xi'}$  and it maps  $Q^{W_{\Xi'}}$  to  $Q^{W_\Xi}$ .*

*Proof.* The independence follows, since  $\frac{1}{x_{\Xi/\Xi'}} \in Q^{W_{\Xi'}}$  by Corollary 5.2. The second part follows, since for any  $v \in W_\Xi$ , and for any set of coset representatives  $W_{\Xi/\Xi'}$ , the set  $vW_{\Xi/\Xi'}$  is again a set of coset representatives.  $\square$

Actually, we will see in Corollary 12.2 that the operator  $C_\Xi$  sends  $S$  to  $S^{W_\Xi}$ .

**Remark 5.6.** The formula for the operator  $C_\Xi$  (with  $\Xi' = \emptyset$ ) had appeared before in related contexts, namely, in discussions around the Becker-Gottlieb transfer for topological complex-oriented theories (see [BE90, (2.1)] and [GR12, §4.1]). The definition of the element  $Y_{\Xi/\Xi'}$  can be viewed as a generalized algebraic analogue of this formula.

**Lemma 5.7** (Composition rule). *Given subsets  $\Xi'' \subseteq \Xi' \subseteq \Xi$  of  $\Pi$  and given sets of representatives  $W_{\Xi/\Xi'}$  and  $W_{\Xi'/\Xi''}$ , take  $W_{\Xi/\Xi''} := \{wv \mid w \in W_{\Xi/\Xi'}, v \in W_{\Xi'/\Xi''}\}$  as the set of representatives of  $W_\Xi/W_{\Xi''}$ . Then*

$$C_{\Xi/\Xi'} \circ C_{\Xi'/\Xi''} = C_{\Xi/\Xi''} \text{ and } Y_{\Xi/\Xi'} Y_{\Xi'/\Xi''} = Y_{\Xi/\Xi''}.$$

*Proof.* We prove the formula for  $Y$ 's, the one for  $C$ 's follows since  $C$  acts as  $Y$ , and the composition of actions corresponds to multiplication. We have  $Y_{\Xi/\Xi'} Y_{\Xi'/\Xi''} =$

$$\left(\sum_{w \in W_{\Xi/\Xi'}} \delta_w \frac{1}{x_{\Xi/\Xi'}}\right) \left(\sum_{v \in W_{\Xi'/\Xi''}} \delta_v \frac{1}{x_{\Xi'/\Xi''}}\right) = \sum_{w \in W_{\Xi/\Xi'}, v \in W_{\Xi'/\Xi''}} \delta_{wv} \frac{1}{v^{-1}(x_{\Xi/\Xi'})x_{\Xi'/\Xi''}}.$$

By Corollary 5.2, we have  $v^{-1}(x_{\Xi/\Xi'}) = x_{\Xi/\Xi'}$ . Therefore,  $v^{-1}(x_{\Xi/\Xi'})x_{\Xi'/\Xi''} = x_{\Xi/\Xi'}x_{\Xi'/\Xi''} = x_{\Xi/\Xi''}$ . We conclude by definition of  $W_{\Xi/\Xi''}$ .  $\square$

The following lemma follows from the definition of  $C_{\Xi/\Xi'}$ .

**Lemma 5.8** (Projection formula). *We have*

$$C_{\Xi/\Xi'}(qq') = qC_{\Xi/\Xi'}(q') \text{ for any } q \in Q^{W_\Xi} \text{ and } q' \in Q.$$

**Lemma 5.9.** *Given a subset  $\Xi$  of  $\Pi$  and  $\alpha \in \Xi$  we have*

- (a)  $Y_\Xi = Y'Y_\alpha = Y_\alpha Y''$  for some  $Y'$  and  $Y'' \in Q_W$ ,
- (b)  $Y_\Xi X_\alpha = X_\alpha Y_\Xi = 0$ ,  $Y_\alpha Y_\Xi = \kappa_\alpha Y_\Xi$  and  $Y_\Xi Y_\alpha = Y_\Xi \kappa_\alpha$ .

*Proof.* (a) The first identity follows from Lemma 5.7 applied to  $\Xi' = \{\alpha\}$  (in this case  $Y' = Y_{\Xi/\Xi'}$ ).

For the second identity, let  ${}^\alpha W_\Xi$  be a set of right coset representatives of  $W_\alpha \backslash W_\Xi$ , thus each  $w \in W_\Xi$  can be written uniquely either as  $w = s_\alpha u$  or as  $w = u$  with  $u \in {}^\alpha W_\Xi$ . Then

$$\begin{aligned} Y_\Xi &= \sum_{u \in {}^\alpha W_\Xi} (1 + \delta_\alpha) \delta_u \frac{1}{x_\Xi} = \sum_{u \in {}^\alpha W_\Xi} (1 + \delta_\alpha) \frac{1}{x_{-\alpha}} x_{-\alpha} \delta_u \frac{1}{x_\Xi} \\ &= \sum_{u \in {}^\alpha W_\Xi} Y_\alpha x_{-\alpha} \delta_u \frac{1}{x_\Xi} = Y_\alpha \sum_{u \in {}^\alpha W_\Xi} \delta_u \frac{u^{-1}(x_{-\alpha})}{x_\Xi}. \end{aligned}$$

(b) then follows from (a) and (3.2).  $\square$

## 6. THE PUSH-PULL OPERATORS ON THE DUAL

We now introduce and study the push-pull operators on the dual of the twisted formal group algebra  $Q_W^*$ .

For  $w \in W$ , we define  $f_w^\Xi := \sum_{v \in wW_\Xi} f_v$ . Observe that  $f_w^\Xi = f_{w'}^\Xi$  if and only if  $wW_\Xi = w'W_\Xi$ . Consider the subring of invariants  $(Q_W^*)^{W_\Xi}$  by means of the  $\bullet$ '-action of  $W_\Xi$  on  $Q_W^*$  and fix a set of representatives  $W_{\Pi/\Xi}$  of  $W/W_\Xi$ . By Lemma 4.2, we then have the following

**Lemma 6.1.** *The family  $\{f_w^\Xi\}_{w \in W_{\Pi/\Xi}}$  forms a basis of  $(Q_W^*)^{W_\Xi}$  as a left  $Q$ -module, and  $f_w^\Xi f_v^\Xi = \delta_{w,v}^{Kr} f_v^\Xi$  for any  $w, v \in W_{\Pi/\Xi}$ .*

In other words,  $\{f_w^\Xi\}_{w \in W_{\Pi/\Xi}}$  is a set of pairwise orthogonal projectors, and the direct sum of their images is  $(Q_W^*)^{W_\Xi}$ .

**Definition 6.2.** Given subsets  $\Xi' \subseteq \Xi$  of  $\Pi$  and a set of representatives  $W_{\Xi/\Xi'}$  we define a  $Q$ -linear operator on  $Q_W^*$  by

$$A_{\Xi/\Xi'}(f) := Y_{\Xi/\Xi'} \bullet f, \quad f \in Q_W^*,$$

and call it the *push-pull operator* with respect to  $W_{\Xi/\Xi'}$ . It is  $Q$ -linear as so is the  $\bullet$ '-action. We set  $A_\Xi = A_{\Xi/\emptyset}$ .

Lemma 5.7 immediately implies:

**Lemma 6.3** (Composition rule). *Given subsets  $\Xi'' \subseteq \Xi' \subseteq \Xi$  of  $\Pi$  and sets of representatives  $W_{\Xi/\Xi'}$  and  $W_{\Xi'/\Xi''}$ , let  $W_{\Xi/\Xi''} = \{wv \mid w \in W_{\Xi/\Xi'}, v \in W_{\Xi'/\Xi''}\}$ . Then we have  $A_{\Xi/\Xi'} \circ A_{\Xi'/\Xi''} = A_{\Xi/\Xi''}$ .*

**Lemma 6.4** (Projection formula). *We have*

$$A_{\Xi/\Xi'}(ff') = fA_{\Xi/\Xi'}(f') \quad \text{for any } f \in (Q_W^*)^{W_\Xi} \text{ and } f' \in Q_W^*.$$

*Proof.* Using (4.2) and Lemma 4.3, we compute

$$\begin{aligned} A_{\Xi/\Xi'}(ff') &= Y_{\Xi/\Xi'} \bullet (ff') = \left( \sum_{w \in W_{\Xi/\Xi'}} \delta_w \frac{1}{x_{\Xi/\Xi'}} \right) \bullet (ff') = \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \frac{1}{x_{\Xi/\Xi'}} \bullet (ff') \\ &= \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \left( f \left( \frac{1}{x_{\Xi/\Xi'}} \bullet f' \right) \right) = \sum_{w \in W_{\Xi/\Xi'}} (\delta_w \bullet f) (\delta_w \bullet \frac{1}{x_{\Xi/\Xi'}} \bullet f') \\ &= f \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \frac{1}{x_{\Xi/\Xi'}} \bullet f' = fA_{\Xi/\Xi'}(f') \quad \square \end{aligned}$$

Here is an analogue of Lemma 5.5.

**Lemma 6.5.** *The operator  $A_{\Xi/\Xi'}$  restricted to  $(Q_W^*)^{W_{\Xi'}}$  is independent of the choices of representatives  $W_{\Xi/\Xi'}$  and it maps  $(Q_W^*)^{W_{\Xi'}}$  to  $(Q_W^*)^{W_\Xi}$ .*

*Proof.* Let  $f \in (Q_W^*)^{W_{\Xi'}}$ . For any  $w \in W$  and  $v \in W_{\Xi'}$ , by Corollary 5.2, we have

$$\left( \delta_{wv} \frac{1}{x_{\Xi/\Xi'}} \right) \bullet f = \left( \delta_w \frac{1}{x_{\Xi/\Xi'}} \delta_v \right) \bullet f = \left( \delta_w \frac{1}{x_{\Xi/\Xi'}} \right) \bullet \delta_v \bullet f = \left( \delta_w \frac{1}{x_{\Xi/\Xi'}} \right) \bullet f.$$

which proves that the action on  $f$  of any factor  $\delta_w \left( \frac{1}{x_{\Xi/\Xi'}} \right)$  in  $Y_{\Xi/\Xi'}$  is independent of the choice of the coset representative  $w$ .

Now if  $v \in W_\Xi$ , we have

$$v(A_{\Xi/\Xi'}(f)) = \delta_v \bullet Y_{\Xi/\Xi'} \bullet f = (\delta_v Y_{\Xi/\Xi'}) \bullet f = A_{\Xi/\Xi'}(f),$$

where the last equality holds since  $\delta_v Y_{\Xi/\Xi'}$  is again an operator  $Y_{\Xi/\Xi'}$  corresponding to the set of coset representatives  $vW_{\Xi/\Xi'}$  (instead of  $W_{\Xi/\Xi'}$ ). This proves the second claim.  $\square$

**Lemma 6.6.** *We have  $A_{\Xi/\Xi'}(f_v) = \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi/\Xi'}} f_{vw^{-1}}$ . In particular,*

$$A_{\Xi/\Xi'}(f_v^{\Xi'}) = \frac{1}{v(x_{\Xi/\Xi'})} f_v^{\Xi}, \quad A_{\Pi/\Xi}(f_v^{\Xi}) = \frac{1}{v(x_{\Pi/\Xi})} \mathbf{1} \quad \text{and} \quad A_{\Pi}(v(x_{\Pi})f_v) = \mathbf{1}.$$

*Proof.* By Lemma 4.2 we get

$$A_{\Xi/\Xi'}(f_v) = \left( \sum_{w \in W_{\Xi/\Xi'}} \delta_w \frac{1}{x_{\Xi/\Xi'}} \right) \bullet f_v = \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \left( \frac{1}{v(x_{\Xi/\Xi'})} f_v \right) = \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi/\Xi'}} f_{vw^{-1}}.$$

In particular

$$\begin{aligned} A_{\Xi/\Xi'}(f_v^{\Xi'}) &= \sum_{w \in W_{\Xi'}} \frac{1}{vw(x_{\Xi/\Xi'})} \sum_{u \in W_{\Xi/\Xi'}} f_{vwu^{-1}} = \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi'}} \sum_{u \in W_{\Xi/\Xi'}} f_{vwu^{-1}} \\ &= \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in vW_{\Xi}} f_w = \frac{1}{v(x_{\Xi/\Xi'})} f_v^{\Xi}, \end{aligned}$$

where the second equality follows from Corollary 5.2.  $\square$

Together with Lemma 6.1 we therefore obtain:

**Corollary 6.7.** *We have  $A_{\Xi/\Xi'}((Q_W^*)^{W_{\Xi'}}) = (Q_W^*)^{W_{\Xi}}$ .*

**Definition 6.8.** We define the *characteristic map*  $c: Q \rightarrow Q_W^*$  by  $q \mapsto q \bullet \mathbf{1}$ .

By the definition of the ‘ $\bullet$ ’ action,  $c$  is an  $R$ -algebra homomorphism given by  $c(q) = \sum_{w \in W} w(q)f_w$ , that is,  $c(q) \in Q_W^*$  is the evaluation at  $q \in Q_W$  via the action (4.3) of  $Q_W$  on  $Q$ . Note that  $c$  is  $Q_W$ -equivariant with respect to this action and the ‘ $\bullet$ ’-action. Indeed,  $c(z \cdot q) = (z \cdot q) \bullet \mathbf{1} = z \bullet (q \bullet \mathbf{1}) = z \bullet c(q)$ . In particular,  $c$  is  $W$ -equivariant.

The following lemma provides an analogue of the push-pull formula of [CPZ, Theorem. 12.4].

**Lemma 6.9.** *Given subsets  $\Xi' \subseteq \Xi$  of  $\Pi$ , we have  $A_{\Xi/\Xi'} \circ c = c \circ C_{\Xi/\Xi'}$ .*

*Proof.* By definition, we have

$$A_{\Xi/\Xi'}(c(q)) = Y_{\Xi/\Xi'} \bullet c(q) = c(Y_{\Xi/\Xi'} \cdot q) = c(C_{\Xi/\Xi'}(q)). \quad \square$$

## 7. RELATIONS BETWEEN BASES COEFFICIENTS

In this section we describe relations between coefficients appearing in decompositions of various elements on the different bases of  $Q_W$  and of  $Q_W^*$ .

Given a sequence  $I = (i_1, \dots, i_m)$ , let  $I^{\text{rev}} := (i_m, \dots, i_1)$ .

**Lemma 7.1.** *Given a sequence  $I$  in  $\{1, \dots, n\}$ , for any  $x, y \in S$  and  $f, f' \in Q_W^*$  we have*

$$C_{\Pi}(\Delta_I(x)y) = C_{\Pi}(x\Delta_{I^{\text{rev}}}(y)) \quad \text{and} \quad A_{\Pi}(B_I(f)f') = A_{\Pi}(fB_{I^{\text{rev}}}(f')).$$

*Similarly, we have*

$$C_{\Pi}(C_I(x)y) = C_{\Pi}(xC_{I^{\text{rev}}}(y)) \quad \text{and} \quad A_{\Pi}(A_I(f)f') = A_{\Pi}(fA_{I^{\text{rev}}}(f')).$$



*Proof.* By Lemma 5.9.(b) we have  $Y_\Pi X_\alpha = 0$  for any  $\alpha \in \Pi$ . By (4.5) we obtain

$$0 = A_\Pi(B_\alpha(s_\alpha(f)f')) = A_\Pi(fB_\alpha(f') - B_\alpha(f)f').$$

Hence,  $A_\Pi(B_\alpha(f)f') = A_\Pi(fB_\alpha(f'))$  and  $A_\Pi(B_I(f)f') = A_\Pi(fB_{I^{\text{rev}}}(f'))$  by iteration.

To prove the corresponding formula involving  $A_I$ , note that  $A_\alpha = \kappa_\alpha - B_\alpha$ , so

$$\begin{aligned} fA_\alpha(f') - A_\alpha(f)f' &= f(\kappa_\alpha \bullet f' - B_\alpha(f')) - (\kappa_\alpha \bullet f - B_\alpha(f))f' \\ &\stackrel{(4.2)}{=} B_\alpha(f)f' - fB_\alpha(f') = B_\alpha(s_\alpha(f')f), \end{aligned}$$

so  $A_\Pi(A_\alpha(f)f') = A_\Pi(fA_\alpha(f'))$  and again  $A_\Pi(A_I(f)f') = A_\Pi(fA_{I^{\text{rev}}}(f'))$  by iteration. The formulas involving  $C$  operators are obtained similarly.  $\square$

**Corollary 7.2.** *Let  $I = (i_1, \dots, i_m)$  be a sequence in  $\{1, \dots, n\}$ . Let*

$$X_I = \sum_{v \in W} a_{I,v}^X \delta_v \quad \text{and} \quad X_{I^{\text{rev}}} = \sum_{v \in W} a_{I,v}^{\prime X} \delta_v \quad \text{for some } a_{I,v}^X, a_{I,v}^{\prime X} \in Q,$$

then  $v(x_\Pi) a_{I,v}^{\prime X} = v(a_{I,v-1}^X) x_\Pi$ . Similarly, let

$$Y_I = \sum_{v \in W} a_{I,v}^Y \delta_v \quad \text{and} \quad Y_{I^{\text{rev}}} = \sum_{v \in W} a_{I,v}^{\prime Y} \delta_v \quad \text{for some } a_{I,v}^Y, a_{I,v}^{\prime Y} \in Q,$$

then  $v(x_\Pi) a_{I,v}^{\prime Y} = v(a_{I,v-1}^Y) x_\Pi$ .

*Proof.* We have

$$\begin{aligned} v(x_\Pi) A_\Pi(B_I(f_e) f_v) &= v(x_\Pi) A_\Pi((X_I \bullet f_e) f_v) = v(x_\Pi) A_\Pi\left(\left(\sum_w w^{-1} (a_{I,w}^X) f_{w-1}\right) f_v\right) \\ &= v(x_\Pi) A_\Pi(v(a_{I,v-1}^X) f_v) \stackrel{6.6}{=} v(a_{I,v-1}^X) \mathbf{1}, \end{aligned}$$

and symmetrically

$$\begin{aligned} x_\Pi A_\Pi(f_e B_{I^{\text{rev}}}(f_v)) &= x_\Pi A_\Pi\left(f_e \sum_w a_{I,w}^{\prime X} \delta_w \bullet f_v\right) \\ &= x_\Pi A_\Pi\left(f_e \sum_w v w^{-1} (a_{I,w}^{\prime X}) f_{vw-1}\right) \\ &= x_\Pi A_\Pi(a_{I,v}^{\prime X} f_e) = a_{I,v}^{\prime X} \mathbf{1}. \end{aligned}$$

Lemma 7.1 then yields the formula by comparing the coefficients of  $X_I$  and  $X_{I^{\text{rev}}}$ . The formula involving  $Y_I$  is obtained similarly.  $\square$

**Lemma 7.3.** *For any sequence  $I$ , we have*

$$A_{I^{\text{rev}}}(x_\Pi f_e) = \sum_{v \in W} v(x_\Pi) a_{I,v}^Y f_v \quad \text{and} \quad B_{I^{\text{rev}}}(x_\Pi f_e) = \sum_{v \in W} v(x_\Pi) a_{I,v}^X f_v.$$

*Proof.* We prove the first formula only. The second one can be obtained using similar arguments. Let  $Y_{I^{\text{rev}}} = \sum_{v \in W} a_{I,v}^{\prime Y} \delta_v$  and  $Y_I = \sum_{v \in W} a_{I,v}^Y \delta_v$  as in Corollary 7.2.

$$\begin{aligned} A_{I^{\text{rev}}}(x_\Pi f_e) &= Y_{I^{\text{rev}}} \bullet x_\Pi f_e = \sum_{v \in W} x_\Pi (a_{I,v}^{\prime Y} \delta_v \bullet f_e) \\ &= \sum_{v \in W} x_\Pi (a_{I,v}^{\prime Y} \bullet f_{v-1}) = \sum_{v \in W} x_\Pi v^{-1} (a_{I,v}^{\prime Y}) f_{v-1} = \sum_{v \in W} x_\Pi v (a_{I,v-1}^Y) f_v. \end{aligned}$$

The formula then follows from Corollary 7.2.  $\square$

Let  $\{X_{I_w}^*\}_{w \in W}$  and  $\{Y_{I_w}^*\}_{w \in W}$  be the  $Q$ -linear bases of  $Q_W^*$  dual to  $\{X_{I_w}\}_{w \in W}$  and  $\{Y_{I_w}\}_{w \in W}$ , respectively, i.e.  $X_{I_w}^*(X_{I_v}) = \delta_{w,v}^{\text{Kr}}$  for  $w, v \in W$ . By Lemma 3.2 we have  $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w} = \sum_{w \leq v} b_{v,w}^Y Y_{I_w}$ . Therefore, by duality we have

$$(7.1) \quad X_{I_w}^* = \sum_{v \geq w} b_{v,w}^X f_v \quad \text{and} \quad Y_{I_w}^* = \sum_{v \geq w} b_{v,w}^Y f_v.$$

**Lemma 7.4.** *We have  $X_{I_e}^* = \mathbf{1}$  and, therefore,  $X_{I_e}^*(z) = z \cdot \mathbf{1}$ ,  $z \in Q_W$  (the action defined in (4.3)). For any sequence  $I$  with  $\ell(I) \geq 1$ , we have  $X_{I_e}^*(X_I) = X_I \cdot \mathbf{1} = 0$  and, moreover, if we express  $X_I = \sum_{v \in W} q_v X_{I_v}$ , then  $q_e = 0$ .*

*Proof.* Indeed, for each  $v \in W$  we have  $X_{I_e}^*(\delta_v) = b_{v,e}^X = \mathbf{1} = \mathbf{1}(\delta_v)$ . Therefore,  $X_{I_e}^* = \mathbf{1}$ . The formula for  $X_{I_e}^*(z)$  then follows by (4.4). Since  $X_\alpha \cdot \mathbf{1} = 0$ , we have  $X_I \cdot \mathbf{1} = 0$ . Finally, we obtain

$$0 = X_I \cdot \mathbf{1} = \sum_{v \in W} q_v X_{I_v} \cdot \mathbf{1} = q_e + \sum_{\ell(v) \geq 1} q_v X_{I_v} \cdot \mathbf{1} = q_e. \quad \square$$

**Lemma 7.5.** *Let  $w_0$  be the longest element in  $W$  of length  $N$ . We have*

$$A_\Pi(X_{I_{w_0}}^*) = (-1)^N \mathbf{1} \quad \text{and} \quad A_\Pi(Y_{I_{w_0}}^*) = \mathbf{1}.$$

*Proof.* Consider the first formula. By Lemma 3.2  $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w}$  with  $b_{w,w}^X = x_w$ , therefore  $X_{I_w}^* = \sum_{v \geq w} b_{v,w}^X f_v$ . Lemma 6.6 yields

$$A_\Pi(X_{I_w}^*) = \sum_{v \geq w} \frac{b_{v,w}^X}{v(x_\Pi)} \mathbf{1}.$$

If  $w = w_0$  is the longest element, then  $A_\Pi(X_{I_{w_0}}^*) = \frac{(-1)^N x_{w_0}}{w_0(x_\Pi)} \mathbf{1} = (-1)^N \mathbf{1}$  by (3.3).

The second formula is obtained similarly using Lemma 3.3 instead.  $\square$

**Lemma 7.6.** *For any reduced sequence  $I$  of an element  $w$  and  $q \in Q$  we have*

$$X_I q = \sum_{v \leq w} \phi_{I,v}(q) X_{I_v} \quad \text{for some } \phi_{I,v}(q) \in Q.$$

*Proof.* For any subsequence  $J$  of  $I$  (not necessarily reduced), we have  $w(J) \leq w$  by [Deo77, Th. 1.1]. Thus, by expanding all products of  $X_i = \frac{1}{x_i}(1 - \delta_i)$ , moving all coefficients to the left, and then using Lemma 3.2 and transitivity of the Bruhat order,

$$X_I q = \sum_{w \leq v} \tilde{\phi}_{I,w}(q) \delta_w = \sum_{w \leq v} \phi_{I,w}(q) X_{I_w}$$

for some coefficients  $\tilde{\phi}_{I,w}(q)$  and  $\phi_{I,w}(q) \in Q$ .  $\square$

## 8. ANOTHER BASIS OF THE $W_\Xi$ -INVARIANT SUBRING

Recall that  $\{f_w^\Xi\}_{w \in W_{\Pi/\Xi}}$  is a basis of the invariant subring  $(Q_W^*)^{W_\Xi}$ . In the present section we construct another basis  $\{X_u^*\}_{u \in W^\Xi}$  of the subring  $(Q_W^*)^{W^\Xi}$ , which generalizes [KK86, Lemma 4.34] and [KK90, Lemma 2.27].

Given a subset  $\Xi$  of  $\Pi$  we define

$$W^\Xi = \{w \in W \mid \ell(ws_\alpha) > \ell(w) \text{ for any } \alpha \in \Xi\}.$$

Note that  $W^\Xi$  is a set of left coset representatives of  $W/W_\Xi$  such that each  $w \in W^\Xi$  is the unique representative of minimal length.

We will extensively use the following fact [Hu90, §1.10]:

$$(8.1) \quad \text{For any } w \in W \text{ there exist unique } u \in W^\Xi \text{ and } v \in W_\Xi$$

$$\text{such that } w = uv \text{ and } \ell(w) = \ell(u) + \ell(v).$$

**Definition 8.1.** Let  $\Xi$  be a subset of  $\Pi$ . We say that the reduced sequences  $\{I_w\}_{w \in W}$  are  $\Xi$ -compatible if for each  $w \in W$  and the unique factorization  $w = uv$  with  $u \in W^\Xi$  and  $v \in W_\Xi$ ,  $\ell(w) = \ell(u) + \ell(v)$  of (8.1) we have  $I_w = I_u \cup I_v$ , i.e.  $I_w$  starts with  $I_u$  and ends with  $I_v$ .

Observe that there always exists a  $\Xi$ -compatible family of reduced sequences. Indeed, one could start with arbitrary reduced sequences  $\{I_u\}_{u \in W^\Xi}$  and  $\{I_v\}_{v \in W_\Xi}$ , and complete it into a  $\Xi$ -compatible family  $\{I_w\}_{w \in W}$  by defining  $I_w$  as the concatenation  $I_u \cup I_v$  for  $w = uv$  with  $u \in W^\Xi, v \in W_\Xi$ .

**Theorem 8.2.** For any  $\Xi$ -compatible choice of reduced sequences  $\{I_w\}_{w \in W}$ , if  $u \in W^\Xi$ , then for any sequence  $I$  in  $W_\Xi$  of length at least 1 (i.e.  $\alpha_i \in \Xi$  for each  $i$  appearing in the sequence  $I$ ), we have

$$X_{I_u}^*(zX_I) = 0 \text{ for all } z \in Q_W.$$

*Proof.* Since  $\{X_{I_w}\}_{w \in W}$  is a basis of  $Q_W$ , we may assume that  $z = X_{I_w}$  for some  $w \in W$ . We decompose  $X_I = \sum_{v \in W_\Xi} q_v X_{I_v}$  with  $q_v \in Q$ . By Lemma 7.4 we may assume  $v \neq e$ .

We proceed by induction on the length of  $w$ . If  $\ell(w) = 0$ , we have  $X_{I_w} = X_{I_e} = 1$ . Since  $W_\Xi \cap W^\Xi = \{e\}$ , for any  $v \in W_\Xi, v \neq e$ , we conclude that  $X_{I_u}^*(X_{I_v}) = 0$ .

The induction step goes as follows: Assume  $\ell(w) \geq 1$ . Since the sequences are  $\Xi$ -compatible, we have

$$X_{I_w} X_I = X_{I_{w'}} X_{I_{v'}} X_I = X_{I_{w'}} X_{I_{I'}}, \text{ where } w' \in W^\Xi, v' \in W_\Xi, I' \in W_\Xi, \text{ and}$$

$\ell(I') \geq \ell(I) \geq 1$ . We can thus assume that  $w \in W^\Xi$ , so that by Lemma 7.6,

$$X_{I_w} X_I = \sum_{v \neq e} (X_{I_w} q_v) X_{I_v} = \sum_{\tilde{w} \leq w, v \neq e} \phi_{I_w, \tilde{w}}(q_v) X_{I_{\tilde{w}}} X_{I_v}.$$

Now  $X_{I_u}^*(X_{I_w} X_{I_v}) = X_{I_u}^*(X_{I_{wv}}) = 0$  since  $wv$  is not a minimal coset representative: indeed, we already have  $w \in W^\Xi$  and  $v \neq e$ . Applying  $X_{I_u}^*$  to other terms in the above summation gives zero by induction.  $\square$

**Remark 8.3.** The proof will not work if we replace  $X$ 's by  $Y$ 's, because constant terms appear (we can not assume  $v \neq e$ ).

**Corollary 8.4.** For any  $\Xi$ -compatible choice of reduced sequences  $\{I_u\}_{u \in W}$ , the family  $\{X_{I_u}^*\}_{u \in W^\Xi}$  is a  $Q$ -module basis of  $(Q_W^*)^{W_\Xi}$ .

*Proof.* For every  $\alpha_i \in \Xi$  we have

$$(\delta_i \bullet X_{I_u}^*)(z) = X_{I_u}^*(z\delta_i) = X_{I_u}^*(z(1 - x_i X_i)) = X_{I_u}^*(z), \quad z \in Q_W,$$

where the last equality follows from Theorem 8.2. Therefore,  $X_{I_u}^*$  is  $W_\Xi$ -invariant.

Let  $\sigma \in (Q_W^*)^{W_\Xi}$ , i.e. for each  $\alpha_i \in \Xi$  we have  $\sigma = s_i(\sigma) = \delta_i \bullet \sigma$ . Then

$$\sigma(zX_i) = \sigma(z \frac{1}{x_{\alpha_i}} (1 - \delta_{\alpha_i})) = \sigma(z \frac{1}{x_{\alpha_i}}) - (\delta_i \bullet \sigma)(z \frac{1}{x_{\alpha_i}}) = (\sigma - \delta_i \bullet \sigma)(z \frac{1}{x_{\alpha_i}}) = 0$$

for any  $z \in Q_W$ . Write  $\sigma = \sum_{w \in W} x_w X_{I_w}^*$  for some  $x_w \in Q$ . If  $w \notin W^\Xi$ , then  $I_w$  ends with some  $i$  such that  $\alpha_i \in \Xi$  which implies that

$$x_w = \sigma(X_{I_w}) = \sigma(X_{I'_w} X_i) = 0,$$

where  $I'_w$  is the sequence obtained by deleting the last entry in  $I_w$ . So  $\sigma$  is a linear combination of  $\{X_{I_u}^*\}_{u \in W^\Xi}$ .  $\square$

**Corollary 8.5.** *If the reduced sequences  $\{I_w\}_{w \in W}$  are  $\Xi$ -compatible, then  $b_{wv,u}^X = b_{w,u}^X$  for any  $v \in W^\Xi$ ,  $u \in W^\Xi$  and  $w \in W$ , where  $b_{wv,u}^X$  are the coefficients of Lemma 3.2.*

*Proof.* From Lemma 3.2 we have  $X_{I_u}^* = \sum_{w \geq u} b_{w,u}^X f_w$ . By Lemma 4.2 we obtain that  $v(X_{I_u}^*) = \sum_{w \geq u} b_{w,u}^X f_{wv^{-1}}$  for any  $v \in W^\Xi$ . Since  $X_{I_u}^*$  is  $W^\Xi$ -invariant by Corollary 8.4 and  $\{f_w\}_{w \in W}$  is a basis of  $Q_W^*$ , this implies that  $b_{wv^{-1},u}^X = b_{w,u}^X$ .  $\square$

## 9. THE FORMAL DEMAZURE ALGEBRA AND THE HECKE ALGEBRA

In the present section we recall the definition and basic properties of the formal (affine) Demazure algebra  $\mathbf{D}_F$  following [HMSZ], [CZZ] and [Zh13].

Following [HMSZ], we define the *formal affine Demazure algebra*  $\mathbf{D}_F$  to be the  $R$ -subalgebra of the twisted formal group algebra  $Q_W$  generated by elements of  $S$  and the Demazure elements  $X_i$  for all  $i \in \{1, \dots, n\}$ . By [CZZ, Lemma 5.8],  $\mathbf{D}_F$  is also generated by  $S$  and all  $X_\alpha$  for all  $\alpha \in \Sigma$ . Since  $\kappa_\alpha \in S$ , the algebra  $\mathbf{D}_F$  is also generated by the  $Y_\alpha$ 's and the elements of  $S$ . Finally, since  $\delta_\alpha = 1 - x_\alpha X_\alpha$ , all elements  $\delta_w$  are in  $\mathbf{D}_F$ , and  $\mathbf{D}_F$  is a sub- $S_W$ -module of  $Q_W$ , both on the left and on the right.

**Remark 9.1.** Since  $\{X_{I_w}\}_{w \in W}$  is a  $Q$ -basis of  $Q_W$ , restricting the action (4.3) of  $Q_W$  onto  $\mathbf{D}_F$  we obtain an isomorphism between the algebra  $\mathbf{D}_F$  and the  $R$ -subalgebra  $\mathcal{D}(\Lambda)_F$  of  $\text{End}_R(S)$  generated by operators  $\Delta_\alpha$  (resp.  $C_\alpha$ ) for all  $\alpha \in \Sigma$ , and multiplication by elements from  $S$ . This isomorphism maps  $X_\alpha \mapsto \Delta_\alpha$  and  $Y_\alpha \mapsto C_\alpha$ . Therefore, for any identity or statement involving elements  $X_\alpha$  or  $Y_\alpha$  there is an equivalent identity or statement involving the operators  $\Delta_\alpha$  or  $C_\alpha$ .

According to [HMSZ, Theorem 6.14] (or [CZZ, 7.9] when the ring  $R$  is not necessarily a domain), in type  $A_n$ , the algebra  $\mathbf{D}_F$  is generated by the Demazure elements  $X_i$ ,  $i \in \{1, \dots, n\}$ , and multiplication by elements from  $S$  subject to the following relations:

- (a)  $X_i^2 = \kappa_i X_i$
- (b)  $X_i X_j = X_j X_i$  for  $|i - j| > 1$ ,
- (c)  $X_i X_j X_i - X_j X_i X_j = \kappa_{ij}(X_j - X_i)$  for  $|i - j| = 1$  and
- (d)  $X_i q = s_i(q) X_i + \Delta_i(q)$ ,

Furthermore, by [CZZ, Prop. 7.7], for any choice of reduced decompositions  $\{I_w\}_{w \in W}$ , the family  $\{X_{I_w}\}_{w \in W}$  (resp. the family  $\{Y_{I_w}\}_{w \in W}$ ) is a basis of  $\mathbf{D}_F$  as a left  $S$ -module.

We show now that for some hyperbolic formal group law  $F_h$ , the formal Demazure algebra can be identified with the classical Iwahori-Hecke algebra.

Recall that the Iwahori-Hecke algebra  $\mathcal{H}$  of the symmetric group  $S_{n+1}$  is a  $\mathbb{Z}[t, t^{-1}]$ -algebra with generators  $T_i$ ,  $i \in \{1, \dots, n\}$ , subject to the following relations:

- (A)  $(T_i + t)(T_i - t^{-1}) = 0$  or, equivalently,  $T_i^2 = (t^{-1} - t)T_i + 1$ ,
- (B)  $T_i T_j = T_j T_i$  for  $|i - j| > 1$  and
- (C)  $T_i T_j T_i = T_j T_i T_j$  for  $|i - j| = 1$ .

(The  $T_i$ 's appearing in the definition of the Iwahori-Hecke algebra [CG10, Def. 7.1.1] correspond to  $tT_i$  in our notation, where  $t = q^{-1/2}$ .)

Following [HMSZ, Def. 6.3] let  $D_F$  denote the  $R$ -subalgebra of  $\mathbf{D}_F$  generated by the elements  $X_i$ ,  $i \in \{1, \dots, n\}$ , only. By [HMSZ, Prop. 7.1], over  $R = \mathbb{C}$ , if  $F = F_a$  (resp.  $F = F_m$ ), then  $D_F$  is isomorphic to the completion of the nil-Hecke algebra (resp. the 0-Hecke algebra) of Kostant-Kumar. The following observation provides another motivation for the study of formal (affine) Demazure algebras.

Let us consider the FGL of example 2.2 with invertible  $\mu_1$ . After normalization we may assume  $\mu_1 = 1$ . Then its formal inverse is  $\frac{x}{x-1}$ , and since  $(1 + \mu_2 x_i x_j) x_{i+j} = x_i + x_j - x_i x_j$ , the coefficient  $\kappa_{ij}$  of relation (c) is simply  $\mu_2$ :

$$(9.1) \quad \kappa_{ij} = \frac{1}{x_i + x_j x_j} - \frac{1}{x_i + x_j x_i} - \frac{1}{x_i x_j} = \frac{x_i + x_j - x_i x_j - x_i + x_j}{x_i x_j x_i + x_j} = \frac{(1 + \mu_2 x_i x_j) x_{i+j} - x_{i+j}}{x_i x_j x_i + x_j} = \mu_2$$

**Proposition 9.2.** *Let  $F_h$  be a normalized (i.e.  $\mu_1 = 1$ ) hyperbolic formal group law over an integral domain  $R$  containing  $\mathbb{Z}[t, t^{-1}]$ , and let  $a, b \in R$ . Then the following are equivalent*

- (1) *The assignment  $T_i \mapsto aX_i + b$ ,  $i \in \{1, \dots, n\}$ , defines an isomorphism of  $R$ -algebras  $\mathcal{H} \otimes_{\mathbb{Z}[t, t^{-1}]} R \rightarrow D_F$ .*
- (2) *We have  $a = t + t^{-1}$  or  $-t - t^{-1}$  and  $b = -t$  or  $t^{-1}$  respectively. Furthermore  $\mu_2(t + t^{-1})^2 = -1$ ; in particular, the element  $t + t^{-1}$  is invertible in  $R$ .*

*Proof.* Assume there is an isomorphism of  $R$ -algebras given by  $T_i \mapsto aX_i + b$ . Then relations (b) and (B) are equivalent and relation (A) implies that

$$0 = (aX_i + b)^2 + (t - t^{-1})(aX_i + b) - 1 = [a^2 + 2ab + a(t - t^{-1})]X_i + b^2 + b(t - t^{-1}) - 1.$$

Therefore  $b = -t$  or  $t^{-1}$  and  $a = t^{-1} - t - 2b = t + t^{-1}$  or  $-t - t^{-1}$  respectively, since 1 and  $X_i$  are  $S$ -linearly independent in  $D_F \subseteq \mathbf{D}_F$ .

Relations (C) and (a) then imply

$$\begin{aligned} 0 &= (aX_i + b)(aX_j + b)(aX_i + b) - (aX_j + b)(aX_i + b)(aX_j + b) \\ &= a^3(X_i X_j X_i - X_j X_i X_j) + (a^2 b + ab^2)(X_i - X_j). \end{aligned}$$

Therefore, by relation (c) and (9.1), we have  $a^3 \mu_2 - a^2 b - ab^2 = 0$  which implies that  $0 = a^2 \mu_2 - ab - b^2 = (t + t^{-1})^2 \mu_2 + 1$ .

Conversely, by substituting the values of  $a$  and  $b$ , it is easy to check that the assignment is well defined, essentially by the same computations. It is an isomorphism since  $a = \pm(t + t^{-1})$  is invertible in  $R$ .  $\square$

**Remark 9.3.** The isomorphism of Proposition 9.2 provides a presentation of the Iwahori-Hecke algebra with  $t + t^{-1}$  inverted in terms of the Demazure operators on the formal group algebra  $R[[\Lambda]]_{F_h}$ .

**Remark 9.4.** In general, the coefficients  $\mu_1$  and  $\mu_2$  of  $F_h$  can be parametrized as  $\mu_1 = \epsilon_1 + \epsilon_2$  and  $\mu_2 = -\epsilon_1 \epsilon_2$  for some  $\epsilon_1, \epsilon_2 \in R$ . In 9.2 it corresponds to  $\epsilon_1 = \frac{t}{t+t^{-1}}$  and  $\epsilon_2 = \frac{t^{-1}}{t+t^{-1}}$  (up to a sign) and in this case [BuHo, Thm. 4.1] implies that  $F_h$  does not correspond to a topological complex oriented cohomology theory (i.e. a theory obtained from complex cobordism by tensoring over the Lazard ring).

Observe that such  $F_h$  still corresponds to an algebraic oriented cohomology theory in the sense of Levine-Morel.

### 10. THE ALGEBRAIC RESTRICTION TO THE FIXED LOCUS ON $G/B$

In the present section we define the algebraic counterpart of the restriction to the  $T$ -fixed locus of  $G/B$ .

Consider the  $S$ -linear dual  $S_W^* = \text{Hom}_S(S_W, S)$  of the twisted formal group algebra. Since  $\{\delta_w\}_{w \in W}$  is a basis for both  $S_W$  and  $Q_W$ ,  $S_W^*$  can be identified with the free  $S$ -submodule of  $Q_W^*$  with basis  $\{f_w\}_{w \in W}$  or, equivalently, with the subset  $\{f \in Q_W^* \mid f(S_W) \subseteq S\}$ .

Since  $\delta_\alpha = 1 - x_\alpha X_\alpha$  for each  $\alpha \in \Sigma$ , there is a natural inclusion of  $S$ -modules  $\eta: S_W \hookrightarrow \mathbf{D}_F$ . The elements  $\{X_{I_w}\}_{w \in W}$  (and, hence,  $\{Y_{I_w}\}_{w \in W}$ ) form a basis of  $\mathbf{D}_F$  as a left  $S$ -module by [CZZ, Prop. 7.7]. Observe that the natural inclusion  $S_W \hookrightarrow Q_W$  factors through  $\eta$ . Tensoring  $\eta$  by  $Q$  we obtain an isomorphism  $\eta_Q: Q_W \xrightarrow{\cong} Q \otimes_S \mathbf{D}_F$ , because both are free  $Q$ -modules and their bases  $\{X_{I_w}\}_{w \in W}$  are mapped to each other.

**Definition 10.1.** Consider the  $S$ -linear dual  $\mathbf{D}_F^* = \text{Hom}_S(\mathbf{D}_F, S)$ . The induced map  $\eta^*: \mathbf{D}_F^* \rightarrow S_W^*$  (composition with  $\eta$ ) will be called the *restriction to the fixed locus*.

**Lemma 10.2.** *The map  $\eta^*$  is an injective ring homomorphism and its image in  $S_W^* \subseteq Q_W^* = Q \otimes_S S_W^*$  coincides with the subset*

$$\{f \in S_W^* \mid f(\mathbf{D}_F) \subseteq S\}.$$

Moreover, the basis of  $\mathbf{D}_F^*$  dual to  $\{X_{I_w}\}_{w \in W}$  is  $\{X_{I_w}^*\}_{w \in W}$  in  $Q_W^*$ .

*Proof.* The coproduct  $\Delta$  on  $Q_W$  restricts to a coproduct on  $\mathbf{D}_F$  by [CZZ, Theorem 9.2] and to the coproduct on  $S_W$  via  $\eta$ . Hence, the map  $\eta^*$  is a ring homomorphism.

There is a commutative diagram

$$\begin{array}{ccc} \mathbf{D}_F^* & \xrightarrow{\eta^*} & S_W^* \\ \downarrow & & \downarrow \\ Q \otimes_S \mathbf{D}_F^* & \xrightarrow[\cong]{\eta_Q^*} & Q \otimes_S S_W^* \end{array}$$

where the vertical maps are injective by freeness of the modules and because  $S$  injects into  $Q$ . The description for the image then follows from the fact that  $\{X_{I_w}\}_{w \in W}$  is a basis for both  $\mathbf{D}_F$  and  $Q_W$ .

The last part of the lemma follows immediately.  $\square$

By Lemma 10.2,  $\sigma \in \mathbf{D}_F^* \subseteq Q_W^*$  means that  $\sigma(\mathbf{D}_F) \subseteq S$ . For any  $X \in \mathbf{D}_F$  we have  $(X \bullet \sigma)(\mathbf{D}_F) = \sigma(\mathbf{D}_F X) \subseteq S$ , so  $X \bullet \sigma \in \mathbf{D}_F^*$ . Hence, the ‘ $\bullet$ ’-action of  $Q_W$  on  $Q_W^*$  induces a ‘ $\bullet$ ’-action of  $\mathbf{D}_F$  on  $\mathbf{D}_F^*$ .

For each  $v \in W$ , we define

$$\tilde{f}_v := x_\Pi \bullet f_v = v(x_\Pi) f_v \in Q_W^*, \quad \text{i.e.} \quad \tilde{f}_v \left( \sum_{w \in W} q_w \delta_w \right) = v(x_\Pi) q_v.$$

**Lemma 10.3.** *We have  $\tilde{f}_v \in \mathbf{D}_F^*$  for any  $v \in W$ .*

*Proof.* We know that  $x_\Pi = w_0(x_{w_0})$ , and by Lemma 3.1.(e),  $\frac{x_{w_0}}{v(x_{w_0})}$  is invertible in  $S$  for any  $v \in W$ , so it suffices to show that  $x_\Pi f_v \in \mathbf{D}_F^*$ . If  $v = w_0$ , by Lemma 3.2, we have

$$X_{I_{w_0}} = \sum_{w \leq w_0} a_{w_0, w}^X \delta_w, \text{ where } a_{w_0, w_0}^X = (-1)^N \frac{1}{x_{w_0}}, \text{ so}$$

$$(x_\Pi f_{w_0})(X_{I_u}) = (x_\Pi f_{w_0}) \left( \sum_{w \leq u} a_{u, w}^X \delta_w \right) = (x_\Pi a_{w_0, w_0}^X) \delta_{u, w_0}^{\text{Kr}} = (-1)^N \frac{x_\Pi}{x_{w_0}} \delta_{u, w_0}^{\text{Kr}} \in S.$$

By Lemma 10.2, we have  $x_\Pi f_{w_0} \in \mathbf{D}_F^*$ . For an arbitrary  $v \in W$ , by Lemma 4.2, we obtain

$$x_\Pi f_v = x_\Pi f_{w_0 w_0^{-1} v} = v^{-1} w_0 (x_\Pi f_{w_0}) = v^{-1} w_0 (x_\Pi f_{w_0}) \in \mathbf{D}_F^*. \quad \square$$

**Corollary 10.4.** *For any  $z \in \mathbf{D}_F$ , we have  $x_\Pi z \in S_W$  and  $z x_\Pi \in S_W$ .*

*Proof.* It suffices to show that for any sequence  $I_v$ ,  $x_\Pi X_{I_v}$  and  $X_{I_v} x_\Pi$  belong to  $S_W$ . Indeed,

$$x_\Pi X_{I_v} = x_\Pi \sum_{w \leq v} a_{v, w}^X \delta_w = \sum_{w \leq v} (x_\Pi a_{v, w}^X) \delta_w = \sum_{w \leq v} (x_\Pi f_w)(X_{I_v}) \delta_w \in S_W,$$

and

$$X_{I_v} x_\Pi = \sum_{w \leq v} a_{v, w}^X \delta_w x_\Pi = \sum_{w \leq v} a_{v, w}^X w(x_\Pi) \delta_w = \sum_{w \leq v} (w(x_\Pi) f_w)(X_{I_v}) \delta_w \in S_W. \quad \square$$

Let  $\zeta: \mathbf{D}_F \rightarrow S_W$  be the multiplication on the *right* by  $x_\Pi$  (it does indeed land in  $S_W$  by Corollary 10.4). The dual map  $\zeta^*: S_W^* \rightarrow \mathbf{D}_F^*$  is the ‘ $\bullet$ ’-action by  $x_\Pi$ , and  $\zeta^*(f_v) = \tilde{f}_v$ .

**Remark 10.5.** In  $T$ -equivariant cohomology, the map  $\zeta^*$  corresponds to the push-forward from the  $T$ -fixed point set of  $G/B$  to  $G/B$  itself, see [CZZ2, Lemma 8.5]. In the topological context, for singular cohomology, it coincides with the map  $i_*$  discussed in [AB84, p.8].

**Lemma 10.6.** *The unique maximal left  $\mathbf{D}_F$ -module (by the  $\bullet$ -action) that is contained in  $S_W^*$  is  $\mathbf{D}_F^*$ .*

*Proof.* Let  $f$  be any element in a given  $\mathbf{D}_F$ -module  $M$  contained in  $S_W^*$ . Then  $X_I \bullet f \in M \subseteq S_W^*$  for any sequence  $I$ , and  $(X_I \bullet f)(\delta_e) = f(X_I) \in S$ . Since the  $X_I$ ’s generate  $\mathbf{D}_F$  as an  $S$ -module, we have  $f(\mathbf{D}_F) \subseteq S$ , and therefore  $f \in \mathbf{D}_F^*$  by Lemma 10.2.  $\square$

Define the  $S$ -module

$$\mathcal{Z} = \{ f \in S_W^* \mid B_i(f) \in S_W^* \text{ for any simple root } \alpha_i \}.$$

Since for an element  $f = \sum_{w \in W} q_w f_w$ ,  $q_w \in S$  we have

$$B_i(f) = X_i \bullet f = \sum_{w \in W} \frac{q_w - q_{w s_i}}{x_{w(\alpha_i)}} f_w = \sum_{w \in W} \frac{q_w - q_{s_w(\alpha_i)w}}{x_{w(\alpha_i)}} f_w,$$

this can be rewritten as

$$\mathcal{Z} = \left\{ \sum_{w \in W} q_w f_w \in S_W^* \mid \frac{q_w - q_{s_\alpha w}}{x_\alpha} \in S \text{ for any root } \alpha \text{ and any } w \in W \right\}.$$

The following theorem provides another characterization of  $\mathbf{D}_F^*$

**Theorem 10.7.** *We have  $\mathbf{D}_F^* \subseteq \mathcal{Z}$ , and under the conditions of Lemma 2.7, we have  $\mathbf{D}_F^* = \mathcal{Z}$ .*

*Proof.* Since  $\mathbf{D}_F^* \subseteq S_W^*$  is a sub- $\mathbf{D}_F$ -module, we have  $\mathbf{D}_F^* \subseteq \mathcal{Z}$ . By Lemma 10.6,  $\mathbf{D}_F^*$  is the unique maximal  $\mathbf{D}_F$ -module contained in  $S_W^*$ , so we only need to prove that  $\mathcal{Z}$  is a  $\mathbf{D}_F$ -submodule.

It suffices to show that for any  $f \in \mathcal{Z}$  and for any simple root  $\alpha_i$ , the element  $X_i \bullet f$  is still in  $\mathcal{Z}$ , or in other words, that for any two simple roots  $\alpha_i$  and  $\alpha_j$ , we still have  $X_i X_j \bullet f \in S_W^*$ . If  $\alpha_i = \alpha_j$ , it follows from  $X_i^2 = \kappa_i X_i$ .

If  $s_j(\alpha_i) = \alpha_i$ , then  $s_i s_j = s_j s_i$ . Let  $f = \sum_{w \in W} q_w f_w$ , then  $X_i \bullet f = \sum_{w \in W} \frac{q_w - q_{ws_i}}{x_w(\alpha_i)} f_w$ . Set  $p_w = \frac{q_w - q_{ws_i}}{x_w(\alpha_i)}$ , then

$$(X_j X_i) \bullet f = \sum_{w \in W} \frac{p_w - p_{ws_j}}{x_w(\alpha_j)} f_w = \sum_{w \in W} \frac{q_w - q_{ws_i} - q_{ws_j} + q_{ws_j s_i}}{x_w(\alpha_i) x_w(\alpha_j)} f_w.$$

Rearranging the numerator, we see that it is divisible by both  $x_w(\alpha_i)$  and  $x_w(\alpha_j)$ , so it is divisible by  $x_w(\alpha_i) x_w(\alpha_j)$  by Lemma 2.7.

Suppose  $s_j(\alpha_i) \neq \alpha_i$ . Then  $s_j(\alpha_i) \neq \alpha_j$ . Since  $X_i \bullet f = \sum_w p_w f_w$  with  $p_w \in S$  as above, we need to prove that the coefficient of  $f_w$  in  $X_j X_i \bullet f$  is in  $S$ , for any  $w$ . This coefficient is

$$\frac{p_w - p_{ws_j}}{x_w(\alpha_j)} = \frac{(q_w - q_{ws_i})x_{ws_j(\alpha_i)} - (q_{ws_j} - q_{ws_j s_i})x_w(\alpha_i)}{x_w(\alpha_i) x_w(\alpha_j) x_{ws_j(\alpha_i)}}.$$

Since the numerator is already divisible by  $x_w(\alpha_i)$  and by  $x_{ws_j(\alpha_i)}$  by assumption, it suffices, by Lemma 2.7, to show that it is divisible by  $x_w(\alpha_j)$ . Setting  $\gamma = w(\alpha_j)$  and  $\nu = w(\alpha_i)$ , it becomes  $(q_w - q_{s_\nu w})x_{s_\gamma(\nu)} - (q_{s_\gamma w} - q_{s_\gamma s_\nu w})x_\nu$ . Using that  $x_{s_\gamma(\nu)} = F(x_\nu, x_{-\langle \nu, \gamma^\vee \rangle \gamma}) \equiv x_\nu \pmod{x_\gamma}$ , the numerator is congruent to (cf. the proof of [HMSZ, Lem. 5.7])

$$((q_w - q_{s_\gamma w}) - (q_{s_\nu w} - q_{s_\gamma s_\nu w}))x_\nu$$

which is 0 mod  $x_\gamma$ , by assumption.  $\square$

**Remark 10.8.** The geometric translation of this theorem ([CZZ2, Theorem 9.2]) generalizes the classical result [Br97, Proposition 6.5.(i)].

**Remark 10.9.** In Theorem 10.7, it is not possible to remove entirely the assumptions on the root system and the base ring, as the following example shows. Take a root datum of type  $G_2$ , and a ring  $R$  in which  $3 = 0$ , with the additive formal group law  $F$  over  $R$ . Then,  $S$  is  $\Sigma$ -regular, and if  $(\alpha_1, \alpha_2)$  is a basis of simple roots, with  $\beta = 2\alpha_2 + 3\alpha_1$  being the longest root, we have  $x_\beta = 2x_{\alpha_2} = -x_{\alpha_2}$ . It is not difficult to check that the element  $f = (\prod_{\alpha \in \Sigma^+, \alpha \neq \beta} x_\alpha) f_e$  is in  $\mathcal{Z}$ , but

$$f(X_{I_{w_0}}) \stackrel{\text{Lem. 3.2}}{=} \left( \prod_{\alpha \in \Sigma^+, \alpha \neq \beta} x_\alpha \right) \cdot \frac{1}{x_{w_0}} = \frac{1}{x_\beta} \notin S,$$

so  $f \notin \mathbf{D}_F^*$ . Therefore,  $\mathcal{Z} \supsetneq \mathbf{D}_F^*$ . Indeed,  $\mathcal{Z}$  is not even a  $\mathbf{D}_F$ -module.

Recall from (7.1) that  $X_{I_w}^* = \sum_{v \geq w} b_{v,w}^X f_v$  and  $Y_{I_w}^* = \sum_{v \geq w} b_{v,w}^Y f_v$ .

**Corollary 10.10.** *For any  $v, w \in W$  and root  $\alpha$ , we have  $x_\alpha \mid (b_{v,w}^X - b_{s_\alpha v, w}^X)$  and  $x_\alpha \mid (b_{v,w}^Y - b_{s_\alpha v, w}^Y)$ .*



**Remark 10.11.** It is not difficult to see that Corollary 10.4 and Corollary 10.10 provide a characterization of elements of  $\mathbf{D}_F$  inside  $Q_W$ . This characterization coincides with the residue description of  $\mathbf{D}_F$  in [ZZ, §4], which generalizes Ginzburg–Kapranov–Vasserot’s construction of certain Hecke algebras in [GKV].

For any  $\Xi \subseteq \Pi$  and  $w \in W$ , define

$$\hat{X}_{I_w}^\Xi = \sum_{v \in W_\Xi} \delta_v \frac{b_{v^{-1}, w}^X}{x_\Xi} \quad \text{and} \quad \hat{Y}_{I_w}^\Xi = \sum_{v \in W_\Xi} \delta_v \frac{b_{v^{-1}, w}^Y}{x_\Xi}.$$

By Lemma 3.2,  $b_{v, e}^X = 1$ , so

$$\hat{X}_\emptyset^\Xi = \sum_{v \in W_\Xi} \delta_v \frac{1}{x_\Xi} = Y_\Xi.$$

Note that  $Y_\Xi$  does not depend on the choice of reduced sequences  $\{I_w\}_{w \in W}$ , but  $\hat{X}_{I_w}^\Xi$  and  $\hat{Y}_{I_w}^\Xi$  do, since  $b_{w, v}^X$  and  $b_{w, v}^Y$  do for  $w$  such that  $\ell(w) \geq 3$ . Moreover, we have

$$(10.1) \quad \hat{X}_{I_w}^\Pi \bullet \tilde{f}_e = X_{I_w}^* \quad \text{and} \quad \hat{Y}_{I_w}^\Pi \bullet \tilde{f}_e = Y_{I_w}^*$$

by a straightforward computation.

**Lemma 10.12.** *For any  $\Xi \subseteq \Pi$  and  $w \in W$ , we have  $\hat{X}_{I_w}^\Xi \in \mathbf{D}_F$  and  $\hat{Y}_{I_w}^\Xi \in \mathbf{D}_F$ .*

*Proof.* The ring  $Q_W$  is functorial in the root datum (i.e. along morphisms of lattices that send roots to roots) and in the formal group law. This functoriality sends elements  $X_\alpha$  (or  $Y_\alpha$ ) to themselves, so it restricts to a functoriality of the subring  $\mathbf{D}_F$ . It also sends the elements  $\hat{X}_{I_w}^\Xi$  (or  $\hat{Y}_{I_w}^\Xi$ ) to themselves. We can therefore assume that the root datum is adjoint, and that the formal group law is the universal one over the Lazard ring, in which all integers are regular, since it is a polynomial ring over  $\mathbb{Z}$ .

Consider the involution  $\iota$  on  $Q_W$  given by  $q\delta_w \mapsto (-1)^{\ell(w)}w^{-1}(q)\delta_{w^{-1}}$ . It satisfies  $\iota(zz') = \iota(z')\iota(z)$ . Since  $\iota(X_\alpha) = Y_{-\alpha}$ , it restricts to an involution on  $\mathbf{D}_F$ .

To show that  $\hat{X}_{I_w}^\Xi \in \mathbf{D}_F$ , it suffices to show that  $\iota(\hat{X}_{I_w}^\Xi) \in \mathbf{D}_F$ . We have

$$\iota(\hat{X}_{I_w}^\Xi) = \sum_{v \in W_\Xi} (-1)^{\ell(v)} \frac{b_{v^{-1}, w}^X}{x_\Xi} \delta_{v^{-1}} = \frac{1}{x_\Xi} \sum_{v \in W_\Xi} (-1)^{\ell(v)} b_{v, w}^X \delta_v.$$

Since the root datum is adjoint, we have  $\mathbf{D}_F = \{f \in Q_W \mid f \cdot S \subseteq S\}$  by [CZZ, Remark 7.8], so it suffices to show that  $\iota(\hat{X}_{I_w}^\Xi) \cdot x \in S$  for any  $x \in S$ . We have

$$\iota(\hat{X}_{I_w}^\Xi) \cdot x = \frac{1}{x_\Xi} \sum_{v \in W_\Xi} (-1)^{\ell(v)} b_{v, w}^X v(x).$$

By Lemma 2.7, it is enough to show that  $\sum_{v \in W} (-1)^{\ell(v)} b_{v, w}^X v(x)$  is divisible by  $x_\alpha$  for any root  $\alpha \in \Sigma_\Xi^-$ . Let  ${}^\alpha W_\Xi = \{v \in W_\Xi \mid \ell(s_\alpha v) > \ell(v)\}$ . Then  $(-1)^{\ell(s_\alpha v)} = -(-1)^{\ell(v)}$  and  $W_\Xi = {}^\alpha W_\Xi \sqcup s_\alpha {}^\alpha W_\Xi$ . So

$$\begin{aligned} \sum_{v \in W_\Xi} (-1)^{\ell(v)} b_{v, w}^X v(x) &= \sum_{v \in {}^\alpha W_\Xi} (-1)^{\ell(v)} (b_{v, w}^X v(x) - b_{s_\alpha v, w}^X s_\alpha v(x)) \\ &= \sum_{v \in {}^\alpha W_\Xi} (-1)^{\ell(v)} (b_{v, w}^X v(x) - b_{v, w}^X s_\alpha v(x) + b_{v, w}^X s_\alpha v(x) - b_{s_\alpha v, w}^X s_\alpha v(x)) \\ &= \sum_{v \in {}^\alpha W_\Xi} (-1)^{\ell(v)} (b_{v, w}^X x_\alpha \Delta_\alpha(v(x)) + (b_{v, w}^X - b_{s_\alpha v, w}^X) s_\alpha v(x)) \end{aligned}$$

which is divisible by  $x_\alpha$  by Corollary 10.10. Therefore  $\hat{X}_{I_w}^\Xi \in \mathbf{D}_F$ . The proof that  $\hat{Y}_{I_w}^\Xi \in \mathbf{D}_F$  is similar.  $\square$

**Theorem 10.13.**  $Q_W^*$  is a free  $Q_W$ -module of rank 1 generated by  $f_w$  for any  $w \in W$ , and  $\mathbf{D}_F^*$  is a free left  $\mathbf{D}_F$ -module of rank 1 generated by  $\tilde{f}_w$  for any  $w \in W$ .

*Proof.* Since  $\delta_v \bullet f_w = f_{wv^{-1}}$ , we have  $Q_W \bullet f_w = Q_W^*$ . Moreover, if  $z = \sum_{v \in W} q_v \delta_v$  such that  $z \bullet f_w = 0$ , then  $\sum_{v \in W} q_v f_{wv^{-1}} = 0$ , so  $q_v = 0$  for all  $v \in W$ , i.e.  $z = 0$ ; the first part is proven.

To prove the second part, note that by Lemma 10.3  $\tilde{f}_e \in \mathbf{D}_F^*$  for any  $w$ . Moreover,  $\{\tilde{f}_e\}$  is  $Q_W$ -linearly independent by the first part of the proof, hence it is  $\mathbf{D}_F$ -linearly independent. On the other hand,  $\mathbf{D}_F \bullet \tilde{f}_e = \mathbf{D}_F^*$  by Lemma 10.12 and (10.1), so  $\tilde{f}_e$  generates  $\mathbf{D}_F^*$  as a left  $\mathbf{D}_F$ -module. Since  $\tilde{f}_w = \frac{x_\Pi}{w^{-1}(x_\Pi)} \delta_{w^{-1}} \bullet \tilde{f}_e$ , and  $\frac{x_\Pi}{w^{-1}(x_\Pi)} \in S$  by Lemma 3.1.(e), the same is true for  $\tilde{f}_w$ .  $\square$

## 11. THE ALGEBRAIC RESTRICTION TO THE FIXED LOCUS ON $G/P$

We now extend the results of the previous section to the relative case of  $W/W_\Xi$ .

For any  $\Xi \subseteq \Pi$ , let  $S_{W/W_\Xi}$  be the free  $S$ -module with basis  $(\delta_{\bar{w}})_{\bar{w} \in W/W_\Xi}$  (it is not necessarily a ring). Let  $Q_{W/W_\Xi} = Q \otimes_S S_{W/W_\Xi}$  be its  $Q$ -localization. There is a left  $S$ -linear coproduct on  $S_{W/W_\Xi}$ , defined on basis elements by the formula  $\delta_{\bar{w}} \mapsto \delta_{\bar{w}} \otimes \delta_{\bar{w}}$ ; it extends by the same formula to a  $Q$ -linear coproduct on  $Q_{W/W_\Xi}$ . The induced products on the  $S$ -dual  $S_{W/W_\Xi}^*$  and the  $Q$ -dual  $Q_{W/W_\Xi}^*$  are given by the formula  $f_{\bar{v}} f_{\bar{w}} = \delta_{\bar{v}, \bar{w}}^{\text{Kr}} f_{\bar{v}}$ .

If  $\Xi' \subseteq \Xi$  and  $\bar{w} \in W/W_{\Xi'}$ , let  $\hat{w}$  be its class in  $W/W_\Xi$ . We consider the projection and the sum over orbit maps

$$p_{\Xi/\Xi'} : S_{W/W_{\Xi'}} \rightarrow S_{W/W_\Xi} \quad \text{and} \quad d_{\Xi/\Xi'} : S_{W/W_\Xi} \rightarrow \sum_{\substack{\bar{v} \in W/W_{\Xi'} \\ \hat{v} = \hat{w}}} \delta_{\bar{v}}.$$

with  $S$ -dual maps

$$p_{\Xi/\Xi'}^* : S_{W/W_\Xi}^* \rightarrow S_{W/W_{\Xi'}}^* \quad \text{and} \quad d_{\Xi/\Xi'}^* : S_{W/W_{\Xi'}}^* \rightarrow S_{W/W_\Xi}^*$$

$$f_{\hat{w}} \mapsto \sum_{\substack{\bar{v} \in W/W_{\Xi'} \\ \hat{v} = \hat{w}}} f_{\bar{v}} \quad \quad \quad f_{\bar{w}} \mapsto f_{\hat{w}}.$$

We use the same notation for maps between the corresponding  $Q$ -localized modules  $Q_{W/W_\Xi}$  and  $Q_{W/W_{\Xi'}}$ , and we write  $p_{\Xi/\Xi'}^*$  and  $d_{\Xi/\Xi'}^*$  for their  $Q$ -dual maps. As usual, when  $\Xi' = \emptyset$ , we omit it, as in  $p_\Xi : S_W \rightarrow S_{W/W_\Xi}$ . Note that the maps  $p_{\Xi/\Xi'}$  preserve the coproduct (the maps  $d_{\Xi/\Xi'}$  don't), and thus the dual maps  $p_{\Xi/\Xi'}^*$  and  $p_{\Xi/\Xi'}^*$  are ring maps. We set  $\mathbf{D}_{F,\Xi} := p_\Xi(\mathbf{D}_F) \subseteq Q_{W/W_\Xi}$ .

The coproduct on  $Q_{W/W_\Xi}$  therefore restricts to a coproduct on  $\mathbf{D}_{F,\Xi}$ . We then have the following commutative diagram of  $S$ -modules which defines the map  $\eta_\Xi$

$$(11.1) \quad \begin{array}{ccccc} S_W & \xrightarrow{\eta} & \mathbf{D}_F & \hookrightarrow & Q_W \\ p_\Xi \downarrow & & p_\Xi \downarrow & & p_\Xi \downarrow \\ S_{W/W_\Xi} & \xrightarrow{\eta_\Xi} & \mathbf{D}_{F,\Xi} & \hookrightarrow & Q_{W/W_\Xi} \end{array}$$

**Lemma 11.1.** *The map  $p_{\Xi/\Xi'} : Q_{W/W_{\Xi'}} \rightarrow Q_{W/W_\Xi}$  restricts to  $\mathbf{D}_{F,\Xi'} \rightarrow \mathbf{D}_{F,\Xi}$ .*

*Proof.* It follows by diagram chase from Diagram (11.1) applied first to  $\Xi$  and then to  $\Xi'$ , using the surjectivity of  $p_{\Xi'} : \mathbf{D}_F \rightarrow \mathbf{D}_{F,\Xi'}$ .  $\square$

**Lemma 11.2.** *We have  $p_{\Xi}(zX_{\alpha}) = 0$  for any  $\alpha \in \Sigma_{\Xi}$  and  $z \in Q_W$ .*

*Proof.* Since  $p_{\Xi}$  is a map of  $Q$ -modules, it suffices to consider  $z = \delta_w$ , in which case  $\delta_w X_{\alpha} = \frac{1}{w(x_{\alpha})} \delta_w - \frac{1}{w(x_{\alpha})} \delta_{ws_{\alpha}}$ , so  $p(\delta_w X_{\alpha}) = \frac{1}{w(x_{\alpha})} (\delta_{\bar{w}} - \delta_{\bar{w}}) = 0$ .  $\square$

For any  $w \in W$ , let  $X_{I_w}^{\Xi}$  (resp.  $Y_{I_w}^{\Xi}$ ) be the element  $p_{\Xi}(X_{I_w}) \in \mathbf{D}_{F,\Xi}$  (resp.  $p_{\Xi}(Y_{I_w}) \in \mathbf{D}_{F,\Xi}$ ).

**Lemma 11.3.** (a) *Let  $\{I_w\}_{w \in W}$  be a family of  $\Xi$ -compatible reduced sequences. If  $w \notin W^{\Xi}$ , then  $X_{I_w}^{\Xi} = 0$ .*

*Let  $\{I_w\}_{w \in W^{\Xi}}$  be a family of reduced sequences of minimal length. Then*

(b) *the family  $\{X_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  forms an  $S$ -basis of  $\mathbf{D}_{F,\Xi}$  and a  $Q$ -basis of  $Q_{W/W^{\Xi}}$ .*

(c) *the family  $\{Y_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  forms an  $S$ -basis of  $\mathbf{D}_{F,\Xi}$  and a  $Q$ -basis of  $Q_{W/W^{\Xi}}$ .*

*Proof.* (a) If  $w \notin W^{\Xi}$ , then  $w = uv$  with  $u \in W^{\Xi}$  and  $e \neq v \in W_{\Xi}$ . By Lemma 11.2, we have  $p_{\Xi}(X_{I_w}) = p_{\Xi}(X_{I_u} X_{I_v}) = 0$ .

(b) Let us complete  $\{I_w\}_{w \in W^{\Xi}}$  to a  $\Xi$ -compatible choice of reduced sequences  $\{I_w\}_{w \in W}$  by choosing reduced decompositions for elements in  $W_{\Xi}$ . Since  $\{X_{I_w}\}_{w \in W}$  is a basis of  $\mathbf{D}_F$ , its image  $\mathbf{D}_{F,\Xi}$  in  $Q_{W/W^{\Xi}}$  is spanned by  $\{X_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  by part (a). Writing  $X_{I_w} = \sum_{v \leq w} a_{w,v}^X \delta_v$  yields  $X_{I_w}^{\Xi} = \sum_{v \leq w} a_{w,v}^X \delta_{\bar{v}}$ . Since  $w \in W^{\Xi}$  is of minimal length in  $wW_{\Xi}$ , the coefficient of  $\delta_{\bar{w}}$  in  $X_{I_w}^{\Xi}$  is  $a_{w,w}^X = (-1)^{\ell(w)} \frac{1}{x_w}$ , invertible in  $Q$ , so the matrix expressing the  $\{X_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  on the basis  $\{\delta_{\bar{w}}\}_{w \in W^{\Xi}}$  is upper triangular with invertible (in  $Q$ ) determinant, hence  $\{X_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  is  $Q$ -linearly independent in  $Q_{W/W^{\Xi}}$  and therefore  $S$ -linearly independent in  $\mathbf{D}_{F,\Xi}$ .

(c) Applying  $p_{\Xi}$  to the expression of Corollary 3.6 for  $v \in W^{\Xi}$  and using (a) we obtain  $Y_{I_v}^{\Xi} = (-1)^{\ell(v)} X_{I_v}^{\Xi} + \sum_{w < v} c_{v,w} X_{I_w}^{\Xi}$ ,  $w \in W^{\Xi}$ . Hence, the matrix expressing  $Y_{I_v}^{\Xi}$ 's in terms of  $X_{I_w}^{\Xi}$ 's is upper triangular with  $\pm 1$  on the main diagonal.  $\square$

Observe in particular that  $\mathbf{D}_{F,\Pi} \simeq S$  with the basis  $\{X_{\emptyset}^{\Pi} = \delta_{\bar{e}}\}$ .

**Definition 11.4.** The dual map  $\eta_{\Xi}^* : \mathbf{D}_{F,\Xi}^* \rightarrow S_{W/W^{\Xi}}^*$  is called the *algebraic restriction to the fixed locus*.

As in Lemma 10.2, and by a similar proof, we obtain:

**Lemma 11.5.** *The map  $\eta_{\Xi}^*$  is an injective ring homomorphism and its image in  $S_{W/W^{\Xi}}^* \subseteq Q_{W/W^{\Xi}}^*$  coincides with the subset*

$$\{f \in S_{W/W^{\Xi}}^* \mid f(\mathbf{D}_{F,\Xi}) \subseteq S\}.$$

*Moreover, the basis of  $\mathbf{D}_{F,\Xi}^*$  dual to  $\{X_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$  (resp. to  $\{Y_{I_w}^{\Xi}\}_{w \in W^{\Xi}}$ ) maps to  $\{(X_{I_w}^{\Xi})^*\}_{w \in W^{\Xi}}$  (resp. to  $\{(Y_{I_w}^{\Xi})^*\}_{w \in W^{\Xi}}$ ) in  $Q_{W/W^{\Xi}}^*$ .*

So far, the situation is summarized in the diagram of  $S$ -linear ring maps

$$(11.2) \quad \begin{array}{ccc} \mathbf{D}_F^* & \xrightarrow{\eta^*} & S_W^* \\ p_{\Xi}^* \uparrow & & \uparrow p_{\Xi}^* \\ \mathbf{D}_{F,\Xi}^* & \xrightarrow{\eta_{\Xi}^*} & S_{W/W^{\Xi}}^* \end{array}$$

in which both columns become injections  $Q_{W/W_\Xi}^* \hookrightarrow Q_W^*$  after  $Q$ -localization. The geometric translation of this diagram is in the proof of Corollary 8.7 in [CZZ2].

**Lemma 11.6.** *For any  $\Xi$ -compatible choice of reduced sequences  $\{I_w\}_{w \in W}$ , the  $W_\Xi$ -invariant subring  $(\mathbf{D}_F^*)^{W_\Xi}$  is a free  $S$ -module with basis  $\{X_{I_w}^*\}_{w \in W_\Xi}$ .*

*Proof.* It follows from Corollary 8.4 since  $(\mathbf{D}_F^*)^{W_\Xi} = (Q_W^*)^{W_\Xi} \cap \mathbf{D}_F^*$ .  $\square$

**Lemma 11.7.** *The injective maps  $p_\Xi^*: S_{W/W_\Xi}^* \rightarrow S_W^*$ ,  $p_\Xi^*: Q_{W/W_\Xi}^* \rightarrow Q_W^*$  and  $p_\Xi^*: \mathbf{D}_{F,\Xi}^* \rightarrow \mathbf{D}_F^*$  have images  $(S_W^*)^{W_\Xi}$ ,  $(Q_W^*)^{W_\Xi}$  and  $(\mathbf{D}_F^*)^{W_\Xi}$ , respectively.*

*Proof.* For any  $w \in W$ , we have  $p_\Xi^*(f_{\bar{w}}) = f_{\bar{w}}^\Xi$ . Thus  $p_\Xi^*(Q_{W/W_\Xi}^*) = (Q_W^*)^{W_\Xi}$  by Lemma 6.1. Similarly,  $p_\Xi^*(S_{W/W_\Xi}^*) = (S_W^*)^{W_\Xi}$ . Take a  $\Xi$ -compatible choice of reduced sequences  $\{I_w\}_{w \in W}$ . Dualizing the fact (Lemma 11.3) that  $p_\Xi(X_{I_w}) = X_{I_w}^\Xi$  is 0 if  $w \notin W_\Xi$  and a basis element otherwise, we obtain that  $p_\Xi^*((X_{I_w}^\Xi)^*) = X_{I_w}^*$  if  $w \in W_\Xi$ , and thus the conclusion for  $\mathbf{D}_{F,\Xi}^*$  by Lemma 11.6.  $\square$

**Remark 11.8.** Note that if  $\{I_w\}_{w \in W}$  is not  $\Xi$ -compatible, then we may not have  $p_\Xi^*((X_{I_w}^\Xi)^*) = X_{I_w}^*$  for all  $w \in W_\Xi$ . Even if it is  $\Xi$ -compatible, in general  $p_\Xi^*((Y_{I_w}^\Xi)^*)$  is not equal to  $Y_{I_w}^*$ , which needs not be fixed by  $W_\Xi$ .

Through the resulting isomorphism  $\mathbf{D}_{F,\Xi}^* \simeq (\mathbf{D}_F^*)^{W_\Xi}$ , we obtain

$$\begin{aligned} \mathbf{D}_{F,\Xi}^* &= \{f \in S_{W/W_\Xi}^* \mid f(\mathbf{D}_{F,\Xi}) \subseteq S\} \\ &\simeq (\mathbf{D}_F^*)^{W_\Xi} = \{f \in (S_W^*)^{W_\Xi} \mid f(\mathbf{D}_F) \subseteq S\} \\ &= \{f \in S_W^* \mid f(\mathbf{D}_F) \subseteq S \text{ and } f(K_\Xi) = 0\} \end{aligned}$$

where  $K_\Xi$  is the kernel of  $p_\Xi$ , i.e. the sub- $S$ -module of  $\mathbf{D}_F$  generated by  $(X_{I_w})_{w \notin W_\Xi}$  for a  $\Xi$ -compatible choice of reduced sequences  $\{I_w\}_{w \in W}$ .

Since  $(\mathbf{D}_F^*)^{W_\Xi} = \mathbf{D}_F^* \cap (S_W^*)^{W_\Xi}$ , an element of  $S_{W/W_\Xi}^*$  is in  $\mathbf{D}_{F,\Xi}^*$  if and only if its image by  $p_\Xi^*$  is in  $\mathbf{D}_F^*$ . Since  $B_\alpha(f) = 0$  when  $f \in (S_W^*)^{W_\Xi}$  and  $\alpha \in W_\Xi$ , Theorem 10.7 then gives:

**Theorem 11.9.** *Under the conditions of Lemma 2.7, an element  $f \in S_{W/W_\Xi}^*$  is in  $\mathbf{D}_{F,\Xi}^*$  if and only if  $B_\alpha \circ p_\Xi^*(f) \in S_W^*$  for any  $\alpha \notin \Sigma_\Xi$ . In other words,  $f = \sum_{\bar{w}} q_{\bar{w}} f_{\bar{w}}$  is in  $\mathbf{D}_{F,\Xi}^*$  if and only if  $x_{w(\alpha)}$  divides  $q_{\bar{w}} - q_{\overline{s_{w(\alpha)}\bar{w}}}$  for any  $\bar{w} \in W/W_\Xi$  and any  $\alpha \notin \Sigma_\Xi$ .*

## 12. THE PUSH-PULL OPERATORS ON $\mathbf{D}_F^*$

In this section we restrict the push-pull operators onto the dual of the formal affine Demazure algebra  $\mathbf{D}_F^*$ , and define a non-degenerate pairing on it.

By Lemma 10.12, we have  $Y_\Xi \in \mathbf{D}_F$ , so

**Corollary 12.1.** *The operator  $Y_\Xi$  (resp.  $A_\Xi$ ) restricted to  $S$  (resp. to  $\mathbf{D}_F^*$ ) defines an operator on  $S$  (resp. on  $\mathbf{D}_F^*$ ). Moreover, we have*

$$C_\Xi(S) \subseteq S^{W_\Xi} \quad \text{and} \quad A_\Xi(\mathbf{D}_F^*) \subseteq (\mathbf{D}_F^*)^{W_\Xi}.$$

*Proof.* Here  $Y_\Xi$  acts on  $S \subseteq Q$  via (4.3). Since  $Y_\Xi \in \mathbf{D}_F \subseteq \{z \in Q_W \mid z \cdot S \subseteq S\}$  by [CZZ, Remark 7.8] and  $Y_\Xi \cdot Q \subseteq (Q)^{W_\Xi}$ , the result follows.

As for  $A_{\Xi}$ , by Lemma 10.2 any  $f \in \mathbf{D}_F^*$  has the property that  $f(\mathbf{D}_F) \subseteq S$ . Therefore,  $(A_{\Xi}(f))(\mathbf{D}_F) = (Y_{\Xi} \bullet f)(\mathbf{D}_F) = f(\mathbf{D}_F Y_{\Xi}) \subseteq S$ , so  $A_{\Xi}(f) \in \mathbf{D}_F^*$ . The result then follows by Lemma 6.4.  $\square$

**Corollary 12.2.** *Suppose that the root datum has no irreducible component of type  $C_n^{sc}$  or that 2 is invertible in  $R$ . Then if  $|W_{\Xi'}|$  is regular in  $R$ , for any  $\Xi' \subseteq \Xi \subseteq \Pi$ , we have*

$$C_{\Xi/\Xi'}(S^{W_{\Xi'}}) \subseteq S^{W_{\Xi}}.$$

*Proof.* Let  $x \in S^{W_{\Xi'}}$ , then  $|W_{\Xi'}| \cdot x = \sum_{w \in W_{\Xi'}} w(x)$ . So we have

$$\begin{aligned} |W_{\Xi'}| \cdot C_{\Xi/\Xi'}(x) &= C_{\Xi/\Xi'}(|W_{\Xi'}| \cdot x) = \sum_{u \in W_{\Xi/\Xi'}} u\left(\frac{|W_{\Xi'}| \cdot x}{x_{\Xi/\Xi'}}\right) \\ &= \sum_{u \in W_{\Xi/\Xi'}} \sum_{v \in W_{\Xi'}} uv\left(\frac{x}{x_{\Xi/\Xi'}}\right) = \sum_{w \in W_{\Xi}} w\left(\frac{xx_{\Xi'}}{x_{\Xi}}\right) \in S^{W_{\Xi}}. \end{aligned}$$

Thus  $|W_{\Xi'}| \cdot C_{\Xi/\Xi'}(x) \in S$ , which implies that  $C_{\Xi/\Xi'}(x) \in S$  by [CZZ, Lemma 3.5]. Besides, it is fixed by  $W_{\Xi}$  by Lemma 5.5.  $\square$

**Corollary 12.3.** *If  $|W|$  is invertible in  $R$ , then  $C_{\Xi/\Xi'}(S^{W_{\Xi'}}) = S^{W_{\Xi}}$ .*

*Proof.* From the proof of Corollary 12.2 we know that for any  $x \in S^{W_{\Xi'}}$ ,  $|W_{\Xi}|x = C_{\Xi} \cdot (xx_{\Xi})$ , so  $C_{\Xi}(S) = S^{W_{\Xi}}$ . The conclusion then follows from the identity  $C_{\Xi/\Xi'} \circ C_{\Xi'} = C_{\Xi}$  of Lemma 5.7.  $\square$

**Theorem 12.4.** *For any  $v, w \in W$ , we have*

$$A_{\Pi}(Y_{I_v}^* A_{I_w^{rev}}(\tilde{f}_e)) = \delta_{w,v}^{Kr} \mathbf{1} = A_{\Pi}(X_{I_v}^* B_{I_w^{rev}}(\tilde{f}_e)).$$

*Consequently, the pairing*

$$A_{\Pi} : \mathbf{D}_F^* \times \mathbf{D}_F^* \rightarrow (\mathbf{D}_F^*)^W \cong S, \quad (\sigma, \sigma') \mapsto A_{\Pi}(\sigma\sigma')$$

*is non-degenerate and has the property that  $(A_{I_w^{rev}}(\tilde{f}_e))_{w \in W}$  is dual to the basis  $(Y_{I_v}^*)_{v \in W}$ , while  $(B_{I_w^{rev}}(\tilde{f}_e))_{w \in W}$  is dual to the basis  $(X_{I_v}^*)_{v \in W}$ .*

*Proof.* We prove the first identity. The second identity is obtained similarly.

Let  $Y_{I_w^{rev}} = \sum_{v \in W} a'_{w,v} \delta_v$  and  $Y_{I_u} = \sum_{v \in W} a_{w,v} \delta_v$ . Let  $\delta_w = \sum_{v \in W} b_{w,v} Y_{I_v}$  so that  $\sum_{v \in W} a_{w,v} b_{v,u} = \delta_{w,u}^{Kr}$  and  $Y_{I_u}^* = \sum_{v \in W} b_{v,u} f_v$ .

Combining the formula of Lemma 7.3 with the formula  $A_{\Pi}(f_v) = \frac{1}{v(x_{\Pi})} \mathbf{1}$  of Lemma 6.6, we obtain

$$A_{\Pi}(Y_{I_u}^* A_{I_w^{rev}}(x_{\Pi} f_e)) = \sum_{v \in W} b_{v,u} v(x_{\Pi}) a_{w,v} A_{\Pi}(f_v) = \sum_{v \in W} b_{v,u} a_{w,v} \mathbf{1} = \delta_{w,u}^{Kr} \mathbf{1}. \quad \square$$

### 13. AN INVOLUTION

In the present section we define an involution on  $\mathbf{D}_F$  and study the relationship between the equivariant characteristic map and the push-pull operators.

We define an  $R$ -linear involution  $\tau : Q_W \rightarrow Q_W$  by

$$\tau(q\delta_w) = w^{-1}(q) \frac{x_{\Pi}}{w^{-1}(x_{\Pi})} \delta_{w^{-1}} = x_{\Pi} \delta_{w^{-1}} q \frac{1}{x_{\Pi}} = \delta_{w^{-1}} q \frac{w(x_{\Pi})}{x_{\Pi}}.$$

**Lemma 13.1.** *We have  $\tau(z_1 z_2) = \tau(z_2) \tau(z_1)$  for any  $z_1, z_2 \in Q_W$ , and we have  $\tau^2 = \text{id}$ , i.e. the map  $\tau$  just defined is indeed an involution. It is the identity on  $Q$  inside  $Q_W$ .*

*Proof.* Clearly, for any  $q \in Q$ , we have  $\tau(q) = q$  and  $\tau(q\delta_w) = \tau(\delta_w)q$ , so it suffices to check that  $\tau(\delta_v\delta_w) = \tau(\delta_w)\tau(\delta_v)$ , which is immediate from the definition of the multiplication in  $Q_W$ . The identity  $\tau^2 = \text{id}$  is obvious on the generators  $q\delta_w$ .  $\square$

Note that  $\frac{x_\Pi}{w^{-1}(x_\Pi)}$  is in  $S$  for any  $w \in W$  by Lemma 3.1.(e), so the involution  $\tau$  restricts to  $S_W$ .

**Corollary 13.2.** *For any simple reflection  $\alpha$ , we have  $\tau(X_\alpha) = X_\alpha$  and  $\tau(Y_\alpha) = Y_\alpha$ . For any sequence  $I$ ,  $\tau(qX_I) = X_{I^{\text{rev}}}q$ . In particular,  $\tau$  induces an involution on  $\mathbf{D}_F$ .*

*Proof.* By Lemma 13.1 it suffices to show that  $\tau(X_\alpha) = X_\alpha$ , which follows from direct computation.  $\square$

Recall that the characteristic map  $c : Q \rightarrow Q_W^*$  introduced in 6.8 has the property that  $q \mapsto \sum_{w \in W} w(q)f_w$ , or in other words,  $c(q)(z) = z \cdot q$  for  $z \in Q_W$ . In particular, we have

$$c(q)(X_I) = \Delta_I(q) \text{ and } c(q)(\delta_w) = w(q), \quad w \in W.$$

**Lemma 13.3.** *For any  $q \in Q$  and  $z \in Q_W$ , we have*

$$A_\Pi \left( (\tau(z) \bullet \tilde{f}_e) c(q) \right) = (z \cdot q) \mathbf{1}.$$

*Proof.* Let  $z = p\delta_w$ ,  $p \in Q$ , then  $\tau(z) \bullet \tilde{f}_e = \delta_{w^{-1}} p \frac{w(x_\Pi)}{x_\Pi} \bullet (x_\Pi f_e) = pw(x_\Pi) f_w$ , so

$$\begin{aligned} A_\Pi \left( (\tau(z) \bullet \tilde{f}_e) c(q) \right) &= A_\Pi \left( (pw(x_\Pi) f_w) \left( \sum_{v \in W} v(q) f_v \right) \right) \\ &= A_\Pi (pw(x_\Pi) w(q) f_w) = pw(q) \mathbf{1} = (z \cdot q) \mathbf{1}. \end{aligned} \quad \square$$

We have the following special cases of Lemma 13.3:

**Corollary 13.4.** *For any sequence  $I$  and  $x \in S$ , we have*

$$A_\Pi(c(q)A_{I^{\text{rev}}}(\tilde{f}_e)) = C_I(q) \mathbf{1} \quad \text{and} \quad A_\Pi(c(q)B_{I^{\text{rev}}}(\tilde{f}_e)) = \Delta_I(q) \mathbf{1}.$$

*Proof.* Letting  $z = Y_I$  (resp.  $z = X_I$ ) in Lemma 13.3, and using  $\tau(Y_I) = Y_{I^{\text{rev}}}$  and  $\tau(X_I) = X_{I^{\text{rev}}}$  from Corollary 13.2 we get the two identities.  $\square$

**Corollary 13.5.** *For any  $z \in Q_W$ , we have  $A_\Pi(\tau(z) \bullet \tilde{f}_e) = (z \cdot 1) \mathbf{1}$ . In particular,  $A_\Pi(q \bullet \tilde{f}_e) = q \mathbf{1}$  and  $A_\Pi(B_I(\tilde{f}_e)) = \Delta_{I^{\text{rev}}}(1) \mathbf{1} = \delta_{I, \emptyset}^K \mathbf{1}$ .*

#### 14. A NON-DEGENERATE PAIRING ON THE SUBRING OF INVARIANTS

In this section, we provide two new  $S$ -module bases of invariants  $(\mathbf{D}_F^*)^{W_\Xi}$  (see Theorem 14.3). We also construct a non-degenerate pairing  $(\mathbf{D}_F^*)^{W_\Xi} \times (\mathbf{D}_F^*)^{W_\Xi} \rightarrow (\mathbf{D}_F^*)^W \cong S$ , corresponding to multiplication and push-forward to the base in the geometric setting.

For any  $w \in W, u \in W^\Xi$  we set

$$d_{w,u}^Y = u(x_{\Pi/\Xi}) \sum_{v \in W_\Xi} a_{w,uv}^Y, \quad d_{w,u}^X = u(x_{\Pi/\Xi}) \sum_{v \in W_\Xi} a_{w,uv}^X, \quad \rho_\Xi = \prod_{w \in W^\Xi} w(x_{\Pi/\Xi})$$

where  $a_{w,uv}^X$  and  $a_{w,uv}^Y$  are the coefficients introduced in Lemma 3.2 and 3.3.

**Lemma 14.1.** *For any  $w \in W$  we have*

$$A_{\Xi}(A_{I_w^{\text{rev}}}(\tilde{f}_e)) = \sum_{u \in W^{\Xi}} d_{w,u}^Y f_u^{\Xi}, \quad A_{\Xi}(B_{I_w^{\text{rev}}}(\tilde{f}_e)) = \sum_{u \in W^{\Xi}} d_{w,u}^X f_u^{\Xi}.$$

*Proof.* We prove the first formula only; the second one is obtained similarly. By Lemma 7.3 and 6.6 we obtain

$$A_{\Xi}(A_{I_w^{\text{rev}}}(x_{\Pi} f_e)) = A_{\Xi}\left(\sum_{v \in W} v(x_{\Pi}) a_{w,v}^Y f_v\right) = \sum_{v \in W} v(x_{\Pi/\Xi}) a_{w,v}^Y f_v^{\Xi}.$$

By (8.1), representing  $v = uv'$ , and using Lemma 5.1, we then express the latter as

$$\sum_{u \in W^{\Xi}, v' \in W^{\Xi}} uv'(x_{\Pi/\Xi}) a_{w,uv'}^Y f_{uv'}^{\Xi} = \sum_{u \in W^{\Xi}, v' \in W^{\Xi}} u(x_{\Pi/\Xi}) a_{w,uv'}^Y f_u^{\Xi}. \quad \square$$

**Lemma 14.2.** *For any  $w \in W, u \in W^{\Xi}$ , we have  $d_{w,u}^Y$  and  $d_{w,u}^X$  belong to  $S$ .*

*Proof.* It follows from Lemma 14.1 and the fact  $\mathbf{D}_F^* \subseteq S_W^*$ .  $\square$

**Theorem 14.3.** *For any choice of reduced sequences  $\{I_w\}_{w \in W^{\Xi}}$ , the two families  $\{A_{\Xi}(A_{I_u^{\text{rev}}}(\tilde{f}_e))\}_{u \in W^{\Xi}}$  and  $\{A_{\Xi}(B_{I_u^{\text{rev}}}(\tilde{f}_e))\}_{u \in W^{\Xi}}$  are  $S$ -module bases of  $(\mathbf{D}_F^*)^{W^{\Xi}}$ .*

*Proof.* Let us first complete our choice of reduced sequence to a  $\Xi$ -compatible one, by choosing sequences  $I_u$  for each  $u \in W^{\Xi}$ . By Corollary 12.1 our families are in the  $S$ -module  $(\mathbf{D}_F^*)^{W^{\Xi}}$ . To show that they are bases, it suffices to show that the respective matrices  $M_{\Xi}^Y$  and  $M_{\Xi}^X$  expressing them in terms of the basis  $\{X_{I_u}^*\}_{u \in W^{\Xi}}$  of Lemma 11.6 have invertible determinants (in  $S$ ).

If  $u' \in W^{\Xi}$  and  $v \in W^{\Xi}$ , we have  $u' \leq u'v$  where the equality holds if and only if  $v = e$ . By Lemma 3.3, we get  $a_{u,u'v}^Y = 0$  unless  $u' \leq u$  and  $a_{u,u'v}^Y = 0$  if  $v \neq e$ . This implies that  $d_{u,u'}^Y = 0$  unless  $u' \leq u$ , and that

$$d_{u,u}^Y = u(x_{\Pi/\Xi}) \sum_{v \in W^{\Xi}} a_{u,uv}^Y = u(x_{\Pi/\Xi}) a_{u,u}^Y = u(x_{\Pi/\Xi}) \frac{1}{x_u}.$$

Hence, the matrix  $D_{\Xi}^Y := (d_{u,u'}^Y)_{u,u' \in W^{\Xi}}$  is lower triangular with determinant  $\rho_{\Xi} \prod_{u \in W^{\Xi}} \frac{1}{x_u}$ . Similarly, the matrix  $D_{\Xi}^X := (d_{u,u'}^X)_{u,u' \in W^{\Xi}}$  is lower triangular with determinant  $\rho_{\Xi} \prod_{u \in W^{\Xi}} \frac{(-1)^{\ell(u)}}{x_u}$ .

On the other hand, for  $u \in W^{\Xi}$ , we have

$$X_{I_u}^* = \sum_{w \in W} b_{w,u}^X f_w = \sum_{u' \in W^{\Xi}} \sum_{v \in W^{\Xi}} b_{u'v,u}^X f_{u'v}.$$

By Corollary 8.5, and because  $X_{I_u}^*$  is fixed by  $W_{\Xi}$ , we have  $b_{u'v,u}^X = b_{u',u}^X$ . Therefore,

$$X_{I_u}^* = \sum_{u' \in W^{\Xi}} b_{u',u}^X \sum_v f_{u'v} = \sum_{u' \in W^{\Xi}} b_{u',u}^X f_{u'}.$$

By Lemma 3.2,  $b_{u',u}^X = 0$  unless  $u' \geq u$ , so the matrix  $E_{\Xi}^X := \{b_{u',u}^X\}_{u',u \in W^{\Xi}}$  is lower triangular with determinant  $\prod_{u \in W^{\Xi}} (-1)^{\ell(u)} x_u$ .

The matrix  $M_{\Xi}^X = (E_{\Xi}^X)^{-1} D_{\Xi}^X$  has determinant

$$\rho_{\Xi} \prod_{u \in W^{\Xi}} \frac{1}{(x_u)^2}$$

which is invertible in  $S$  by Lemma 14.5 below. Since the determinant of  $M_{\Xi}^Y = (E_{\Xi}^X)^{-1} D_{\Xi}^Y$  differs by sign only, it is invertible as well.  $\square$

Recall the definition of  $\Sigma_{\Xi}$  from the beginning of section 5, and let  $w_{0,\Xi}$  be the longest element of  $W_{\Xi}$ .

**Lemma 14.4.** *For any  $w \in W_{\Xi}$ , we have  $x_w x_{w w_{0,\Xi}} = w_{0,\Xi}(x_{\Xi})$ . In particular, if  $\Xi = \Pi$  we have  $x_w x_{w w_0} = x_{w_0}$ .*

*Proof.* Recall from Lemma 3.3 that  $b_{w,w}^Y = x_w = \prod_{w\Sigma^- \cap \Sigma^+} x_{\alpha}$ . By (3.3), it also equals  $\prod_{w\Sigma_{\Xi}^- \cap \Sigma_{\Xi}^+} x_{\alpha}$ . Since  $w_{0,\Xi}\Sigma_{\Xi}^- = \Sigma_{\Xi}^+$ , we have  $w w_{0,\Xi}\Sigma_{\Xi}^- \cap \Sigma_{\Xi}^+ = w\Sigma_{\Xi}^+ \cap \Sigma_{\Xi}^+$ . Moreover,

$$(w\Sigma_{\Xi}^- \cap \Sigma_{\Xi}^+) \cap (w\Sigma_{\Xi}^+ \cap \Sigma_{\Xi}^+) \subset w\Sigma_{\Xi}^- \cap w\Sigma_{\Xi}^+ = w(\Sigma_{\Xi}^- \cap \Sigma_{\Xi}^+) = \emptyset$$

and their union is  $\Sigma_{\Xi}^+$ .  $\square$

**Lemma 14.5.** *For any  $\Xi \subset \Pi$  the product  $\rho_{\Xi} \prod_{u \in W_{\Xi}} \frac{1}{x_u^2}$  is an invertible element in  $S$ .*

*Proof.* We already know that this product is in  $S$ , since it is the determinant of the matrix  $M_{\Xi}^X$  whose coefficients are in  $S$ . Consider the  $R$ -linear involution  $u \mapsto \bar{u}$  on  $S = R[[\Lambda]]_F$  induced by  $\lambda \mapsto -\lambda$ ,  $\lambda \in \Lambda$ . Observe that it is  $W$ -equivariant.

For any  $\alpha \in \Xi$ , we have

$$x_{\Xi} = s_{\alpha}(x_{\Xi})x_{-\alpha}x_{\alpha}^{-1} = s_{\alpha}(x_{\Xi})\bar{x}_{\alpha}x_{\alpha}^{-1}$$

and, therefore, by induction  $x_{\Xi} = w(x_{\Xi})\bar{x}_v x_v^{-1}$  for any  $v \in W_{\Xi}$ . In particular,  $x_{\Pi} = w(x_{\Pi})\bar{x}_w x_w^{-1}$  for any  $w \in W$ . Then

$$x_{\Xi}^{|W_{\Xi}|} = \prod_{v \in W_{\Xi}} v(x_{\Xi})\bar{x}_v x_v^{-1} \quad \text{and} \quad x_{\Pi}^{|W|} = \prod_{w \in W} w(x_{\Pi})\bar{x}_w x_w^{-1}.$$

If  $w = uv$  with  $\ell(w) = \ell(u) + \ell(v)$ , by Lemma 3.1, part (d),  $x_{uv} = x_u u(x_v)$  and  $\bar{x}_{uv} = \bar{x}_u u(\bar{x}_v)$ . Hence

$$\begin{aligned} x_{\Pi}^{|W|} &= \prod_{w \in W} w(x_{\Pi})\bar{x}_w x_w^{-1} = \prod_{u \in W_{\Xi}} \prod_{v \in W_{\Xi}} uv(x_{\Pi/\Xi} x_{\Xi})\bar{x}_{uv} x_{uv}^{-1} \\ &\stackrel{5.2}{=} \prod_{u \in W_{\Xi}} u(x_{\Pi/\Xi}^{|W_{\Xi}|}) \prod_{v \in W_{\Xi}} uv(x_{\Xi})\bar{x}_u u(\bar{x}_v) x_u^{-1} u(x_v^{-1}) \\ (14.1) \quad &= \rho_{\Xi}^{|W_{\Xi}|} \prod_{u \in W_{\Xi}} (\bar{x}_u x_u^{-1})^{|W_{\Xi}|} u \left( \prod_{v \in W_{\Xi}} v(x_{\Xi})\bar{x}_v x_v^{-1} \right) \\ &= \rho_{\Xi}^{|W_{\Xi}|} \prod_{u \in W_{\Xi}} (\bar{x}_u x_u^{-1})^{|W_{\Xi}|} u(x_{\Xi})^{|W_{\Xi}|} \end{aligned}$$

On the other hand, by Lemma 14.4,

$$\bar{x}_{\Xi}^{|W_{\Xi}|} = w_{0,\Xi}(x_{\Xi})^{|W_{\Xi}|} = \prod_{v \in W_{\Xi}} x_v x_{v w_{0,\Xi}} = \prod_{v \in W_{\Xi}} x_v^2$$

and, in particular,  $\bar{x}_{\Pi}^{|W|} = \prod_{w \in W} x_w^2$ . So, we obtain

$$\begin{aligned} \bar{x}_{\Pi}^{|W|} &= \prod_{w \in W} x_w^2 = \prod_{u \in W_{\Xi}} \prod_{v \in W_{\Xi}} x_{uv}^2 = \prod_{u \in W_{\Xi}} \prod_{v \in W_{\Xi}} x_u^2 u(x_v^2) \\ &= \left( \prod_{u \in W_{\Xi}} x_u^{2|W_{\Xi}|} \right) \left( \prod_{u \in W_{\Xi}} u \left( \prod_{v \in W_{\Xi}} x_v^2 \right) \right) = \left( \prod_{u \in W_{\Xi}} x_u^2 \right)^{|W_{\Xi}|} \left( \prod_{u \in W_{\Xi}} u(\bar{x}_{\Xi}) \right)^{|W_{\Xi}|}. \end{aligned}$$



Combining this with equation (14.1), we obtain

$$\left(\rho_{\Xi}^{-1} \prod_{u \in W^{\Xi}} x_u^2\right)^{|W^{\Xi}|} = \bar{x}_{\Pi}^{|W|} x_{\Pi}^{-|W|} \left(\prod_{u \in W^{\Xi}} u(\bar{x}_{\Xi}^{-1} x_{\Xi}) \bar{x}_u x_u^{-1}\right)^{|W^{\Xi}|}$$

which is an element of  $S$ , since it is a product of elements of the form  $x_{\alpha} x_{-\alpha}^{-1} \in S$ . Therefore  $\rho_{\Xi} \prod_{u \in W^{\Xi}} \frac{1}{x_u^2}$  is invertible, since so is its  $|W^{\Xi}|$ -th power.  $\square$

**Corollary 14.6.** *Given  $\Xi' \subseteq \Xi \subseteq \Pi$  we have  $A_{\Xi}(\mathbf{D}_F^*) = (\mathbf{D}_F^*)^{W^{\Xi}}$ . For any set of coset representatives  $W_{\Xi/\Xi'}$  the operator  $A_{\Xi/\Xi'}$  induces a surjection  $(\mathbf{D}_F^*)^{W_{\Xi/\Xi'}} \rightarrow (\mathbf{D}_F^*)^{W^{\Xi}}$  (independent of the choices of  $W_{\Xi/\Xi'}$  by Lemma 6.5).*

*Proof.* By Corollary 12.1 and Theorem 14.3, we obtain the first part. To prove the second part, let  $\sigma \in (\mathbf{D}_F^*)^{W_{\Xi/\Xi'}}$ . By the first part, there exists  $\sigma' \in \mathbf{D}_F^*$  such that  $\sigma = A_{\Xi/\Xi'}(\sigma')$ , so by Lemma 6.3 we have

$$A_{\Xi/\Xi'}(\sigma) = A_{\Xi/\Xi'}(A_{\Xi'}(\sigma')) = A_{\Xi}(\sigma') \in (\mathbf{D}_F^*)^{W^{\Xi}}.$$

Hence,  $A_{\Xi/\Xi'}$  restricts to  $A_{\Xi/\Xi'} : (\mathbf{D}_F^*)^{W_{\Xi/\Xi'}} \rightarrow (\mathbf{D}_F^*)^{W^{\Xi}}$ . Since  $A_{\Xi}(\mathbf{D}_F^*) = (\mathbf{D}_F^*)^{W^{\Xi}}$ , we also have  $A_{\Xi/\Xi'}((\mathbf{D}_F^*)^{W_{\Xi/\Xi'}}) = (\mathbf{D}_F^*)^{W^{\Xi}}$ .  $\square$

The following result can be viewed as an algebraic version of the Poincaré pairing.

**Theorem 14.7.** *Assume that the choice of reduced sequences  $\{I_w\}_{w \in W}$  is  $\Xi$ -compatible. If  $u \in W^{\Xi}$ , then*

$$A_{\Pi/\Xi}(X_{I_u}^* A_{\Xi}(B_{I_w^{\text{rev}}}(x_{\Pi} f_e))) = \delta_{w,u}^{\text{Kr}} \mathbf{1}.$$

Consequently, the pairing

$$(\mathbf{D}_F^*)^{W^{\Xi}} \times (\mathbf{D}_F^*)^{W^{\Xi}} \rightarrow (\mathbf{D}_F^*)^W \cong S, \quad (\sigma, \sigma') \mapsto A_{\Pi/\Xi}(\sigma \sigma')$$

is non-degenerate,  $\{A_{\Xi}(B_{I_u^{\text{rev}}}(x_{\Pi} f_e))\}_{u \in W^{\Xi}}$  and  $\{X_{I_u}^*\}_{u \in W^{\Xi}}$  being dual  $S$ -bases of  $(\mathbf{D}_F^*)^{W^{\Xi}}$ .

*Proof.* By Corollary 14.6, the pairing is well-defined (i.e. it does map into  $S$ ). By the projection formula (Lemma 6.4), the composition rule (Lemma 6.3) and Theorem 12.4, we obtain

$$\begin{aligned} A_{\Pi/\Xi}(X_{I_u}^* A_{\Xi}(B_{I_w^{\text{rev}}}(x_{\Pi} f_e))) &= A_{\Pi/\Xi}(A_{\Xi}(X_{I_u}^* B_{I_w^{\text{rev}}}(x_{\Pi} f_e))) \\ &= A_{\Pi}(X_{I_u}^* B_{I_w^{\text{rev}}}(x_{\Pi} f_e)) = \delta_{w,u}^{\text{Kr}} \mathbf{1}. \end{aligned} \quad \square$$

## 15. ALGEBRAIC PUSH-FORWARDS AND THE PAIRING

In the present section, we construct an algebraic version of the push-forward map and investigate its properties with respect to pull-backs, the Hecke action and the non-degenerate pairing.

For any  $\Xi \subseteq \Pi$ , the  $W^{\Xi}$  invariant subring  $S^{W^{\Xi}}$  (resp.  $Q^{W^{\Xi}}$ ) acts by multiplication on the right on  $S_{W/W^{\Xi}}$  (resp.  $Q_{W/W^{\Xi}}$ ) by the formula  $(\sum_{\bar{w}} q_{\bar{w}} \delta_{\bar{w}}) \cdot q' = \sum_{\bar{w}} q_{\bar{w}} w(q') \delta_{\bar{w}}$  (note that  $w(q')$  does not depend on the choice of a representative  $w$  of  $\bar{w}$ ). When  $q \in S^{W^{\Xi}}$  (resp.  $Q^{W^{\Xi}}$ ) and  $f \in S_{W/W^{\Xi}}^*$  (resp.  $f \in Q_{W/W^{\Xi}}^*$ ), we write  $q \bullet f$  for the map dual to the multiplication on the right by  $q$ .

Recall that  $d_{\Xi}^* : Q_{W/W_{\Xi'}}^* \rightarrow Q_{W/W^{\Xi}}^*$  was defined at the beginning of section 11, and that it sends  $f_{\bar{w}}$  to  $f_{\bar{w}}$ . By Corollary 5.2 we know that  $\frac{1}{x_{\Xi/\Xi'}} \in (Q)^{W_{\Xi'}}$ .

We define  $\mathcal{A}_{\Xi/\Xi'}: Q_{W/W_{\Xi'}}^* \rightarrow Q_{W/W_{\Xi}}^*$  by  $\mathcal{A}_{\Xi/\Xi'}(f) := d_{\Xi/\Xi'}^*((1/x_{\Xi/\Xi'}) \bullet f)$ . The left commutative diagram

$$\begin{array}{ccc} Q_{W/W_{\Xi'}} & \xleftarrow{p_{\Xi'}} & Q_W \\ \cdot \frac{1}{x_{\Xi/\Xi'}} \circ d_{\Xi/\Xi'} \uparrow & & \uparrow \cdot Y_{\Xi/\Xi'} \\ Q_{W/W_{\Xi}} & \xleftarrow{p_{\Xi}} & Q_W \end{array} \quad \begin{array}{ccc} Q_{W/W_{\Xi'}}^* & \xrightarrow{p_{\Xi'}^*} & Q_W^* \\ \mathcal{A}_{\Xi/\Xi'} \downarrow & & \downarrow \mathcal{A}_{\Xi/\Xi'} \\ Q_{W/W_{\Xi}}^* & \xrightarrow{p_{\Xi}^*} & Q_W^* \end{array}$$

in which  $\cdot 1/x_{\Xi/\Xi'}$  and  $\cdot Y_{\Xi/\Xi'}$  mean multiplication on the right, dualizes as the right one. Since  $p_{\Xi}^*$  restricts to a ring isomorphism  $\mathbf{D}_{F,\Xi}^* \xrightarrow{\sim} (\mathbf{D}_F^*)^{W_{\Xi}}$  by Lemma 11.7 and since  $\mathcal{A}_{\Xi/\Xi'}$  restricts to a map  $(\mathbf{D}_F^*)^{W_{\Xi'}} \rightarrow (\mathbf{D}_F^*)^{W_{\Xi}}$  by Corollary 14.6, we obtain:

**Lemma 15.1.** *The map  $\mathcal{A}_{\Xi/\Xi'}$  restricts to  $\mathbf{D}_{F,\Xi}^* \rightarrow \mathbf{D}_{F,\Xi'}^*$  and the diagram*

$$\begin{array}{ccc} \mathbf{D}_{F,\Xi'}^* & \xrightarrow[p_{\Xi'}^*]{\simeq} & (\mathbf{D}_F^*)^{W_{\Xi'}} \\ \mathcal{A}_{\Xi/\Xi'} \downarrow & & \downarrow \mathcal{A}_{\Xi/\Xi'} \\ \mathbf{D}_{F,\Xi}^* & \xrightarrow[p_{\Xi}^*]{\simeq} & (\mathbf{D}_F^*)^{W_{\Xi}} \end{array}$$

commutes. The composition rule of Lemma 6.3 translates as

$$\mathcal{A}_{\Xi/\Xi'} \circ \mathcal{A}_{\Xi'/\Xi''} = \mathcal{A}_{\Xi/\Xi''} \quad \text{for any } \Xi'' \subseteq \Xi' \subseteq \Xi \subseteq \Pi$$

and the projection formula of Lemma 6.4 translates as

$$\mathcal{A}_{\Xi/\Xi'}(p_{\Xi/\Xi'}^*(f)f') = f\mathcal{A}_{\Xi/\Xi'}(f') \quad \text{for any } f \in \mathbf{D}_{F,\Xi}^* \text{ and } f' \in \mathbf{D}_{F,\Xi'}^*.$$

**Remark 15.2.** The map  $\mathcal{A}_{\Xi/\Xi'}$  corresponds to a push-forward in the geometric context, see [CZZ2, Diagram (8.3)].

**Lemma 15.3.** *Within  $Q_{W/W_{\Xi}}$ , we have  $\mathbf{D}_{F,\Xi}x_{\Pi/\Xi} \subseteq S_{W/W_{\Xi}}$ . So the right multiplication by  $x_{\Pi/\Xi}$  induces a map  $\mathbf{D}_{F,\Xi} \rightarrow S_{W/W_{\Xi}}$ . Consequently, it defines a map  $S_{W/W_{\Xi}}^* \rightarrow \mathbf{D}_{F,\Xi}^*$ ,  $f \mapsto x_{\Pi/\Xi} \bullet f$ .*

*Proof.* By Lemma 11.3 we know that  $\{X_{I_w}^{\Xi}\}_{w \in W_{\Xi}}$  is a basis of  $\mathbf{D}_{F,\Xi}$ , so it suffices to show that  $X_{I_w}^{\Xi}x_{\Pi/\Xi} \in S_{W/W_{\Xi}}$ . We have

$$X_{I_w}^{\Xi}x_{\Pi/\Xi} = \sum_{u \in W_{\Xi}} \left( \sum_{v \in W_{\Xi}} a_{w,uv}^X \right) \delta_{\bar{u}} x_{\Pi/\Xi} = \sum_{u \in W_{\Xi}} \left( \sum_{v \in W_{\Xi}} u(x_{\Pi/\Xi}) a_{w,uv}^X \right) \delta_{\bar{u}} = \sum_{u \in W_{\Xi}} d_{w,u}^Y \delta_{\bar{u}},$$

which belongs to  $S_{W/W_{\Xi}}$  by Lemma 14.2.  $\square$

The geometric translation of the map  $S_{W/W_{\Xi}}^* \rightarrow \mathbf{D}_{F,\Xi}^*$  is the push-forward map from the  $T$ -fixed points of  $G/P_{\Xi}$  to  $G/P_{\Xi}$ , see [CZZ2, Diagram (8.1)].

**Example 15.4.** Note that in general  $x_{\Pi/\Xi} \mathbf{D}_{F,\Xi} \not\subseteq S_{W/W_{\Xi}}$ . For example, let the root datum be of type  $A_2^{ad}$  and  $\Xi = \{\alpha_2\}$ , then  $x_{\Pi/\Xi} = x_{-\alpha_1}x_{-\alpha_1-\alpha_2}$ . Let  $w = s_2s_1 \in W_{\Xi}$ , then

$$X_{21} = \frac{1}{x_{\alpha_1}x_{\alpha_2}} \delta_e - \frac{1}{x_{\alpha_2}x_{\alpha_1+\alpha_2}} \delta_{s_2} - \frac{1}{x_{\alpha_1}x_{\alpha_2}} \delta_{s_1} + \frac{1}{x_{\alpha_2}x_{\alpha_1+\alpha_2}} \delta_{s_2s_1}.$$

Then  $X_{21}^{\Xi}x_{\Pi/\Xi} \in S_{W/W_{\Xi}}$  but  $x_{\Pi/\Xi}X_{21}^{\Xi} \notin S_{W/W_{\Xi}}$ .

One easily checks that the diagram on the left below is commutative, and it restricts as the one on the right by Lemma 15.3.

$$\begin{array}{ccc}
 Q_{W/W_{\Xi'}} & \xrightarrow{\cdot x_{\Pi/\Xi'}} & Q_{W/W_{\Xi'}} \\
 \cdot \frac{1}{x_{\Xi/\Xi'}} \circ d_{\Xi/\Xi'} \uparrow & & \uparrow d_{\Xi/\Xi'} \\
 Q_{W/W_{\Xi}} & \xrightarrow{\cdot x_{\Pi/\Xi}} & Q_{W/W_{\Xi}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{D}_{F,\Xi'} & \xrightarrow{\cdot x_{\Pi/\Xi'}} & S_{W/W_{\Xi'}} \\
 \cdot \frac{1}{x_{\Xi/\Xi'}} \circ d_{\Xi/\Xi'} \uparrow & & \uparrow d_{\Xi/\Xi'} \\
 \mathbf{D}_{F,\Xi} & \xrightarrow{\cdot x_{\Pi/\Xi}} & S_{W/W_{\Xi}}
 \end{array}$$

By  $S$ -dualization, one obtains the commutative diagram

$$\begin{array}{ccc}
 \mathbf{D}_{F,\Xi'}^* & \xleftarrow{x_{\Pi/\Xi'} \bullet} & S_{W/W_{\Xi'}}^* \\
 \mathcal{A}_{\Xi/\Xi'} \downarrow & & \downarrow d_{\Xi/\Xi'}^* \\
 \mathbf{D}_{F,\Xi}^* & \xleftarrow{x_{\Pi/\Xi} \bullet} & S_{W/W_{\Xi}}^*
 \end{array}$$

whose geometric interpretation in terms of push-forwards is given in [CZZ2, Diagram (8.3)]

Theorems 12.4 and 14.7 then immediately translate as:

**Theorem 15.5.** *The pairing  $\mathbf{D}_F^* \times \mathbf{D}_F^* \rightarrow \mathbf{D}_{F,\Pi}^* \simeq S$  defined by sending  $(\sigma, \sigma')$  to  $\mathcal{A}_{\Pi}(\sigma\sigma')$  is non degenerate;  $\{B_{I_w}^{\text{rev}}(x_{\Pi}f_e)\}_{w \in W}$  and  $\{X_{I_v}^*\}_{v \in W}$  are dual bases and so are  $\{A_{I_w}^{\text{rev}}(x_{\Pi}f_e)\}_{w \in W}$  and  $\{Y_{I_v}^*\}_{v \in W}$ .*

**Theorem 15.6.** *The pairing  $\mathbf{D}_{F,\Xi}^* \times \mathbf{D}_{F,\Xi}^* \rightarrow \mathbf{D}_{F,\Pi}^* \simeq S$  defined by sending  $(\sigma, \sigma')$  to  $\mathcal{A}_{\Pi/\Xi}(\sigma\sigma')$  is non degenerate;  $\{\mathcal{A}_{\Xi}(B_{I_w}^{\text{rev}}(x_{\Pi}f_e))\}_{w \in W^{\Xi}}$  and  $\{(X_{I_v}^{\Xi})^*\}_{v \in W^{\Xi}}$  are dual bases and so are  $\{\mathcal{A}_{\Xi}(A_{I_w}^{\text{rev}}(x_{\Pi}f_e))\}_{w \in W^{\Xi}}$  and  $\{(Y_{I_v}^{\Xi})^*\}_{v \in W^{\Xi}}$ .*

*Proof.* For any choice of  $\{I_w\}_{w \in W^{\Xi}}$ , we complete it into a  $\Xi$ -compatible family  $\{I_w\}_{w \in W}$ , then by Lemma 11.6  $\{(X_{I_v}^{\Xi})^*\}_{v \in W^{\Xi}}$  and  $\{(Y_{I_v}^{\Xi})^*\}_{v \in W^{\Xi}}$  are bases of  $\mathbf{D}_{F,\Xi}^*$ .

We prove the result for  $Y$ 's only (the proof for  $X$ 's is the same). We decompose  $p_{\Xi}^*((Y_{I_v}^{\Xi})^*) = \sum_{w \in W} e_{w,v} Y_{I_w}^*$  for some  $e_{w,v} \in S$ . By definition we have  $p_{\Xi}^*((Y_{I_v}^{\Xi})^*)(Y_{I_w}) = (Y_{I_v}^{\Xi})^*(Y_{I_w}^{\Xi})$ . So if  $w \in W^{\Xi}$ , then  $e_{w,v} = (Y_{I_v}^{\Xi})^*(Y_{I_w}^{\Xi}) = \delta_{v,w}^{Kr}$ .

As in the proof of Theorem 14.7, but using the projection formula and the composition rule of Lemma 15.1 with Theorem 15.6, we obtain

$$\begin{aligned}
 \mathcal{A}_{\Pi/\Xi}((Y_{I_v}^{\Xi})^* \mathcal{A}_{\Xi}(A_{I_w}^{\text{rev}}(x_{\Pi}f_e))) &= \mathcal{A}_{\Pi/\Xi} \mathcal{A}_{\Xi}(p_{\Xi}^*((Y_{I_v}^{\Xi})^*) A_{I_w}^{\text{rev}}(x_{\Pi}f_e)) \\
 &= \mathcal{A}_{\Pi} \left( \left( \sum_{u \in W} e_{u,v} Y_{I_u}^* \right) A_{I_w}^{\text{rev}}(x_{\Pi}f_e) \right) = \delta_{v,w}^{Kr}. \quad \square
 \end{aligned}$$

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