arXiv:1609.06929v1 [math.AG] 22 Sep 2016

RELATIVE EQUIVARIANT MOTIVES VERSUS MODULES

BAPTISTE CALMÈS, ALEXANDER NESHITOV, AND KIRILL ZAINOULLINE

Abstract. The purpose of the present notes is to introduce a language relating various motivic categories of G-varieties (G is a semisimple linear algebraic group over a field) and categories of certain D_G-modules, where D_G is the Hecke-type ring associated to G.

Contents

1. Introduction 1
2. Relative equivariant motives and modules 2
3. The Weyl group action on cohomology 4
4. Motives vs. modules 7
5. Nilpotency for equivariant oriented theories 8
6. The endomorphisms of $\mathbf{D}^{\star}_{F,P}$ 13
7. Applications to Chow motives 15
8. Endomorphisms of $\mathbf{Q}^{*}_{W/W_P}$ 21
References 24

1. Introduction

The theory of Chow motives is an important tool of study of algebraic varieties. Motivic decompositions of Pfister quadric played an essential role in the proof of the Milnor conjecture by Voevodsky, and the motivic decompositions of norm varieties were used to prove the Bloch-Kato conjecture by Rost, Suslin and Voevodsky. Another application to the theory of quadratic forms can be found in the works of Vishik, Karpenko and Merkurjev. Let G be a semisimple algebraic group over a base field k. Our primary objects of interest are projective homogeneous G-varieties. Motivic decompositions of such varieties were intensively investigated in the last two decades. The case of split varieties was established by Köck [25], who showed that in this case the motive decomposes as a sum of Tate motives. The results of Chernousov-Gille-Merkurjev [11] and Brosnan [5] give decompositions of motives of isotropic homogeneous varieties into direct sums of motives of smaller anisotropic varieties. Rost [39] established the motivic decomposition of a Pfister quadric as a sum of twisted copies of an indecomposable motive R called the Rost motive. The case of Severi-Brauer varieties was studied by Karpenko in [24].

2010 *Mathematics Subject Classification.* 14F43, 14M15, 20C08, 14C15.

Key words and phrases. linear algebraic group, torsor, flag variety, equivariant oriented cohomology, motivic decomposition, Hecke algebra.

Motivic decompositions of generically split projective homogeneous varieties were studied in [38].

In the present paper we consider the case of a versal inner form of a projective homogeneous variety, i.e. a variety of the form E/P where E is a versal (i.e. a generic) torsor of a split semisimple group G and P is a parabolic subgroup (not necessarily special). Note that the groups G and P are uniquely determined by combinatorial data: the root system of G, the character lattice T^* of its split maximal torus T, and the subset subset of simple roots of G defining P.

The main aim of the present paper is to describe the motivic decompositions of E/P in terms of these combinatorial data. We work in a bit more general situation than the theory of Chow motives. Namely we consider an oriented cohomology theory $\mathtt{h}$ in the sense of Levine-Morel [33] and the theory of $\mathtt{h}$-motives. In the case $\mathtt{h} = CH$ it coincides with the classical category of Chow motives.

It is convenient to use the notion of equivariant motives, which we introduce in Section 2. The main result of the present paper 5.8 establishes a $1-1$ correspondence between the motivic decomposition of E/P and G-equivariant motivic decomposition of the split variety G/P. The corollary 4.3 provides an injection of endomorphism ring of the G-equivariant motive of G/P into the endomorphism ring of the $\mathbf{D}_F$-module $\mathbf{D}_{F,P}^\star$. In the case when P is special, this injection is an isomorphism. Here $\mathbf{D}_F$ is the graded formal affine Demazure algebra introduced in [22]. Note that $\mathbf{D}_F$ and $\mathbf{D}_{F,P}^\star$ allow a combinatorial description in terms of character lattice and root datum of the group G.

Acknowledgements. This paper is based on the PhD thesis [35] of the second author, where the case of a special parabolic subgroup was treated. The results presented here generalize several results of the thesis to the case of an arbitrary parabolic subgroup. The first author acknowledges the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005. The second author is grateful to the University of Ottawa and to the Ontario Trillium graduate scholarship program for the support. The last author is grateful to the NSERC Discovery Grant for the support.

2. Relative equivariant motives and modules

In the present section we introduce categories of relative equivariant motives and modules.

Fix a smooth group scheme G over a field k and its closed algebraic subgroup H. Consider a category G-Var_k of smooth projective (left) G-varieties over k with G-equivariant morphisms. Let $\mathtt{h}$ be a G-equivariant oriented cohomology theory on G-Var_k in the sense of [21].

We define a category of *relative equivariant correspondences* G/H-$Corr_k$ with respect to the inclusion $H \hookrightarrow G$ as follows. Its objects are from G-Var_k and the morphisms are defined by

$$Mor_{G/H\text{-}Corr_k}(X,Y) = \operatorname{im}\big(res_{G/H} \colon \mathtt{h}_G(X\times_k Y) \longrightarrow \mathtt{h}_H(X\times_k Y)\big),$$

with the composition given by the usual correspondence product.

We define a category of *relative equivariant motives* G/H-Mot_k as the pseudo-abelian completion of G/H-$Corr_k$, i.e., objects are pairs (X,p), where $X \in G$-Var_k, p is an idempotent in $\operatorname{End}_{G/H\text{-}Corr_k}(X)$ and morphisms preserve idempotents. We denote by $[X]$ the class of a G-variety X in $G/H\text{-}Mot_k$ and call it the relative equivariant motive of X.

Example 2.1. If $G = H$ is a trivial group and $\mathtt{h} = CH$ is the (equivariant) Chow theory, we obtain the classical non-graded version of the category of Grothendieck's Chow motives over S (e.g. see [46]). If $G = H$ is a split semi-simple linear algebraic group over k, we obtain its G-equivariant version studied in [36] and [37].

We fix a smooth projective G-variety Z over k (call it a base object) and consider the endomorphism ring

$$\mathbf{D}^{Z}_{G/H} = \mathrm{End}_{G/H\text{-}Mot_k}([Z]).$$

Observe that $\mathbf{D}^{Z}_{G/H}$ is a (non-commutative) algebra over a commutative ring

$$\mathbf{S}_{G/H} = \mathrm{im}\,\big(res_{G/H} \colon \mathtt{h}_G(k) \longrightarrow \mathtt{h}_H(k)\big).$$

Consider a full additive subcategory of left $\mathbf{D}^{Z}_{G/H}$-modules generated by modules

$$M^{Z}_{G/H}(X) = Mor_{G/H\text{-}Mot_k}([X],[Z]) \quad \text{for all } X \in G\text{-}Var_k.$$

We denote such subcategory by $G/H\text{-}Mod_Z$ and call it a category of *relative equivariant modules over* Z. Observe that $M^{Z}_{G/H}(Z) = \mathbf{D}^{Z}_{G/H}$ as a left module over itself.

The assignment $[X] \mapsto M^{Z}_{G/H}(X)$ defines a contravariant functor

$$\mathfrak{F}^{Z}_{G/H} \colon G/H\text{-}Mot_k \longrightarrow G/H\text{-}Mod_Z,$$

where

$$f^{Z}_{G/H} \colon Mor_{G/H\text{-}Mot_k}([X],[Y]) \longrightarrow Hom_{\mathbf{D}^{Z}_{G/H}}(M^{Z}_{G/H}(Y), M^{Z}_{G/H}(X))$$

is induced by composing with $[X] \to [Y]$.

Observe that the category $G/H\text{-}Mot_k$ is anti-equivalent to itself via the transposition functor

$$\tau \colon Mor_{G/H\text{-}Mot_k}([X],[Y]) \xrightarrow{\simeq} Mor_{G/H\text{-}Mot_k}([Y],[X])$$

induced by the switch map $X \times_k Y \to Y \times_k X$ with $\tau(\alpha \circ \beta) = \tau(\beta) \circ \tau(\alpha)$. In particular, it defines an involution τ (anti-automorphism of order 2) on the endomorphism ring $\mathbf{D}^{Z}_{G/H}$.

In the present paper we will deal with a case when G is a split semisimple linear algebraic group over k, $H = T$ is its maximal torus and Z is either $pt = Spec\,(k)$ or a variety of Borel subgroups G/B of G containing T.

Observe that if $Z = pt$, then $\mathbf{D}^{pt}_{G/T} = \mathbf{S}_{G/T}$ and $G/T\text{-}Mod_{pt}$ is a category of left $\mathbf{S}_{G/T}$-modules with

$$M^{pt}_{G/T}(X) = \mathrm{im}\,\big(res_{G/T} \colon \mathtt{h}_G(X) \longrightarrow \mathtt{h}_T(X)\big).$$

The functor $\mathfrak{F}^{pt}_{G/T}$ respects the tensor products, the map $f^{pt}_{G/T}$ is the usual motivic realization map and we have the following

Lemma 2.2. *(cf. [36]) If X and Y are G-equivariant cellular spaces over k, then $f^{pt}_{G/T}$ is a Künneth isomorphism.*

In other words, the functor $\mathfrak{F}^{pt}_{G/T}$ induces an anti-equivalence of categories if restricted to a full subcategory of $G/T\text{-}Mot_k$ generated by G-equivariant cellular spaces.

We will often omit the upper index pt, when dealing with the case $Z = pt$.

If $Z = G/B$, then

$$\mathbf{D}^{G/B}_{G/T} = \mathrm{im}\big(res_{G/T} \colon \mathtt{h}_G(G/B \times_k G/B) \longrightarrow \mathtt{h}_T(G/B \times_k G/B)\big).$$

Since the restriction map is injective by [36, Lemma 4.5], we can identify $\mathbf{D}^{G/B}_{G/T}$ with the convolution ring $\mathtt{h}_G(G/B \times_S G/B) \simeq \mathtt{h}_T(G/B)$ of [36, §4].

3. The Weyl group action on cohomology

In this section we recall several facts concerning action of the Weyl group on cohomology rings of various flag varieties and their products.

Let G be a split semi-simple linear algebraic group over a field k. We fix a parabolic subgroup P, Borel subgroup B and a split maximal torus T so that $P \supset B \supset T$. Let X be an arbitrary (left) G-variety. There is a natural (left) action of W on $\mathtt{h}_T(X)$. It can be either realized by pull-backs induced by a right action of W on each step of the Borel construction

$$U \times^T X = U \times X/(u,x) \sim (ut, t^{-1}x), \quad t \in T$$

given by

$$(u,x)T \cdot \sigma T = (u\sigma, \sigma^{-1}x)T, \quad \sigma \in N_G(T)$$

where U is taken to have a right G-action; or through the natural isomorphism $\mathtt{h}_T(X) \simeq \mathtt{h}_G(G/T \times X)$ and a G-equivariant right action of W on the variety $G/T \times_k X$ given by the formula $(gT, x) \cdot \sigma T = (gT\sigma, x)$.

Let W_P denote the Weyl group of the Levi-part of P. We identify the set of T-fixed points of G/P with a finite constant scheme W/W_P with trivial T-action. In the induced pullback $\mathtt{h}_T(G/P) \to \mathtt{h}_T(W/W_P)$ we identify $\mathtt{h}_T(W/W_P)$ with the ring of all set-theoretic maps $\mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ (see [10])

$$\mathtt{h}_T(W/W_P) = \mathtt{h}(U \times^T W/W_P) = \oplus_{xW_P \in W/W_P} \mathtt{h}_T(k) = \mathrm{Maps}(W/W_P, \mathtt{h}_T(k)),$$

where the class of $(u, xW_P)T$ maps to $xW_P \mapsto [uT]$.

Lemma 3.1. *Consider the left W-action on $\mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ given by*

$$(w \cdot f)(x) = w \cdot f(w^{-1}x), \quad x \in W/W_P,\ f \in \mathrm{Maps}(W/W_P, \mathtt{h}_T(k)).$$

Then the pullback map

$$\mathtt{h}_T(G/P) \to \mathtt{h}_T(W/W_P) = \mathrm{Maps}(W/W_P, \mathtt{h}_T(k)) \text{ is } W\text{-equivariant.}$$

Proof. Recall that the action of W on $U \times^T G/P$ is given by $(u, gP)T \cdot \sigma T = (u\sigma, \sigma^{-1}gP)T$ for any U in the Borel construction. Restricted to $U \times^T W/W_P$ the action of $w = \sigma T$ gives a map

$$U \times^T W/W_P \to U \times^T W/W_P, \quad (u, xW_P)T = (u\sigma, \sigma^{-1}xW_P)T.$$

So its pullback defines an endomorphism of $\mathtt{h}_T(W/W_P) = \mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ given by $f \mapsto w \cdot f$, where $w \cdot f \colon x \mapsto wf(w^{-1}x)$. □

Remark 3.2. By the very definition, the W-action on $\mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ and, hence, its restriction on $\mathtt{h}_T(G/P)$ coincides with the $\odot$-action of W on $\mathbf{S}^\star_W$ (resp. on $\mathbf{D}^\star_{F,P}$) of the next section.

Definition 3.3. We call a parabolic subgroup P of G to be $\mathtt{h}$-special, if the natural map $\mathtt{h}_P(k) \to \mathtt{h}_T(k)^{W_P}$ is surjective.

Example 3.4. Recall that a group is called special if all its principal bundles are locally trivial in Zariski topology. For a special parabolic subgroup P and $\mathtt{h}_P(-) = CH(-;\mathbf{R})$ there is an isomorphism $\phi\colon CH_P(k;\mathbf{R}) \xrightarrow{\simeq} CH_T(k;\mathbf{R})^{W_P}$ by [15] or [16, Prop. 6], hence, any special P is $CH(-;\mathbf{R})$-special.

Observe that there are exist non special P for which ϕ is either surjective (e.g. the Levi part of P is PGL_2) or not (e.g. for $Spin_{12}$, see [18]). Observe also that the surjectivity of ϕ depends on the coefficient ring $\mathbf{R}$ as in general $CH_T(k;\mathbf{R})^{W_P} \neq CH_T(k)^{W_P} \otimes \mathbf{R}$.

Lemma 3.5. *If P is $\mathtt{h}$-special, then the pull-back $\mathtt{h}_T(G/P)^W \to \mathtt{h}_T(W/W_P)^W = \mathtt{h}_T(pt)^{W_P}$ is an isomorphism. In particular, the restriction $\mathtt{h}_G(G/P) \to \mathtt{h}_T(G/P)$ is surjective.*

Proof. The composition of restriction homomorphism $\mathtt{h}_G(G/P) \to \mathtt{h}_T(G/P)$ with the isomorphism $\mathtt{h}_T(G/P) \simeq \mathtt{h}_G(G/T \times G/P)$ is a pullback induced by projection on the second factor. So the image is contained in $\mathtt{h}_G(G/T \times G/P)^W$.

Recall that T-fixed points of G/P are given by the natural embedding $W/W_P \hookrightarrow G/P$. Here we consider W/W_P as a finite constant scheme with trivial T-action. Take any U in the Borel construction. There is a commutative diagram

$$\begin{array}{ccc} U \times^G G/P & \longleftarrow & U \times^T G/P \\ \simeq\uparrow & & \uparrow \\ U/P & \xleftarrow{f} & U \times^T W/W_P \end{array}$$

The leftmost arrow is a scheme isomorphism given by $uP \to (u, P)$, the upper horizontal arrow is the projection, and the rightmost arrow arises from the fixed-point embedding $W/W_P \to G/P$. Then the bottom arrow f is given by

$$f\colon (u, wW_P)T \mapsto u\sigma \cdot P \text{ where } \sigma \in N_G(T) \text{ such that } \sigma T W_P = wW_P.$$

Since the diagram is compatible with the embedding $U = U_i \to U_{i+1}$ in the Borel construction, it induces the commutative diagram of equivariant pullbacks:

$$\begin{array}{ccc} \mathtt{h}_G(G/P) & \longrightarrow & \mathtt{h}_T(G/P) \\ \downarrow\simeq & & \downarrow \\ \mathtt{h}_P(k) & \xrightarrow{f^*} & \mathtt{h}_T(W/W_P) \end{array}$$

By lemma 3.1 the rightmost map is W-equivariant, so we have a diagram

$$\begin{array}{ccc} \mathtt{h}_G(G/P) & \longrightarrow & \mathtt{h}_T(G/P)^W \\ \downarrow\simeq & & \downarrow \\ \mathtt{h}_P(k) & \xrightarrow{f^*} & \mathtt{h}_T(W/W_P)^W \end{array} \qquad (*)$$

Recall that $\mathtt{h}_T(W/W_P) = \mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ and by definition of W-action on this set we have

$$\mathrm{Maps}(W/W_P, \mathtt{h}_T(k))^W = \mathrm{Maps}_W(W/W_P, \mathtt{h}_T(k)) = \mathtt{h}_T(k)^{W_P}.$$

By the construction of f we see that the map $f^*\colon \mathtt{h}_P(k) \to \mathrm{Maps}(W/W_P, \mathtt{h}_T(k))$ is given by $x \mapsto f_x$, $f_x(w) = w \cdot \pi^*(x)$ where $\pi^*\colon \mathtt{h}_P(k) \to \mathtt{h}_T(k)$ is the restriction map. Thus, via the identification

$$f^*\colon \mathtt{h}_P(k) \to Maps(W/W_P, \mathtt{h}_T(k))^W = \mathtt{h}_T(k)^{W_P}$$

the map f^* is given by the usual restriction map $\mathtt{h}_P(k) \to \mathtt{h}_T(k)^{W_P}$ which is surjective since P is $\mathtt{h}$-special. The fixed-point pullback $\mathtt{h}_T(G/P) \to \mathtt{h}_T(W/W_P)$ is injective by [10, Theorem 8.11]. Thus in the diagram $(*)$ the rightmost arrow is injective and the bottom arrow is surjective, then the rightmost arrow is surjective as well. □

For any $\rho \in \mathtt{h}_G(X \times X)$, set $\mathtt{h}_G(X, \rho) = \rho(\mathtt{h}_G(X))$ to be the image of the realization map of ρ.

Lemma 3.6. *Suppose X is a G-variety such that $\mathtt{h}_G(X) \to \mathtt{h}_T(X)^W$ is surjective. Then for any idempotent $\rho \in \mathtt{h}_G(X \times X)$ the map*

$$\mathtt{h}_G(X, \rho) \to \mathtt{h}_T(X, \bar\rho)^W \text{ is surjective}$$

where $\bar\rho$ is the image of ρ in $\mathtt{h}_T(X \times X)$.

Proof. We have a direct sum decomposition $\mathtt{h}_G(X) = \mathtt{h}_G(X, \rho) \oplus \mathtt{h}_G(X, id - \rho)$. Let $\bar\rho$ denote the image of ρ in $\mathtt{h}_T(X \times X)$. Then the decomposition $\mathtt{h}_T(X) = \mathtt{h}_T(X, \bar\rho) \oplus \mathtt{h}_T(X, id - \bar\rho)$ is W-equivariant. So the surjection $\mathtt{h}_G(X) \to \mathtt{h}_T(X)^W$ for $X = G/P$ is given by a diagonal matrix:

$$\mathtt{h}_G(X,\rho)\oplus\mathtt{h}_G(X,id-\rho) \to (\mathtt{h}_T(X,\bar\rho)\oplus\mathtt{h}_T(X,id-\bar\rho))^W = \mathtt{h}_T(X,\bar\rho)^W\oplus\mathtt{h}_T(X,id-\bar\rho)^W$$

and each of the maps $\mathtt{h}_G(X, \rho) \to \mathtt{h}_T(X, \bar\rho)^W$ and $\mathtt{h}_G(X, id - \rho) \to \mathtt{h}_T(X, id - \bar\rho)^W$ is surjective. □

Definition 3.7. We say that two parabolic subgroups P and P' are $\mathtt{h}$-degenerate with respect to each other if for any $w \in W_P\backslash W/W_{P'}$ the parabolic subgroup $P_w = R_uP \cdot (P \cap {}^wP')$ is $\mathtt{h}$-special. We say that a family of parabolic subgroups is $\mathtt{h}$-degenerate, if any two of them are $\mathtt{h}$-degenerate with respect to each other. Observe that P is $\mathtt{h}$-degenerate with respect to itself ($P = P'$) implies that P is $\mathtt{h}$-special.

Example 3.8. Any family of special parabolic subgroups is $CH(-;\mathbf{R})$-degenerate.

Consider a full additive subcategory $\mathcal{M}ot_{sp}$ (resp. its subcategory $\mathcal{M}ot_{dg}$) of G/T-Mot_k generated by relative equivariant motives of projective homogeneous G-varieties G/P where P runs through all $\mathtt{h}$-special parabolic subgroups of G (resp. through a given $\mathtt{h}$-degenerate family of parabolic subgroups). Observe that the difference between $\mathcal{M}ot_{sp}$ and $\mathcal{M}ot_{dg}$ is that by [12] the subcategory $\mathcal{M}ot_{dg}$ is closed under the tensor product.

Combining Lemma 3.5 and Lemma 3.6 we obtain

Corollary 3.9. *For any motive M in $\mathcal{M}ot_{sp}$ and, hence, in $\mathcal{M}ot_{dg}$ we have*

$$\mathtt{h}_G(M) \longrightarrow \mathtt{h}_T(M)^W \text{ is surjective.}$$

4. Motives vs. modules

Let $\mathcal{M}od_{par}$ (resp. $\mathcal{M}od_{dg}$) denote a full additive subcategory of $G/B\text{-}Mod_{G/T}$ generated by left $\mathbf{D}_{G/T}^{G/B}=\mathtt{h}_T(G/B)$-modules $M_{G/T}^{G/B}(G/P)=\mathtt{h}_T(G/P)$ for all (resp. for a given $\mathtt{h}$-degenerate family of) parabolic subgroups. We claim that

Theorem 4.1. *The functor $\mathfrak{F}_{G/T}^{G/B}$ is faithful if restricted to $\mathcal{M}ot_{par}\to\mathcal{M}od_{par}$. Moreover, it induces an equivalence if restricted to $\mathcal{M}ot_{dg}\to\mathcal{M}od_{dg}$.*

Proof. Let $X=G/P$ and $Y=G/P'$ for some parabolic subgroups P and P' of G. By definition, $M_{G/T}^{G/B}(X)=Mor_{G/T\text{-}Mot_k}([X],[G/B])=$

$$=\operatorname{im}\big(res_{G/T}\colon \mathtt{h}_G(X\times_k G/B)\longrightarrow \mathtt{h}_T(X\times_k G/B)\big).$$

Since $X\times_k G/B$ is a G-equivariant cellular space over G/B via the filtration introduced in [12], the map $res_{G/T}$ is injective. Indeed, it is a map of free modules induced by the injective map on coefficients $\mathtt{h}_G(G/B)\to\mathtt{h}_T(G/B)$. Hence, $M_{G/T}^{G/B}(X)=\mathtt{h}_G(X\times_k G/B)$. Therefore, we have

$$Hom_{\mathbf{D}_{G/T}^{G/B}}(M_{G/T}^{G/B}(Y),M_{G/T}^{G/B}(X))=Hom_{(\mathtt{h}_T(G/B),\circ)}\big(\mathtt{h}_T(Y),\mathtt{h}_T(X)\big)$$

which is a W-invariant submodule of

$$Hom_{\mathbf{D}_{T/T}^{pt}}(M_{T/T}^{pt}(Y),M_{T/T}^{pt}(X))=Hom_{\mathtt{h}_T(pt)}\big(\mathtt{h}_T(X),\mathtt{h}_T(Y)\big).$$

Consider a commutative diagram of induced maps

$$\begin{array}{ccc} Mor_{G/T\text{-}Mot_k}([X],[Y]) & \xrightarrow{f_{G/T}^{G/B}} & Hom_{\mathbf{D}_{G/T}^{G/B}}(M_{G/T}^{G/B}(Y),M_{G/T}^{G/B}(X)) \\ \big\downarrow & & \big\downarrow \\ Mor_{T/T\text{-}Mot_k}([X],[Y]) & \xrightarrow{f_{T/T}^{pt}} & Hom_{\mathbf{D}_{T/T}^{pt}}(M_{T/T}^{pt}(Y),M_{T/T}^{pt}(X)) \end{array} \tag{1}$$

Observe that the leftmost arrow is injective by definition. Since $\mathcal{M}ot_{par}$ is generated by motives of T-equivariant cellular spaces, by Lemma 2.2 the realization map $f_{T/T}^{pt}$ restricted to $\mathcal{M}ot_{par}$ is an isomorphism by the Künneth theorem. Therefore, the map $f_{G/T}^{G/B}$ is injective and the functor is faithful.

To prove the equivalence, observe that by Corollary 3.9 $Mor_{G/T\text{-}Mot_k}([X],[Y])$ can be identified with W-invariants $Mor_{T/T\text{-}Mot_k}([X],[Y])^W$ for all $[X],[Y]\in\mathcal{M}ot_{dg}$. Hence, the map $f_{G/T}^{G/B}$ is a restriction to W-invariants of the isomorphism $f_{T/T}^{pt}$, so it is an isomorphism. □

We now identify the category $\mathcal{M}od_{dg}$ with the category of certain modules over a Hecke-type algebra. We follow notation of [8], [9] and [10]. Let $\mathbf{R}=\mathtt{h}(pt)$ and $\mathbf{S}=\mathtt{h}_T(pt)$ be coefficient rings and let $\mathbf{S}_W=\mathbf{S}\#\mathbf{R}[W]$ be a twisted group algebra of the Weyl group W. By [7] $\mathbf{S}$ can be identified with the formal group algebra $\mathbf{R}[[T^*]]_F$ corresponding to the formal group law F of the theory $\mathtt{h}$. Let $\mathbf{Q}$ be the localization of $\mathbf{S}=\mathbf{R}[[T^*]]_F$ at all variables x_α corresponding to roots and let $\mathbf{Q}_W=\mathbf{Q}\#\mathbf{R}[W]$ denote the respective localized twisted group algebra. The subalgebra of $\mathbf{Q}_W$ generated by the Demazure elements $X_\alpha=\frac{1}{x_\alpha}-\frac{1}{x_\alpha}\delta_\alpha$ and multiplications by $\mathbf{S}$ is called the *formal affine Demazure algebra* and is denoted by $\mathbf{D}_F$. We define $\mathbf{D}_{F,P}$ to be the image of $\mathbf{D}_F$ under $p\colon\mathbf{Q}_W\to\mathbf{Q}_{W/W_P}$, where $\mathbf{Q}_{W/W_P}$ is a free $\mathbf{Q}$-module on the basis given by cosets W/W_P.

The main result of [10] says that the cohomology ring

$$\mathtt{h}_T(G/P) = \mathtt{h}_G(G/T \times_k G/P)$$

can be identified with the dual algebra $\mathbf{D}^\star_{F,P} = Hom_{\mathbf{S}}(\mathbf{D}_{F,P}, \mathbf{S})$. Moreover, by [36, §5] the convolution algebra $h_G(G/B \times_k G/B)$ can be identified with the formal affine Demazure algebra $\mathbf{D}_F$.

Consider the $\odot$-action of $\mathbf{Q}_W$ on $\mathbf{Q}^*_W = Hom_{\mathbf{Q}}(\mathbf{Q}_W, \mathbf{Q})$ introduced in [30] as

$$q\delta_w \odot pf_v = qw(p)f_{wv}.$$

By [30] this action restricts to the action of $\mathbf{D}_F$ on $\mathbf{D}^\star_{F,P}$ and via the mentioned identifications it coincides with an action of the convolution algebra $\mathbf{D}^{G/B}_{G/T} = \mathtt{h}_G(G/B\times_k G/B)$ on $M^{G/B}_{G/T}(G/P) = h_G(G/P \times_k G/B)$. Combining these identifications, we obtain the following

Theorem 4.2. *The category $\mathcal{M}ot_{dg}$ is equivalent to a subcategory generated by $\mathbf{D}_F$-modules $\mathbf{D}^\star_{F,P}$ for a given $\mathtt{h}$-degenerate family of parabolic subgroups.*

As an immediate consequence of Theorems 4.1 and 4.2 we obtain the following key

Corollary 4.3. *For any parabolic subgroup P there is an inclusion*

$$End_{G/T\text{-}Mot_k}([G/P]) \hookrightarrow End_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P}).$$

Moreover, if P is $\mathtt{h}$-degenerate with respect to itself, i.e., P_w is $\mathtt{h}$-special for all $w \in W_P\backslash W/W_P$, then the inclusion is an isomorphism.

Remark 4.4. It follows from the corollary, from the definition of $\mathbf{D}^\star_{F,P}$, of the $\odot$-action and of the $\bullet$-action of [9] that for a special parabolic subgroup P

$$\mathtt{h}_G(G/P \times G/P) \simeq \mathtt{h}_P(G/P) \simeq {}^{W_P}(\mathbf{D}^\star_F)^{W_P}$$

where the left W_P acts via '$\odot$' and the right W_P acts via '$\bullet$'.

5. Nilpotency for equivariant oriented theories

In the present section we extend the Nilpotency Theorem of [46] to algebraic cobordism Ω of Levine-Morel and apply it to identify the direct sum decompositions of $[G/P]$ in G/G-Mot_k and in G/T-Mot_k. We follow closely arguments and notation of [46].

Let $\mathbf{Sch}_k$ denote the category of reduced schemes of finite type over a field and let $\mathbf{Sm}_k$ denote its subcategory of smooth schemes over k. Let $\Omega(-)$ denote the algebraic cobordism functor of Levine-Morel [33].

Lemma 5.1. *Let $X \in \mathbf{Sch}_k$ and let $E \to X$ be a rank d vector bundle with a zero section $z\colon X \to E$. Then the following diagram commutes*

$$\begin{array}{ccc} \Omega_*(\mathbb{P}(E\oplus 1)) & \xrightarrow{\widetilde{c}_d(q^*E\otimes\mathcal{O}(1))\cap -} & \Omega_{*-d}(\mathbb{P}(E\oplus 1)) \\ \downarrow & & \downarrow{\scriptstyle q_*} \\ \Omega_*(E) & \xrightarrow{\quad z^* \quad} & \Omega_{*-d}(X) \end{array}$$

Proof. There is a global section of the sheaf $q^*E \otimes \mathcal{O}(1) = \underline{Hom}(\mathcal{O}(-1), q^*E)$, given by an element $s \in Hom(\mathcal{O}(-1), q^*E)$ that is the composition of the natural embedding and projection

$$\mathcal{O}(-1) \to q^*(E \oplus 1) \to q^*E.$$

The zero set $Z(s)$ consists of points of $\mathbb{P}(E \oplus 1)$ that correspond to additional lines 1 in $E_x \oplus 1$, over every point of $x \in X$. So X is the zero subscheme of s with regular embedding $\bar{s}\colon X \to \mathbb{P}(E \oplus 1)$ given by

$$X \xrightarrow{z,1} (E \oplus 1) \setminus (Z(E), 0) \to \mathbb{P}(E \oplus 1).$$

By [33, Lemma 6.6.7], the operator $\widetilde{c}_d(q^*E \otimes \mathcal{O}(1)) \cap -$ on $\Omega(\mathbb{P}(E \oplus 1))$ is given by $\bar{s}_*\bar{s}^*$. Then the right-down pass is given by

$$q_*\bar{s}_*\bar{s}^* = \bar{s}^*\colon \Omega_*(\mathbb{P}(E \oplus 1)) \to \Omega_*(E) \xrightarrow{z^*} \Omega_{*-d}(X).$$ □

Lemma 5.2. *Let $X \in \mathbf{Sch}_k$ and $E \to X$ be a vector bundle of rank d. Then for any point $x \in X$ there is an open subscheme U of X with $x \in U$, and a projective morphism $p\colon X' \to X$ such that*

- *p^*E has a filtration by subbundles with linear subsequent quotients.*
- *There is an open subset U' such that $p\colon U' \to U$ is an isomorphism.*

Proof. Consider the projection $p\colon Fl(E) \to X$ where $Fl(E)$ is the variety of complete flags of the vector bundle E over X. Then p^*E has a filtration by tautological sub-bundles. For any $x \in X$ there exists a neighborhood U of x such that the bundle E trivializes, so $Fl(E)|_U \simeq Fl \times U$ and there is a section $s\colon U \to p^{-1}(U)$. Take X' to be the closure of $s(U)$ in $Fl(E)$. Restriction $p\colon X' \to X$ satisfies the desired properties. □

Lemma 5.3. *Let $X, Y \in \mathbf{Sch}_k$ and $p\colon X \to Y$ be a projective birational morphism. Then $p_*\colon \Omega_*(X) \to \Omega_*(Y)$ is surjective.*

Proof. First, consider the case when X and Y are smooth. Then for any $\alpha \in \Omega_*(Y)$ we have $p_*(p^*\alpha) = \alpha \cdot p_*(1_X)$ and by the degree formula $p_*(1_X) = 1_Y + a$ where $a \in \mathbb{L}\cdot\Omega^{>0}(Y)$, hence a is nilpotent in the ring $\Omega^*(Y)$, therefore $p_*(1_X)$ is invertible.

Now consider the general case $p\colon X \to Y$ with $X, Y \in \mathbf{Sch}_k$. Take an element $\beta \in \Omega_*(Y)$. Since algebraic cobordism is detected by smooth schemes by [33, Lemma 2.4.15], there is a smooth scheme Y' and projective morphism $q\colon Y' \to Y$ and an element $\beta' \in \Omega_*(Y')$ such that $q_*(\beta') = \beta$. Let $X' = X \times_Y Y'$. Then the morphism $P\colon X' \to Y$ is projective birational. Take X'' to be a resolution of singularities of X':

$$\begin{array}{ccccc} X'' & \xrightarrow{F} & X' & \xrightarrow{P} & Y' \\ & & \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle q} \\ & & X & \xrightarrow{p} & Y \end{array}$$

Then $F\colon X'' \to X$ is projective birational. Thus $X'', Y' \in \mathbf{Sm}_k$ and the map $P \circ F$ is projective birational. Then by the first case there is $\alpha' \in \Omega_*(X'')$ such that $P_*F_* = \beta'$. Then $\beta = q_*\beta' = q_*P_*F_*(\alpha') = p_*Q_*F_*$ where $Q\colon X' \to X$ is the projection. □

Definition 5.4. Let $X \in \mathbf{Sch}_k$ and Z be a closed subset of X. We say that an element $\alpha \in \Omega_*(X)$ is supported on Z if α lies in the image $\Omega_*(Z) \to \Omega_*(X)$.

Lemma 5.5. *Let $X, X' \in \mathbf{Sch}_k$, $f\colon X' \to X$ be a smooth morphism of schemes, E be a vector bundle of rank d on X and $E' = f^*E$ and Z_i, $i = 1 \dots m$ be irreducible closed subsets of X and $Z_i' = f^{-1}(Z_i)$.*

Then there are closed subsets $\widetilde{Z}_i \to Z_i$ of codimension d such that for any $\alpha \in \Omega_*(X')$ supported on Z_i' the element $\widetilde{c}_d(E') \cap \alpha$ is supported on $f^{-1}(\widetilde{Z}_i)$.

Proof. First, consider the case when E is a line bundle. We have $E \cong E_1 \otimes E_2^\vee$ for some very ample line bundles E_1, E_2. Then by [33, Lemma 2.3.10] we have $\widetilde{c}_1(E) = \widetilde{c}_1(E_1) -_F \widetilde{c}_1(E_2)$. By [33, Lemma 6.6.7] the operator $\widetilde{c}_1(E_1') \cap -$ is given by s_*s^*, where $s\colon f^{-1}(D_1) \to X'$ and D_1 in X is the divisor of the very ample line bundle E_1. Thus, $\widetilde{c}_1(E_1') \cap \alpha$ is supported on $f^{-1}(Z_i \cap D_1)$. Similarly, $\widetilde{c}_1(E_2') \cap \alpha$ is supported on $f^{-1}(Z_i \cap D_2)$, where D_2 is a divisor of E_2. Since E_1, E_2 are very ample, we may choose D_1 and D_2 to intersect each Z_i by codimension 1, thus $\widetilde{c}_1(E_1') -_F \widetilde{c}_1(E_2') \cap (\alpha)$ is supported on some $f^{-1}(\widetilde{Z}_i)$ of codimension 1 for each i.

Consider the general case. For each $i = 1..m$ there is a projective map $p_i\colon Y_i \to X$ given by Lemma 5.2 such that an open subset of Y_i is isomorphic to some open neighbourhood of the generic point of Z_i. Let $W_i = \overline{(p_i^{-1}(U \cap Z_i))}$ be the proper transform of Z_i. Then $W_i \to Z_i$ is projective birational. Let $p_i'\colon Y_i' \to X'$ denote the pullback of p_i along f and $W_i' = f^{-1}(W_i)$. By Lemma 5.3 for any $\alpha \in \Omega_*(X')$ supported on Z_i' we may found a preimage $\alpha' \in \Omega_*(Y_i')$ supported on W_i'. Now by Whitney formula [33, Def. 1.1.2] we have $\widetilde{c}_d((p_i')^*E') \cap - = \prod_{j=1}^d \widetilde{c}_1(E_j')$, where E_j are linear subsequent quotients of Lemma 5.2. Now, applying inductively the case $d = 1$ for α' we can find the subset $\widetilde{W}_i$ of codimension d in W_i, such that $\widetilde{c}_d((p_i')^*E') \cap \alpha' = (\prod_{j=1}^d \widetilde{c}_1(E_j)) \cap \alpha'$ is supported on $f^{-1}(\widetilde{W}_i)$, then its pushforward is equal to $\widetilde{c}_d(E') \cap \alpha$ and is supported on $f^{-1}(\widetilde{Z}_i) = f^{-1}(p(\widetilde{W}_i))$. □

Lemma 5.6. *(cf. [46, Lemma 6.3]) Let $V \to B \leftarrow T$ be closed embeddings with regular f and smooth quasi-projective B. Let $\varepsilon\colon W \to B$ be a smooth morphism. Consider two Cartesian diagrams:*

$$
\begin{array}{ccccc}
W_V & \xrightarrow{f_W} & W & \xleftarrow{g_W} & W_T \\
\downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow \\
V & \xrightarrow{f} & B & \xleftarrow{g} & T
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
T & \xrightarrow{g} & B \\
{\scriptstyle \tilde{f}}\uparrow & & \uparrow{\scriptstyle f} \\
\tilde{T} & \xrightarrow{\tilde{g}} & V
\end{array}
$$

Then there exists a closed embedding $h\colon Z \to V$ such that $\operatorname{codim} h \geqslant \operatorname{codim} g$ and $\operatorname{im}(f_W^* \circ g_{W*}) \subseteq \operatorname{im}(h_{W*})$ inside $\Omega_*(V)$.

Proof. Consider the Cartesian square

$$
\begin{array}{ccc}
W_T & \xrightarrow{g_W} & W \\
{\scriptstyle \widetilde{f}_W}\uparrow & & \uparrow{\scriptstyle f_W} \\
W_{\widetilde{T}} & \xrightarrow{\widetilde{g}_W} & W_V
\end{array}
$$

By [33, Proposition 6.6.3] $f_W^* \circ g_{W*} = \widetilde{g}_{W*} \circ f_W^!$ where the refined pullback $f_W^!$ is given by the composition [33, 6.6.2]

$$\Omega_*(W_T) \to \Omega_*(C_W) \to \Omega_*(N_W) \to \Omega_{*-d}(W_{\widetilde{T}})$$

where C_W is the normal cone of $\widetilde{f}_W$ and N_W is the the normal bundle pullback $\widetilde{g}_W^*(N_{f_W})$ and d is the codimension of f.

Let N be the pullback of the normal bundle $\widetilde{g}^*(N_f)$ and $q\colon \mathbb{P}(N \oplus 1) \to \widetilde{T}$ be the projection and E denote the vector bundle $q^*N \otimes \mathcal{O}(1)$. Consider the closed subscheme $\mathbb{P}(C \oplus 1)$ inside $\mathbb{P}(N \oplus 1)$ where C is the normal cone of the map $\widetilde{f}\colon \widetilde{T} \to T$. Applying Lemma 5.5 to the vector bundle E and irreducble components of $\mathbb{P}(C \oplus 1)$ we get a closed subset Z' of codimension at least d in $\mathbb{P}(C \oplus 1)$ such that for every cobordism class x supported on $\mathbb{P}(C_W \oplus 1)$ the class $x \cap \widetilde{c}_d(\varepsilon^* E)$ is supported on $\varepsilon^{-1}(Z')$. Thus in view of Lemma 5.1 the image of the composition $\Omega_*(C_W) \to \Omega_*(N_W) \to \Omega_{*-d}(W_{\widetilde{T}})$ is supported on $\varepsilon^{-1}(Z)$ where $Z = q(Z')$. So $h\colon Z \to V$ is the desired embedding. □

Let $\mathtt{h}$ be an oriented cohomology theory that is generically constant and satisfies the localization property, so the canonical map from algebraic cobordism $\Omega(X) \otimes_{\mathbb{L}} \mathtt{h}(k) \to \mathtt{h}(X)$ is surjective for every $X \in \mathbf{Sm}_k$.

Lemma 5.7. *Let $\pi\colon Y \to X$ be a smooth morphism and let X be a smooth quasi-projective variety. Let $i_1\colon Z_1 \hookrightarrow X$ and $i_2\colon Z_2 \hookrightarrow X$ be closed embeddings.*

Then there exists a closed embedding $i_3\colon Z_3 \hookrightarrow X$ such that

$$\operatorname{codim} Z_3 \geqslant \operatorname{codim} Z_1 + \operatorname{codim} Z_2 \text{ and } \operatorname{im}(i_1')_* \cdot \operatorname{im}(i_2')_* \subseteq im(i_3')_* \text{ in } \mathtt{h}(Y),$$

where $i'\colon Y_{Z_j} \to Y$, $j = 1, 2, 3$ is obtained from the respective Cartesian square.

Proof. The diagonal embedding $Y \to Y \times Y$ factors as $Y \xrightarrow{\phi} Y \times_X Y \xrightarrow{f_W} Y \times Y$. By Lemma 5.6 applied to $B = X \times X$, $V = X$, $f\colon \Delta_X$, $T = Z_1 \times Z_2$ and $W = Y \times Y$ we obtain a closed embedding $h\colon Z \to X$ such that

$$\operatorname{codim} Z \geqslant \operatorname{codim} Z_1 + \operatorname{codim} Z_2 \text{ and } \operatorname{im}(f_W^* \circ (i_1' \times i_2')_*) \subseteq \operatorname{im}(h_W)_*.$$

There is the Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\ \phi\ } & Y \times_X Y \\ \uparrow{\scriptstyle h'} & & \uparrow{\scriptstyle h_W} \\ Y_Z & \xrightarrow{\ \phi_Z\ } & (Y \times_X Y)_Z \end{array}$$

According to [33, Proposition 6.6.3] we have $\phi^* \circ h_{W*} = h'_* \circ \phi_Z^!$. So for Ω we have

$$\operatorname{im}(i_1')_* \cdot \operatorname{im}(i_2')_* \subseteq \operatorname{im} \Delta_Y^* \circ (i_1' \times i_s')_* \subseteq \operatorname{im}(h'_*).$$

Finally, the natural map $\Omega_*(-) \otimes_{\mathbb{L}} \mathtt{h}(k) \to \mathtt{h}(-)$ is surjective and compatible with push-frowards and intersection product, so we can replace Ω by $\mathtt{h}$. □

Let E be a versal G-torsor over k, and let $E \subseteq V$ be the ambient G-representation. Consider the twisted flag E/P and the composite of P^2-equivariant maps

$$E \times E \xrightarrow{\simeq} E \times G \to V \times G \to G,$$

where action of P^2 on the $E \times E$ is given by $(e_1, e_2) \cdot (p_1, p_2) = (e_1 p_1, e_2 p_2)$ and on G is given by $g \cdot (p_1, p_2) = p_1^{-1} g p_2$. Its induced pullback gives a surjective map

$$\gamma\colon \mathtt{h}_G(G/P \times G/P) = \mathtt{h}_{P^2}(G) \longrightarrow \mathtt{h}_{P^2}(E \times E) = \mathtt{h}(E/P \times E/P).$$

Proposition 5.8. *The map*

$$\gamma\colon End_{G/G\text{-}Mot_k}([G/P]) \longrightarrow End_{Mot_k}([E/P]).$$

lifts idempotents and isomorphisms strictly. Hence, the direct sum decompositions of the usual h-motive $[E/P]$ are in 1-1 correspondence with the direct sum decompositions of $[G/P]$ in the equivariant motivic category $G/G\text{-}Mot_k$.

Proof. Consider a sequence U_i from the Borel construction for a G-equivariant theory $\mathtt{h}_G$. Then $\mathtt{h}^2_P(G) = \lim \mathtt{h}((U_i^2 \times G)/P^2)$ and the morphism γ is a limit of pullbacks

$$\gamma_i\colon \mathtt{h}((U_i^2{\times}G)/P^2 \overset{\simeq}{\to} \mathtt{h}((U_i^2{\times}V{\times}G)/P^2) \to \mathtt{h}((U_i^2{\times}E{\times}G)/P^2) \overset{\simeq}{\to} \mathtt{h}((U_i^2{\times}E{\times}E)/P^2)$$

By the localization sequence each element in the kernel of γ_i lies in the image of $\mathtt{h}((U_i^2 \times Z \times G)/P^2)$, where Z is a closed complement of E in V.

Consider the maps

$$p_{j,j'}\colon (G^n \times U_i^{n+1})/P^{n+1} \longrightarrow (G \times U_i^2)/P^2$$

which descend from the standard projections $p_{j,j'}\colon U_i^{n+1} \to U_i^2$, where the maps $\pi_{j,j'}\colon G^n \to G$ are defined on points by

$$\pi_{j,j'}(g_1,\ldots,g_n) = g_j^{j-1} g_{j'-1} \text{ for } 1 \leqslant j \leqslant n,\ 2 \leqslant j \leqslant n+1 \text{ and } g_0 = 1.$$

Then the n-fold convolution product on $\mathtt{h}((U_i^2 \times G)/P^2)$ is given by

$$x_1 * \ldots * x_n = (p_{1,n})_*(\prod_{j=1}^{n} p^*_{j,j+1}(x_i)).$$

For every j there is the Cartesian square

$$\begin{array}{ccc} (Z \times_k G^n \times_k U_i^{n+1})/P^{n+1} & \xrightarrow{j} & (V \times_k G^n \times_k U_i^{n+1})/P^{n+1} \\ \downarrow & & \downarrow{\scriptstyle p_{j,j+1}} \\ (Z \times_k G \times_k U_i^2)/P^2 & \longrightarrow & (V \times_k G \times_k U_i^2)/P^2 \end{array}$$

Thus, for every $x \in \mathtt{h}(V \times_k G \times_k U_i^2)/P^2)$ supported on $(Z \times_k G \times_k U_i^2)/P^2$ the element $p^*_{j,j+1}(x)$ is supported on $(Z \times_k G^n \times_k U_i^{n+1})/P^{n+1}$. Then by Lemma 5.7 applied to

$$f = p_{12}\colon (V \times_k G^n \times_k U_i^{n+1})/P^{n+1} \to (V \times_k G \times_k U_i^2)/P^2$$

we get that $\prod_{i=1}^n \operatorname{im}(j_*) = 0$, hence, $x^{*n} = 0$ for $n > \dim(V)/\operatorname{codim}(Z)$. Thus γ is a limit of surjective maps with nilpotent kernels. Moreover, the inclusion $U_i \to U_{i+1}$ induce a surjective homomorphism $\ker(\gamma_{i+1}) \to \ker(\gamma_i)$ by [35, Proposition 6.2.1], thus γ lifts idempotents and isomorphisms strictly by [35, Lemma 4.3.4]. □

Corollary 5.9. *There is a 1-1 correspondence between direct sum decompositions of $[G/P]$ in $G/G\text{-}Mot_k$ and in $G/T\text{-}Mot_k$.*

Proof. In the commutative diagram

$$\begin{array}{ccc} \mathtt{h}_G(G/P \times_k G/P) & \xrightarrow{\gamma} & \mathtt{h}(E/P \times_k E/P) \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathtt{h}_T(G/P \times_k G/P) & \longrightarrow & \mathtt{h}(G/P \times_k G/P) \end{array}$$

the map ϕ has nilpotent kernel by the Rost Nilpotence theorem for $\mathtt{h}$ (here one uses the Rost Nilpotence for Chow groups [11] and extends it to an arbitrary $\mathtt{h}$ using the Vishik-Yagita [45] correspondence between Ω- and Chow motives). Hence, the restriction *res* is a limit of surjective maps with nilpotent kernels by the lemma. □

Remark 5.10. It would be interesting to have a direct proof (without using RN of [11]) of the fact that the map *res* has nilpotent kernel.

6. The endomorphisms of $\mathbf{D}^\star_{F,P}$

In the present section we study idempotents in the endomorphism ring $End_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ in the Chow group case.

By definition (following [9]) $\mathbf{Q}_{W/W_P}$ is the free $\mathbf{Q}$-module spanned by $\delta_{\bar{w}}$, where $\bar{w}$ is the class of w in W/W_P. By [9, Lemma 11.2] $p(zX_\alpha)=0$ for any z if $s_\alpha\in W_P$. By [9, Lemma 11.3] if $\{I_w\}$ is W_P-compatible, then $p(X_{I_w})=0$, if w is not in W^P (the set of minimal coset representatives) and, moreover, $p(X_{I_W})$, $w\in W^P$ form an $\mathbf{S}$-basis of $\mathbf{D}_{F,P}$.

The map p induces an injection $p^*\colon \mathbf{D}^\star_{F,P}\hookrightarrow \mathbf{D}^\star_F$ with the image being the subring of W_P-invariants (with respect to the Hecke action $\bullet$). By [9, Thm 14.3] there is an $\mathbf{S}$-basis of $\mathbf{D}^\star_{F,P}$ given by the classes

$$\xi_w=\xi_{I_w}([pt])=Y_P\bullet(X_{I_w^{-1}}\bullet[pt]),\quad w\in W^P.$$

We now restrict to the case of Chow groups. By [30, Example 4.8] we have:

$$X_j\odot\xi_v=\begin{cases}\xi_{s_jv} & \text{if } l(s_jv)\ge l(v) \text{ and } s_jv\in W^P\\ 0 & \text{otherwise}\end{cases}$$

In other words, the action by X_i's corresponds to the weak Bruhat order on W^P. An endomorphism $\phi\in End_{\mathbf{S}}(\mathbf{D}^\star_{F,P})$ is uniquely determined by its values on basis elements ξ_w, $w\in W^P$. That is $\phi(\xi_w)=\sum_v a_{v,w}\xi_v$, where $(a_{w,v})$ is the respective matrix of coefficients from $\mathbf{S}$. Since $\mathbf{D}_F$ is generated by X_j's and elements of $\mathbf{S}$, ϕ is a homomorphism of $\mathbf{D}_F$-modules if it satisfies

$$X_j\odot\phi(\xi_w)=\phi(X_j\odot\xi_w) \text{ for all } j \text{ and } w\in W^P.$$

So we have for $v,w\in W^P$

$$\begin{aligned}X_j\odot\sum_v a_{v,w}\xi_v &= \sum_v (s_j(a_{v,w})X_j+\Delta_j(a_{v,w}))\odot\xi_v\\ &= \sum_{s_jv\in W^P,\ l(s_jv)\ge l(v)} s_j(a_{v,w})\xi_{s_jv}+\sum_v\Delta_j(a_{v,w})\xi_v\\ &= \sum_{v'\in W^P,\ l(s_jv')\le l(v')} s_j(a_{s_jv',w})\xi_{v'}+\sum_v\Delta_j(a_{v,w})\xi_v\end{aligned}$$

and

$$\phi(X_j\odot\xi_w)=\begin{cases}\sum_v a_{v,s_jw}\xi_v & \text{if } l(s_jw)\ge l(w) \text{ and } s_jw\in W^P,\\ 0 & \text{otherwise.}\end{cases}$$

Combining, we get the following recurrent formulas:

Lemma 6.1. *If $l(s_jw) \geq l(w)$ and $s_jw \in W^P$, then*

$$a_{v,s_jw} = \begin{cases} s_j(a_{s_jv,w}) + \Delta_j(a_{v,w}) & \text{if } l(s_jv) \leq l(v), \\ \Delta_j(a_{v,w}) & \text{otherwise.} \end{cases}$$

If $l(s_jv) \leq l(v)$, then (observe that $s_j\Delta_j = \Delta_j$ for Chow groups)

$$a_{s_jv,w} = \begin{cases} s_j(a_{v,s_jw}) - \Delta_j(a_{v,w}) & \text{if } l(s_jw) \geq l(w) \text{ and } s_jw \in W^P, \\ -\Delta_j(a_{v,w}) & \text{otherwise.} \end{cases}$$

Corollary 6.2. *The $\mathbf{D}_F$-module map ϕ is uniquely determined by its value on $\xi_1 = [pt]$, i.e. by coefficients $a_v = a_{v,1}$, $v \in W^P$, such that for all $s_j \in W_P$*

$$\Delta_j(a_v) = \begin{cases} -a_{s_jv} & \text{if } l(s_jv) \leq l(v) \\ 0 & \text{otherwise} \end{cases}$$

Proof. By the recurrent formulas it follows immediately that ϕ is uniquely determined by its value on ξ_1. Comparing $X_j \odot \phi(\xi_1)$ and $\phi(X_j \odot \xi_1)$ we obtain the desired expression for $\Delta_j(a_v)$. □

Assume that an endomorphism ϕ has degree 0, i.e., preserves degrees. Then each $a_{v,w}$ is a polynomial of degree $l(v) - l(w)$. So the matrix $(a_{v,w})$ is lower triangular.

Example 6.3. We have the following formulas for coefficients of lower degree:

Degree 0: $a_1 = a_{1,1} \in \mathbf{R}$. Let $s_k, s_j \in W^P$, i.e., $s_k, s_j \notin W_P$. Then (setting $v = s_k$ and $w = 1$)

$$a_{s_k,s_j} = \begin{cases} a_1 + \Delta_j(a_{s_k}) & \text{if } s_j = s_k \\ \Delta_j(a_{s_k}) & \text{if } s_j \neq s_k. \end{cases}$$

Let $s_is_j, s_ks_l \in W^P$, $s_i \neq s_j$ and $s_k \neq s_l$ (in particular, $s_j, s_l \notin W_P$). Then

$$a_{s_is_j,s_ks_l} = \begin{cases} s_k(a_{s_j,s_l}) + \Delta_k(a_{s_ks_j,s_l}) & \text{if } s_k = s_i \\ s_k(a_{s_i,s_l}) + \Delta_k(a_{s_is_j,s_l}) & \text{if } s_k = s_j \text{ and } s_is_j = s_js_i \\ \Delta_k(a_{s_is_j,s_l}) & \text{otherwise} \end{cases}$$

Degree 1: Let $s_ks_l, s_j \in W^P$, $k \neq l$, i.e., $s_l, s_j \notin W_P$ and $s_ks_l \neq s_ls_k$ if $s_k \in W_P$. Then (setting $v = s_ks_l$ and $w = 1$)

$$a_{s_ks_l,s_j} = \begin{cases} s_j(a_{s_js_ks_l,s_j}) + \Delta_j(a_{s_ks_l}) & \text{if } l(s_js_ks_l) = 1 \\ \Delta_j(a_{s_ks_l}) & \text{otherwise.} \end{cases}$$

Assume that ϕ is an idempotent (in particular, it has degree 0) so that $(a_{v,w})$ is an idempotent matrix. By definition, it satisfies $\phi(\phi([pt])) = \phi([pt])$ which leads to

$$\sum_v a_v\xi_v = \phi(\sum_u a_u\xi_u) = \sum_u a_u\phi(\xi_u) = \sum_{u,v} a_ua_{v,u}\xi_v$$

that is

$$\sum_{u,\ l(u)\leq l(v)} a_ua_{v,u} = a_v, \quad v, u \in W^P.$$

Example 6.4. In degree 0 it gives $a_1^2 = a_1$. In degree 1 it gives for each $s_k \notin W_P$

$$2a_1a_{s_k} + \sum_{s_j\notin W_P} a_{s_j}\Delta_j(a_{s_k}) = a_{s_k}.$$

7. Applications to Chow motives

The purpose of the present section is to demonstrate how the techniques of Hecke-type (Demazure/divided-difference) operators can be applied to show irreducibility of certain $\mathbf{D}_F$-modules $\mathbf{D}^\star_{F,P}$ and, hence, of motives $[E/P]$. We restrict to the case of Chow groups only, i.e. $\mathbf{R}=\mathbb{Z}$ and F is an additive formal group law.

Projective spaces. Let $G=PGL_{n+1}$ be adjoint group of type A_n and let P be of type A_{n-1}, i.e., $G/P=\mathbb{P}^n$ is a projective space. By definition $W=\langle s_1,\ldots,s_n\rangle$ is the symmetric group on $n+1$ elements, $W_P=\langle s_2,s_3,\ldots,s_n\rangle$ is its subgroup and the set of minimal left coset representatives is $W^P=\{1,s_1,s_2s_1,\ldots,s_ns_{n-1}..s_1\}$. The algebra $\mathbf{D}_F$ is then the usual nil affine Hecke algebra over the polynomial ring $\mathbf{S}=\mathbb{Z}[\alpha_1,\ldots,\alpha_n]$ in simple roots. Consider an idempotent $\phi\in End^{(0)}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ and the associated matrix $(a_{v,w})$.

For simplicity, set $c_{i,j}:=a_{s_i..s_1,s_j..s_1}$, $c_{i,0}:=a_{s_i..s_1,1}$, $c_{0,j}=a_{1,s_j..s_1}$ for $i,j\geq 1$ and $c_{0,0}=a_{1,1}$. So $(c_{i,j})=(a_{v,w})$ is a lower-triangular idempotent $(n+1)\times(n+1)$-matrix with polynomial coefficients of degrees $\deg c_{i,j}=i-j$.

The recurrent formulas of Lemma 6.1 turn into:

$$c_{i,i}=s_i(c_{i-1,i-1})+\Delta_i(c_{i,i-1})\ \text{ if } i\geq 1,\quad c_{i,j}=\Delta_j(c_{i,j-1})\ \text{ if } i\neq j,\ j\geq 1,$$

$$\text{and}\quad c_{i,0}=(-1)^{n-i}\Delta_{i+1,..,n}(c_{n,0})\ \text{ for } n>i\geq 1.$$

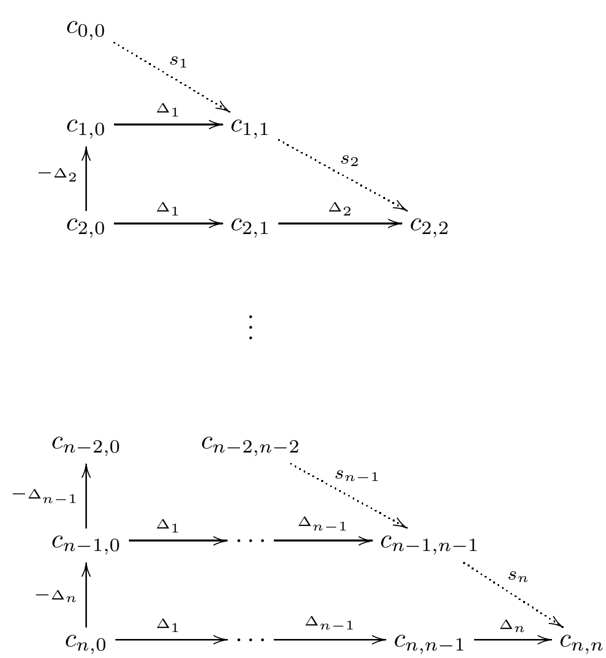
So if $i>j\geq 1$ (under the diagonal), we obtain

$$c_{i,j}=\Delta_{j,..,1}(c_{i,0})=(-1)^{n-i}\Delta_{j,..,1,i+1,..,n}(c_{n,0})$$

and on the diagonal we get

$$c_{i,i}=s_i(c_{i-1,i-1})+(-1)^{n-i}\Delta_{i,..,1,i+1,..,n}(c_{n,0}).$$

In other words, we have the following diagram of operators



Using the relations for the Demazure elements and the fact that $\Delta_j(c_{n,0}) = 0$ for all $j \neq 1, n$ (Corollary 6.2) we get

$$\Delta_k \circ \Delta_{i-1,..,1,i+1,..,n}(c_{n,0}) = 0, \qquad \text{for all } k \neq i.$$

So $c_{i,i-1} \in \mathbf{S}^{W_i}$, where W_i is generated by all simple reflections except the i-th one. Since $c_{i,i-1}$ has degree 1, we can express it as $c_{i,i-1} = b_1\alpha_1 + \ldots + b_n\alpha_n$, $b_i \in \mathbb{Z}$. Then $c_{i,i-1} \in \mathbf{S}^{W_i}$ is equivalent to

$$b_2 = 2b_1, b_3 = 3b_1, \ldots, b_i = ib_1 = (n+1-i)b_n, \ldots, b_{n-2} = 3b_n, b_{n-1} = 2b_n.$$

Assume $n+1 = p^r$ for some prime p and $r \geq 1$. Then

$$p \mid \Delta_i(c_{i,i-1}) = 2b_i - b_{i-1} - b_{i+1} = b_1 + b_n.$$

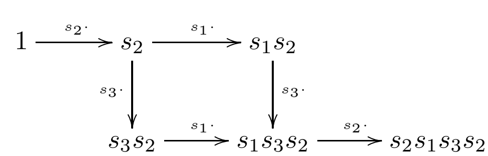
Since ϕ is an idempotent, all diagonal elements $c_{i,i}$ are idempotent as well, i.e., $c_{i,j} = 0, 1$. The recurrent formulas and the fact that $p \mid \Delta_i(c_{i,i-1})$ then imply that $c_{i,i} = c_{i-1,i-1}$ for all i, i.e. that there are no nontrivial idempotents. So we obtain

Proposition 7.1. *Let G be adjoint of type A_n where $n = p^r - 1$ for some r. Let P be the maximal parabolic subgroup generated by all simple reflections exept the very first one. Let $\mathbf{D}_F$ be a formal affine Demazure algebra corresponding to the additive formal group law F over $\mathbb{Z}$ (or $\mathbb{Z}/p\mathbb{Z}$) and to the root lattice of G.*

Then the $\mathbf{D}_F$-module $\mathbf{D}^\star_{F,P}$ is irreducible.

Remark 7.2. In view of Proposition 5.8, Corollaries 5.9 and 4.3, this fact is equivalent (and, indeed, provides a different proof) to the celebrated theorem by Karpenko on indecomposability of the motive of a Severi-Brauer variety of a generic algebra.

The Klein quadric. Let G be a group of type A_3 with $T^* = \langle \alpha_1, \alpha_2, \alpha_3, \omega_2 \rangle$ and let P be of type $A_1 \times A_1$, i.e., $G/P = Gr(2,4)$ is a 4-dimensional split smooth projective quadric. By definition, the Weyl group $W = \langle s_1, s_2, s_3 \rangle$, $W_P = \langle s_1, s_3 \rangle$ and the set of minimal coset representatives W^P is given by the Hasse diagram

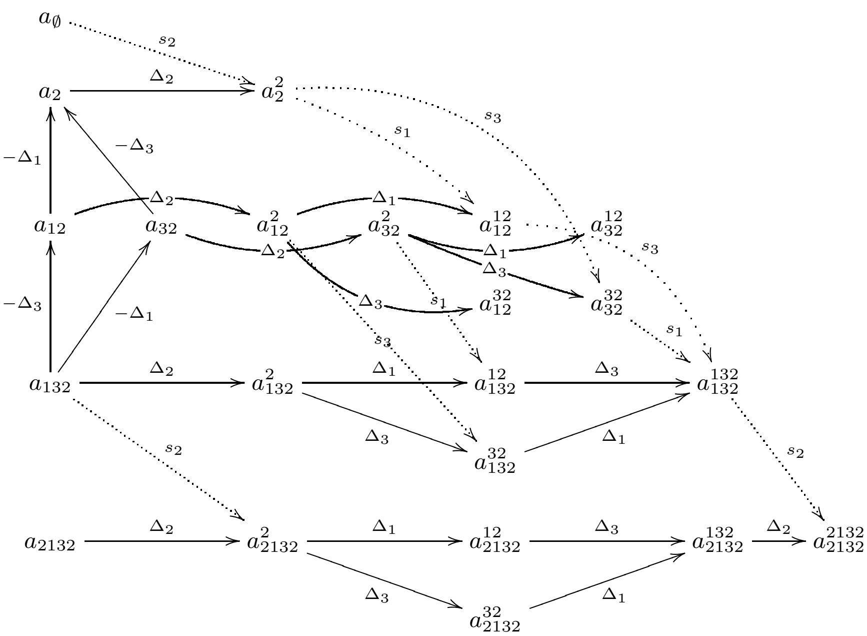


Consider an idempotent $\phi \in End^{(0)}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ and the associated matrix $(a_{v,w})$.

For simplicity, set $a_{ijk..} = a_{s_is_js_k..}$ and $a_\emptyset = a_1$. By the recurrent formulas of Lemma 6.1 and of Corollary 6.2 we obtain:

$$\begin{aligned}
a_{2,2} &= a_\emptyset + \Delta_{2,1,3}(a_{132}),\\
a_{12,12} &= a_{2,2} - \Delta_{1,2,3}(a_{132}) \text{ and } a_{32,32} = a_{2,2} - \Delta_{3,2,1}(a_{132}),\\
a_{32,12} &= -\Delta_{1,2,1}(a_{132}) \text{ and } a_{12,32} = -\Delta_{3,2,3}(a_{132}),\\
a_{132,132} &= a_{12,12} + \Delta_3(\Delta_{1,2} - \Delta_{2,1})(a_{132}) = a_{32,32} + \Delta_1(\Delta_{3,2} - \Delta_{2,3})(a_{132}),\\
a_{2132,2132} &= a_{132,132} + \Delta_{2,1,3,2}(a_{2132}) + \Delta_{2,1,3}s_2(a_{132}).
\end{aligned}$$

which can be also expressed as a diagram of operators (here we denote $a_{ijk..,lmn..}$ by $a_{ijk..}^{lmn..}$)



Lemma 7.3. *For any polynomial g of degree 3 we have*

$$\Delta_{3,2,1}(g) \equiv \Delta_{1,2,3}(g) \equiv \Delta_{3,2,3}(g) \equiv \Delta_{1,2,1}(g) \quad \textit{and}$$
$$\Delta_{2,1,3}(g) \equiv \Delta_{3,1,2}(g) \equiv \Delta_{1,3,2}(g) \equiv 0 \mod 2.$$

Proof. As for the first chain of congruences, since $\Delta_1(g) \equiv \Delta_3(g) \equiv 0$ for any polynomial g which does not contain α_2, and the computations are symmetric with respect to α_1 and α_3, it is enough to check it only on monomials $\alpha_2^2\alpha_1$ and $\alpha_2\omega_2\alpha_1$. Direct computations modulo 2 then give

$$\Delta_{3,2,1}(\alpha_2^2\alpha_1) \equiv \Delta_{3,2}(\alpha_1^2) \equiv \Delta_3(\alpha_2) \equiv 1, \text{ and}$$
$$\Delta_{1,2,3}(\alpha_2^2\alpha_1) \equiv \Delta_{1,2}(\alpha_1\alpha_3) \equiv \Delta_1(\alpha_1+\alpha_2+\alpha_3) \equiv 1;$$
$$\Delta_{3,2,1}(\alpha_2\omega_2\alpha_1) \equiv \Delta_{3,2}(\omega_2\alpha_1) \equiv \Delta_3(\alpha_1+\alpha_2-\omega_2) \equiv 1, \text{ and}$$
$$\Delta_{1,2,3}(\alpha_2\omega_2\alpha_1) \equiv \Delta_{1,2}(\omega_2\alpha_1) \equiv \Delta_1(\alpha_1+\alpha_2-\omega_2) \equiv 1.$$

Similarly, we get $\Delta_{3,2,3}(g) \equiv \Delta_{1,2,1}(g)$ and $\Delta_{3,2,3}(g) \equiv \Delta_{1,2,3}(g)$.

As for the second, it is enough to verify that $\Delta_{1,3}(h) \equiv 0$ for any quadratic h and $\Delta_{2,1,3}(\alpha_2^3) \equiv 0$. Indeed, for quadratic h it reduces to $\Delta_{1,3}(\alpha_2^2) \equiv \Delta_1(\alpha_3) \equiv 0$ and

$$\Delta_{2,1,3}(\alpha_2^3) \equiv \Delta_{2,1}(\alpha_2^2+\alpha_2\alpha_3+\alpha_3^2) \equiv \Delta_2(\alpha_1+\alpha_3) \equiv 0. \qquad \square$$

If ϕ is an idempotent, then all the diagonal entires of the matrix $(a_{v,w})$ are idempotents as well. In particular, the matrix $M = \begin{pmatrix} a_{12,12} & a_{32,12} \\ a_{12,32} & a_{32,32} \end{pmatrix}$ is an idempotent matrix. Since $a_{12,32} \equiv a_{32,12}$, the matrix M modulo 2 is either a trivial matrix or an identity, which implies that

$$a_{12,32} \equiv a_{32,12} \equiv \Delta_{1,2,3}(a_{132}) \equiv \Delta_{3,2,1}(a_{132}) \equiv 0 \mod 2.$$

From the recurrent formulas for the diagonal entries we obtain

$$a_\emptyset \equiv a_{2,2} \equiv a_{12,12} \equiv a_{32,32} \equiv a_{132,132} \equiv a_{2132,2132} \equiv 0,1 \mod 2.$$

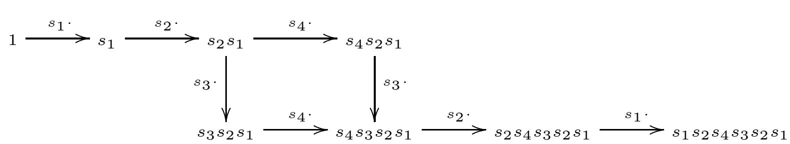
So, we have proven the following

Proposition 7.4. *Let G be a group of type A_3 and let P be the maximal parabolic subgroup generated by the first and the third simple reflections. Let $\mathbf{D}_F$ be a formal affine Demazure algebra corresponding to the additive formal group law F over $\mathbb{Z}$ (or even $\mathbb{Z}/2\mathbb{Z}$) and to the lattice $T^* = \langle \alpha_1, \alpha_2, \alpha_3, \omega_2\rangle$ (observe that T^* modulo the root lattice is $\mathbb{Z}/2\mathbb{Z}$).*

Then the $\mathbf{D}_F$-module $\mathbf{D}^\star_{F,P}$ is irreducible.

Remark 7.5. Again in view of 5.8, 5.9 and 4.3, this fact implies indecomposability of the motive of a generic 4-dimensional quadric.

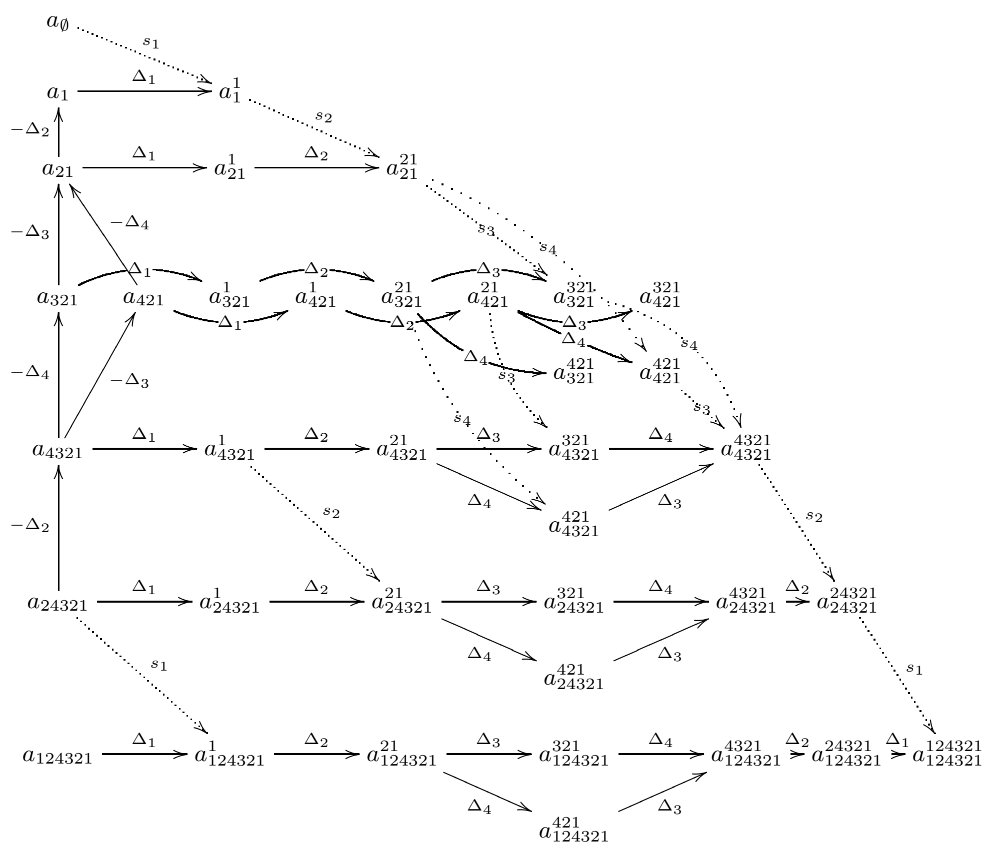
The D_4-case. Let G be of type D_4 and let P be of type A_3, i.e., G/P is a 6-dimensional smooth projective quadric. By definition, the Weyl group $W = \langle s_1, s_2, s_3, s_4\rangle$, $W_P = \langle s_2, s_3, s_4\rangle$ and the set of minimal coset representatives W^P is given by the Hasse diagram



By the recurrent formulas of Lemma 6.1 and of Corollary 6.2 we obtain (as before we set $a_{ijk..} = a_{s_i s_j s_k..}$ and $a_\emptyset = a_1$):

$$\begin{aligned}
a_{1,1} &= a_\emptyset + \Delta_{1,2,3,4,2}(a_{24321}),\\
a_{21,21} &= a_{1,1} - \Delta_{2,1,3,4,2}(a_{24321}),\\
a_{321,321} &= a_{21,21} + \Delta_{3,2,1,4,2}(a_{24321}) \text{ and } a_{421,421} = a_{21,21} + \Delta_{4,2,1,3,2}(a_{24321}),\\
a_{321,421} &= \Delta_{4,2,1,4,2}(a_{24321}) \text{ and } a_{421,321} = \Delta_{3,2,1,3,2}(a_{24321}),\\
a_{4321,4321} &= a_{421,421} + \Delta_{3,2,1,4,2}(a_{24321}) - \Delta_{3,4,2,1,2}(a_{24321})\\
&= a_{321,321} + \Delta_{4,2,1,3,2}(a_{24321}) - \Delta_{4,3,2,1,2}(a_{24321}),\\
a_{24321,24321} &= a_{4321,4321} + (\Delta_{2,4,3,2,1} - \Delta_{2,4,3}s_2\Delta_{1,2})(a_{24321}),\\
a_{124321,124321} &= a_{24321,24321} + \Delta_{1,2,4,3,2}(s_1(a_{24321}) + \Delta_1(a_{124321})).
\end{aligned}$$

which can be also expressed as the following diagram of operators (here we denote $a_{ijk..,lmn..}$ by $a_{ijk..}^{lmn..}$)



Let $\Delta^d_{i_1,i_2,..}$ denote the image $\Delta_{i_1,i_2,..}(S^d(T^*))$ modulo 2.

Assume G corresponds to the root lattice, i.e., $G = PGO_8$.

Lemma 7.6. *We have* $\Delta^2_{3,4} = 0$, $\Delta^3_{3,4} = \langle \alpha_3 + \alpha_4\rangle$,

$$\Delta^4_{3,4} = \langle (\alpha_3+\alpha_4)\alpha_1, (\alpha_3+\alpha_4)\alpha_3, (\alpha_3+\alpha_4)\alpha_4\rangle, \quad \Delta^4_{2,3,4} = \langle \alpha_3+\alpha_4\rangle.$$

Moreover, it holds for any permutation of the set of subscripts $\{1,3,4\}$.

Proof. Follows from the fact that Δ_3 and Δ_4 are trivial mod 2 on all simple roots except α_2 and that $\Delta_{3,4}(\alpha_2^2) \equiv \Delta_{3,4}(\alpha_2^4) \equiv 0$, $\Delta_{3,4}(\alpha_2^3) \equiv \alpha_3+\alpha_4$. □

From the lemma we immediately obtain

$$\begin{aligned} a_{124321,124321}, a_{1,1} &: \Delta^4_{1,2,3,4} = \Delta_1(\Delta^4_{2,3,4}) = 0 \\ a_{24321,24321}, a_{21,21} &: \Delta^3_{2,3,4} = 0,\ \Delta_{2,1}(\Delta^4_{3,4}) = 0 \\ a_{321,321}, a_{421,421} &: \Delta^4_{3,2,1,4} = 0,\ \Delta^4_{4,2,1,3} = 0 \\ a_{4321,4321} &: \Delta^4_{4,2,1,3} = 0,\ \Delta^2_{3,4} = 0 \end{aligned}$$

All this assuming that ϕ is an idempotent gives

$$a_\emptyset \equiv a_{1,1} \equiv a_{21,21} \equiv a_{321,321} \equiv a_{421,421} \equiv a_{4321,4321} \equiv a_{24321,24321} \equiv a_{124321,124321}.$$

So there are no non-trivial idempotents and we have proven the following

Proposition 7.7. *Let G be an adjoint group of type D_4 and let P be the maximal parabolic subgroup generated by all simple reflections except the first one. Let $\mathbf{D}_F$ be a formal affine Demazure algebra corresponding to the additive formal group law F over $\mathbb{Z}$ (or even $\mathbb{Z}/2\mathbb{Z}$) and to the root lattice.*

Then the $\mathbf{D}_F$-module $\mathbf{D}^{\star}_{F,P}$ is irreducible.

Remark 7.8. In view of 5.8, 5.9 and 4.3, this fact implies indecomposability of the motive of a generic twisted form of a 6-dimensional split quadric.

Assume that $\omega_1 \in T^*$, that is $G = SO_8$. Then the lemma 7.6 turns into

Lemma 7.9. *We have $\Delta^2_{3,4} = 0$, $\Delta^3_{3,4} = \langle \alpha_3 + \alpha_4 \rangle$,*

$$\Delta^4_{3,4} = \langle (\alpha_3+\alpha_4)\alpha_1, (\alpha_3+\alpha_4)\alpha_3, (\alpha_3+\alpha_4)\alpha_4, (\alpha_3+\alpha_4)\omega_1 \rangle, \quad \Delta^4_{2,3,4} = \langle \alpha_3+\alpha_4 \rangle.$$

So we obtain

$$a_{124321,124321}, a_{1,1} : \Delta^4_{1,2,3,4} = \Delta_1(\Delta^4_{2,3,4}) = 0$$

$$a_{24321,24321}, a_{21,21} : \Delta^3_{2,3,4} = 0, \ \Delta_{2,1}(\Delta^4_{3,4}) = 0$$

which gives only that

$$a_\emptyset \equiv a_{1,1} \equiv a_{21,21} \quad \text{and} \quad a_{4321,4321} \equiv a_{24321,24321} \equiv a_{124321,124321}.$$

Since $\Delta_3(f) \equiv \Delta_4(f)$ for any linear f, we have

$$\Delta_{4,2,1,3,2} \equiv \Delta_{3,2,1,3,2}.$$

Moreover, direct computations show that

$$\Delta_{3,2,3}(\alpha_2^2\alpha_3) \equiv \Delta_{3,2,3}(\alpha_2^2\alpha_4) \equiv 1.$$

So that $\Delta_{3,2,3}(g) \equiv \Delta_{4,2,4}(g)$. Combining, we obtain

$$\Delta_{4,2,1,3} = \Delta_{4,2,3,1} \equiv \Delta_{3,2,3,1} \equiv \Delta_{4,2,4,1} \equiv \Delta_{3,2,4,1} = \Delta_{3,2,1,4}$$

So $a_{21,21} \equiv a_{321,321} \equiv a_{421,421} \equiv a_{4321,4321}$. Hence, there are no non-trivial idempotents as well.

Proposition 7.10. *Let G be a special orthogonal group of type D_4 and let P be the maximal parabolic subgroup generated by all simple reflections except the first one. Let $\mathbf{D}_F$ be a formal affine Demazure algebra corresponding to the additive formal group law F over $\mathbb{Z}$ (or even $\mathbb{Z}/2\mathbb{Z}$) and to the lattice T^*.*

Then the $\mathbf{D}_F$-module $\mathbf{D}^{\star}_{F,P}$ is irreducible.

Remark 7.11. In view of 5.8, 5.9 and 4.3, this fact implies indecomposability of the motive of a generic 6-dimensional quadric.

Assume $\omega_4 \in T^*$ that is $G = HSpin_8$. Then $\{\alpha_2, \alpha_3, \alpha_4, \omega_4\}$ generates T^*.

We claim that $\Delta_{1,2,3,4,2}(a_{24321}) \equiv 0$. Indeed, let $f = \Delta_{2,3,4,2}(a_{24321}) \in S^1(T^*)$. Then

$$\Delta_3(f) = \Delta_{3,2,3,4,2}(a_{24321}) = \Delta_{2,3,2,4,2}(a_{24321}) = \Delta_{2,3,4,2}(\Delta_4(a_{24321})) = 0.$$

Now for $f = a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + b\omega_4$ we get $\Delta_1(f) \equiv a_2 \mod 2$ but $\Delta_3(f) \equiv a_2 \mod 2$ as well.

Similarly, $\Delta_{3,2,1,4,2}(a_{24321}) \equiv 0$. In this case denote $f = \Delta_{2,1,4,2}(a_{24321})$. Then

$$\Delta_1(f) = \Delta_{1,2,1,4,2}(a_{24321}) = \Delta_{2,1,2,4,2}(a_{24321}) = \Delta_{2,1,4,2}(\Delta_4(a_{24321})) = 0.$$

And $\Delta_3(f) \equiv \Delta_1(f) \equiv 0$.

By the same arguments, $\Delta_{3,2,1,3,2}(a_{24321}) \equiv \Delta_{3,4,2,1,2} \equiv 0$.

Consider now $\Delta_{2,1,3,4,2}(a_{24321})$. Let $g = \Delta_{3,4,2}(a_{24321})$. We then have $\Delta_{3,2}(g) = 0$. Let $g = \sum_{2\le i\le j} c_{ij}\alpha_i\alpha_j + \sum_{2\le i} b_i\omega_4\alpha_i + d\omega_4^2$. Then

$$\Delta_2(g) \equiv c_{22}(\alpha_2+\alpha_3+\alpha_4) + \alpha_2(c_{23}+c_{24}) + \omega_4(b_3+b_4).$$

The fact that $\Delta_{3,2}(g) \equiv 0$ implies that $c_{22}+c_{23}+c_{24} \equiv 0$. But

$$\Delta_1(g) \equiv c_{22}\alpha_1 + c_{23}\alpha_3 + c_{24}\alpha_4 + b_2\omega_4.$$

So that $\Delta_{2,1}(g) \equiv (c_{22}+c_{23}+c_{24}) \equiv 0$. Combining we obtain that

$$a_\emptyset \equiv a_{1,1} \equiv a_{21,21} \equiv a_{321,321} \equiv a_{421,321}$$

and

$$a_{421,421} \equiv a_{321,421} \equiv a_{4321,4321} \equiv a_{24321,24321} \equiv a_{124321,124321}.$$

Then the $\mathbf{D}_F$-module $\mathbf{D}^\star_{F,P}$ is either irreducible or splits into two irreducible direct summands with a generating function $1+t+t^2+t^3$ (over $\mathbf{S}$) each.

Remark 7.12. In view of 5.8, 5.9 and 4.3, this fact implies that the motive M of a $HSpin_8$-generic involution variety is either indecomposable or splits as a direct sum of motives $M = N \oplus N(3)$, where N is indecomposable with a generating function $1+t+t^2+t^3$. Using know result on motives of quadratic forms (e.g. that after splitting the algebra, the motive of a $Spin_8$-generic quadratic form splits into 2-fold Rost motives) it follows that the second decomposition is impossible, i.e., M has to be indecomposable.

8. Endomorphisms of $\mathbf{Q}^*_{W/W_P}$

In the present section we investigate the endomorphism ring $End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$.

Consider a standard basis $\{f_{\bar w}\}$, $w \in W^P$, of the free $\mathbf{Q}$-module $\mathbf{Q}^*_{W/W_P}$, where $f_{\bar w}$ is dual to $\delta_{\bar w}$. Since the $\mathbf{Q}_W$-module $\mathbf{Q}^*_{W/W_P}$ is generated by $f_{\bar 1}$, any endomorphism $\phi \in End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$ is determined by it value on $f_{\bar 1}$ that is

$$\phi(f_{\bar 1}) = \sum_{w\in W^P} a_{\bar w} f_{\bar w},\ a_{\bar w} \in \mathbf{Q}. \tag{2}$$

Since ϕ is $\mathbf{Q}_W$-linear, it has to satisfy $v \odot \phi(f_{\bar 1}) = \phi(v \odot f_{\bar 1}) = \phi(f_{\bar 1})$ for all $v \in W_P$ which translates as

$$\sum_{w\in W^P} v(a_{\bar w}) f_{\overline{vw}} = \sum_{w\in W^P} a_{\bar w} f_{\bar w} \quad \text{for all } v \in W_P.$$

The latter is equivalent to

$$v(a_{\bar w}) = a_{\overline{vw}} \quad \text{for all } v \in W_P. \tag{3}$$

Observe that (3) implies that a coefficient $a_{\bar u}$ is uniquely determined by the coefficient $a_{\bar w}$, where w is the minimal representative of the double W_P-coset containing u. In particular, if $w = 1$, then the respective double coset is W_P and we obtain the condition $a_{\bar 1} \in \mathbf{Q}^{W_P}$.

Example 8.1. Consider G of type A_n and a parabolic subgroup P of type A_{n-1}, i.e. $G/P = \mathbb{P}^n$. We have $W = \langle s_1, \ldots, s_n \rangle$, $W_P = \langle s_2, s_2, \ldots, s_n \rangle$ and $W^P = \{1, v_1, v_2, \ldots, v_n\}$, where s_i denotes the i-th simple reflection and $v_i = s_i s_{i-1} \ldots s_1$.

For any $\phi \in End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$ presentation (2) can be written as

$$\phi(f_{\bar{1}}) = c_0 f_{\bar{1}} + c_1 f_{\bar{v}_1} + \ldots + c_n f_{\bar{v}_n}, \text{ where } c_i \in \mathbf{Q}.$$

Here we have only two double cosets: $W_P = W_P \cdot 1 \cdot W_P$ and $W_P \cdot s_1 \cdot W_P$. So the conditions (3) turn into

$$c_0 \in \mathbf{Q}^{W_P} \text{ and } c_j = \tfrac{v_j}{v_1}(c_1),\ 1 \le j \le n.$$

In other words, ϕ is determined only by two coefficients $c_0 \in \mathbf{Q}^{W_P}$ and $c_1 \in \mathbf{Q}$.

An endomorphism $\phi \in End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$ is an idempotent means that $\phi(\phi(f_{\bar{1}})) = \phi(f_{\bar{1}})$, i.e.,

$$\sum_{\overline{wv} = \bar{u}} a_{\bar{w}} w(a_{\bar{v}}) = a_{\bar{u}}, \quad w, v, u \in W^P. \tag{4}$$

Example 8.2. In the notation of the previous example we have the following multiplication table on W^P:

For $j \ge 1$ (assuming $v_0 = 1$).

$$v_i \cdot v_j = \begin{cases} v_j & \text{if } i < j \\ v_{j-1} & \text{otherwise} \end{cases}$$

In particular, for $n = 5$ it can be represented as a matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 3 & 4 & 5 \\ 3 & 0 & 1 & 2 & 4 & 5 \\ 4 & 0 & 1 & 2 & 3 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

where the coefficient at the position (i, j) is the index of $v_i \cdot v_j$ (we number columns and rows starting from 0). Observe that from the matrix it follows that each number r, $0 \le r \le n$ appears exactly $n+1$ times. That is at the r-th column up to $(r-1)$-th row (r times), then at the zero column and r-th row (1 times) and $r+1$-th column up to n-th row ($n - r$ times).

The condition (4) then becomes: For each r, $0 \le r \le n$ we have

$$\sum_{i=0}^{r-1} c_i \cdot v_i(c_r) + c_r \cdot v_r(c_0) + \sum_{i=r+1}^{n} c_i \cdot v_i(c_{r+1}) = c_r \tag{5}$$

In particular, for $r = 0$, we obtain

$$c_0^2 + \sum_{i=1}^{n} c_i v_i(c_1) = c_0$$

and for $r = 1$ we get

$$c_0 \cdot c_1 + c_1 \cdot v_1(c_0) + \sum_{i=2}^{n} c_i \cdot v_i(s_2(c_1)) = c_1.$$

Example 8.3. An idempotent $\phi \in \mathrm{End}_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$ for the group G of type A_2 and P of type A_1 is determined by

$$\phi(f_{\bar{1}}) = c_0 f_{\bar{1}} + c_1 f_{\bar{s}_1} + s_2(c_1) f_{\bar{s}_2 s_1}, \quad c_0 \in \mathbf{Q}^{W_P},\ c_1 \in \mathbf{Q}$$

where c_0 and c_1 satisfy the following 2 equations (for $r = 0$ and $r = 1$; the equation for $r = 2$ is obtained from the one for $r = 1$ by applying s_2)

$$\begin{cases} c_0^2 + c_1 s_1(c_1) + s_2(c_1 s_1(c_1)) = c_0 \\ c_0 c_1 + c_1 s_1(c_0) + s_2(c_1) s_2 s_1(s_2(c_1)) = c_1. \end{cases}$$

Observe that tensoring with $\mathbf{Q}$ induces an embedding

$$\mathrm{End}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P}) \hookrightarrow \mathrm{End}_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P}).$$

We now investigate its image.

Recall that $\mathbf{D}^\star_{F,P} \subset \mathbf{S}^\star_{W/W_P} \subset \mathbf{Q}^*_{W/W_P}$ is a $\mathbf{D}_F$-module generated by the class of a point $[pt] = x_{\Pi/P} f_{\bar{1}} \in \mathbf{S}^\star_{W/W_P}$, where $x_{\Pi/P} = x_\Pi / x_P$, $x_P = \prod_{\alpha\in\Sigma_P^-} x_\alpha$ and $x_\Pi = \prod_{\alpha\in\Sigma^-} x_\alpha$. Therefore, any $\phi \in \mathrm{End}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ is determined by its value on $[pt]$. On the other hand, $\phi([pt])$ belongs to $\mathbf{D}^\star_{F,P}$ as an element of $\mathbf{S}^\star_{W/W_P} \subset \mathbf{Q}^*_{W/W_P}$ if it satisfies the criteria of [9, Thm. 11.9]. Combining these together we obtain that $\phi \in \mathrm{End}_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$ comes from $\mathrm{End}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ if and only if the coefficients $a_{\bar{w}} \in \mathbf{Q}$ satisfy

$$b_{\bar{w}} = x_{\Pi/P} a_{\bar{w}} \in \mathbf{S} \text{ and } x_{w(\alpha)} \mid (b_{\bar{w}} - b_{\overline{s_{w(\alpha)} w}}) \text{ for all } \alpha \notin \Sigma_P. \tag{6}$$

Expressing the coefficients $a_{\bar{w}}$ in terms of $b_{\bar{w}} \in \mathbf{S}$ and combining all the conditions we (3), (4) and (6) we obtain that ϕ is an idempotent in $\mathrm{End}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ if and only if

$$\begin{cases} v(b_{\bar{w}}) = b_{\overline{vw}} \quad \text{ for all } v \in W_P \\ \sum_{\overline{wv}=\bar{u}} w(x_P) b_{\bar{w}} w(b_{\bar{v}}) = x_\Pi b_{\bar{u}}, \quad w, v, u \in W^P \\ x_{w(\alpha)} \mid (b_{\bar{w}} - b_{\overline{s_{w(\alpha)} w}}) \text{ for all } \alpha \notin \Sigma_P. \end{cases}$$

Example 8.4. In the case (A_2, A_1) and Chow groups let $\Pi = \{\alpha, \beta\}$ denote the set of simple roots. We have $\Sigma_P = \{\pm\beta\}$, $x_{\Pi/P} = x_\alpha x_{-\alpha} x_{\alpha+\beta} x_{-(\alpha+\beta)} = \alpha^2(\alpha+\beta)^2$, $x_P = x_\beta x_{-\beta} = -\beta^2$. Set $\tilde{c}_0 = x_{\Pi/P} c_0$, $\tilde{c}_1 = x_{\Pi/P} c_1$ and $\tilde{c}_2 = x_{\Pi/P} c_2$. Then the polynomials $\tilde{c}_0, \tilde{c}_1 \in \mathbf{S} = \mathbb{Z}[\alpha, \beta]$ define an idempotent in $\mathrm{End}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ if and only if

$$\begin{cases} -\beta^2 \tilde{c}_0^2 - (\alpha+\beta)^2 \tilde{c}_1 s_1(\tilde{c}_1) - \alpha^2 s_2(\tilde{c}_1 s_1(\tilde{c}_1)) = \alpha^2\beta^2(\alpha+\beta)^2 \tilde{c}_0 \\ -\beta^2 \tilde{c}_0 \tilde{c}_1 - (\alpha+\beta)^2 \tilde{c}_1 s_1(\tilde{c}_0) - \alpha^2 s_2(\tilde{c}_1) s_2 s_1(s_2(\tilde{c}_1)) = \alpha^2\beta^2(\alpha+\beta)^2 \tilde{c}_1 \\ \alpha \mid \tilde{c}_0 - \tilde{c}_1. \end{cases}$$

Consider the endomorphism ring $End_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$. By definition there is an inclusion

$$End_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P}) \hookrightarrow End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P}).$$

We will identify the endomorphism rings $End_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$ and $End_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$ as subrings of $End_{\mathbf{Q}_W}(\mathbf{Q}^*_{W/W_P})$.

Observe that the conditions for being an $\mathbf{S}_W$-homomorphism and an idempotent in $End_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$ are the same as (3) and (4) except that all the coefficients have to be in $\mathbf{S}$.

Consider an endomorphism $\phi \in End_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$ with $\phi(f_{\bar{1}}) = \sum a_{\bar{w}} f_{\bar{w}}$. Its image lies in $\mathbf{D}^\star_{F,P}$ if the coefficients $a_w \in \mathbf{S}$ satisfy the criteria in [9, Thm. 11.9].

Hence, ϕ maps $[pt] = \frac{x_\Pi}{x_P}\tilde{f}_1 \in \mathbf{D}^\star_{F,P}$ to an element in $\mathbf{D}^\star_{F,P}$, i.e. comes from $End_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$, if

$$x_{w(\alpha)} \mid \tfrac{x_\Pi}{x_P}(a_w - a_{\overline{s_{w(\alpha)}w}}) \text{ for all } \alpha \notin \Sigma_P$$

(here $ws_\alpha = s_{w(\alpha)}w$, Σ_P is the root subsystem corresponding to P, x_Π is the product over all negative roots and x_P is the subproduct over all negative roots of Σ_P). Since $x_{w(\alpha)} \mid \frac{x_\Pi}{x_P}$ if $w(\alpha) \notin \Sigma_P$, it becomes equivalent by (3) to

$$x_{w(\alpha)} \mid \tfrac{x_\Pi}{x_P}(a_w - s_{w(\alpha)}(a_w)), \tag{7}$$

for all α and $w \in W^P$ such that $\alpha \notin \Sigma_P$ and $w(\alpha) \in \Sigma_P$.

Example 8.5. Divisibility holds, i.e. any endomorphism over $\mathbf{S}_W$ gives rise to an endomorphism over $\mathbf{D}_F$, if W_P is normal in W.

We consider now only endomorphisms of degree 0 (those that preserve the natural grading). In this case, coefficients a_w in the presentation of $\phi \in End^{(0)}_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$ have degree 0, i.e. (for Chow groups) they are from $\mathbf{R}$.

The divisibility condition (7) then always hold as either $w(\alpha) \notin \Sigma_P$ in which case $x_{w(\alpha)} \mid \frac{x_\Pi}{x_P}$, or $w(\alpha) \in W_P$ in which case $a_w - s_{w(\alpha)}(a_w) = 0$ as W acts trivially on $\mathbf{R}$. Hence, we obtain

Lemma 8.6. *There is an embedding*

$$End_{\mathbf{R}[W]}(\mathbf{R}[W/W_P]) = End^{(0)}_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P}) \hookrightarrow End^{(0)}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P}).$$

In the opposite direction if we have $\phi \in End^{(0)}_{\mathbf{D}_F}(\mathbf{D}^\star_{F,P})$, then it is given by $\phi(\frac{x_\Pi}{x_P}\tilde{f}_1) = \sum_{w\in W^P} b_w \tilde{f}_w$, where $b_w \in \mathbf{S}$ satisfies the divisibility condition and $\deg b_w = \dim G/P$. Observe that ϕ comes from $End^{(0)}_{\mathbf{S}_W}(\mathbf{S}^\star_{W/W_P})$ if and only if each b_w is divisible by $\frac{x_\Pi}{x_P}$. We have

$$\phi(\tilde{f}_1) = \sum_{w\in W^P} \tfrac{1}{x_\Sigma} a_w \tilde{f}_w,$$

where $x_\Sigma = x_\Pi x_{-\Pi}$ and $a_w = x_P x_{-\Pi} b_w$, $\deg a_w = \deg x_\Sigma$. Recall that by (4) ϕ is an idempotent if and only if

$$\sum_{\bar{w}\bar{v}=\bar{u}} a_w w(a_v) = x_\Sigma a_u,$$

References

[1] J. Ayoub, L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I. J. für die Reine und angew. Math. 693 (2014), 1–149.

[2] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28, 1–26, 1973.

[3] P. Bressler, S. Evens, The Schubert calculus, braid relations and generalized cohomology. Trans. Amer. Math. Soc. 317 (1990), no.2, 799–811.

[4] P. Bressler, S. Evens, Schubert calculus in complex cobordisms. Trans. Amer. Math. Soc. 331 (1992), no.2, 799–813.

[5] P. Brosnan. On motivic decompositions arising from the method of Bialynicki-Birula, *Invent. Math.* 161, 91-111, 2005.

[6] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline, Chow motives of twisted flag varieties. Compositio Math. 142 (2006), no.4, 1063–1080.

[7] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and cohomology of complete flags, Ann. Sci. École Norm. Sup. (4) 46 (2013), no.3, 405–448.

[8] B. Calmès, K. Zainoulline, C. Zhong, A coproduct structure on the formal affine Demazure algebra, to appear in Math. Z. (arXiv:1209.1676).
[9] B. Calmès, K. Zainoulline, C. Zhong, Push-pull operators on the formal affine Demazure algebra and its dual, arXiv:1312.0019.
[10] B. Calmès, K. Zainoulline, C. Zhong, Equivariant oriented cohomology of flag varieties. Documenta Math. Extra Volume: Alexander S. Merkurjev's 60th Birthday (2015), 113–144.
[11] V. Chernousov, S. Gille, A. Merkurjev, Motivic decomposition of isotropic projective homogeneous varieties, Duke Math. J. 126 (2005), no.1, 137–159.
[12] V. Chernousov, A. Merkurjev, Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. Transform. Groups 11 (2006), no.3, 371–386.
[13] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion. Invent. Math. 21 (1973), 287–301.
[14] M. Demazure, Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.
[15] D. Edidin, W. Graham, Characteristic classes in the Chow ring, J. Algebraic Geom. 6 (1997), no.3, 431–443.
[16] D. Edidin, W. Graham, Equivariant intersection theory. Invent. Math. 131 (1998), no.3, 595–634.
[17] R. Elman, N. Karpenko, A. Merkurjev, Algebraic and geometric theory of quadratic forms. AMS Colloquium Publ. 56, AMS, Providence, RI, 2008. 435 pp.
[18] M. Feshbach, The image of $H^*(BG,\mathbb{Z})$ in $H^*(BT,\mathbb{Z})$ for G a compact Lie group with maximal torus T. Topology 20 (1981), 93–95.
[19] S. Gille, K. Zainoulline, Equivariant pretheories and invariants of torsors, Transf. Groups 17 (2012), no.2, 471–498.
[20] V. Ginzburg, Geometric methods in the representation theory of Hecke algebras and quantum groups, Representation Theories and Algebraic Geometry Nato ASI Series Volume 514, 1998, pp 127-183
[21] J. Heller, J. Malagón-López, Equivariant algebraic cobordism, J. Reine Angew. Math. 684 (2013), 87–112.
[22] A. Hoffnung, J. Malagón-López, A. Savage, K. Zainoulline, Formal Hecke algebras and algebraic oriented cohomology theories. Selecta Math. 20 (2014), no.4, 1213–1245.
[23] V. Kac, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. Invent. Math. 80 (1985), 69–79.
[24] N. Karpenko. Grothendieck Chow motives of Severi-Brauer varieties, *St. Petersburg Math. J.* 7, 649-661, 1996.
[25] B. Köck. Chow motif and higher Chow theory of G/P, *Manuscripta Math.* 70, 363-372, 1991.
[26] B. Kostant, S. Kumar, T-Equivariant K-theory of generalized flag varieties. J. Differential Geometry 32 (1990), 549–603.
[27] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. Advances in Math. 62 (1986), 187–237.
[28] A. Lauda, A categorification of quantum $sl(2)$. Advances in Math. 225 (2010), 3327–3424.
[29] C. Lenart, K. Zainoulline, Towards generalized cohomology Schubert calculus via formal root polynomials, to appear in Math. Research Letters, 18pp.
[30] C. Lenart, K. Zainoulline, C. Zhong. Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant oriented cohomology, Preprint (2016), arXiv:1608.06554.
[31] M. Levine, Comparison of cobordism theories, J. Algebra 322 (2009), no.9, 3291–3317.
[32] M. Levine, Oriented cohomology, Borel-Moore homology, and algebraic cobordism, Michigan Math. J. 57 (2008), 523–572.
[33] M. Levine, F. Morel, Algebraic cobordism, Springer Monographs in Math., Springer, Berlin, 2007. xii+244pp.
[34] A. Nenashev, K. Zainoulline, Oriented cohomology and motivic decomposition of relative cellular spaces, J. of Pure and Applied Algebra 205 (2006), no.2, 323–340.
[35] A. Neshitov, Motivic decompositions and Hecke-type algebras, PhD Thesis, University of Ottawa (2016).
[36] A. Neshitov, V. Petrov, N. Semenov, K. Zainoulline, Motivic decompositions of twisted flag varieties and representations of Hecke-type algebras. Preprint (2015), arXiv:1505.07083.
[37] V. Petrov, N. Semenov, Rost motives, affine varieties, and classifying spaces Preprint (2015), arXiv:1506.07788.

[38] V. Petrov, N. Semenov, K. Zainoulline, J-invariant of linear algebraic groups. Ann. Sci. École Norm. Sup. (4) 41 (2008), no.6, 1023–1053.
[39] M. Rost. The motive of a Pfister form, *preprint* http://www.mathematik.uni-bielefeld.de/ rost/data/motive.pdf, 1998.
[40] R. Rouquier, 2-Kac-Moody algebras, Preprint arxiv:0812.5023v1.
[41] N. Semenov, M. Zhykhovich, Integral motives, relative Krull-Schmidt principle, and Maranda-type theorems, to appear in Math. Ann. (2015).
[42] T. Springer, Schubert varieties and generalizations, Representation theories and algebraic geometry (Montreal, PQ, 1997), 413–440, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.
[43] B. Totaro, The Chow ring of a classifying space. Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.
[44] Ch. Weibel, The K-book: an introduction to algebraic K-theory, Graduate Studies in Math., AMS, vol. 145 (2013).
[45] A. Vishik, N. Yagita, Algebraic cobordism of a Pfister quadric, J. Lond. Math. Soc. (2) 76 (2007), no.3, 586–604.
[46] A. Vishik, K. Zainoulline, Motivic splitting lemma, Documenta Math. 13 (2008), 81–96.

Baptiste Calmès, Université d'Artois, France

Alexander Neshitov, University of Southern California, USA

Kirill Zainoulline, University of Ottawa, Canada