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WITT GROUPS OF GRASSMANN VARIETIES

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Abstract

We compute the Witt groups of split Grassmann varieties, over any regular base X. The answer is that the total Witt group of the Grassmannian is a free module over the total Witt ring of X. We provide an explicit basis for this free module, which is indexed by a special class of Young diagrams, that we call *even* Young diagrams.

Contents

Introduction

At first glance, it might be surprising for the non-specialist that more than thirty years after the definition of the Witt group of a scheme, by Knebusch [14], the Witt group of such a well-known variety as a Grassmannian has not been computed yet. This is especially striking since analogous results for ordinary cohomologies, for K-theory and for Chow groups, have been settled for even longer. The goal of this article is to solve this problem and explain what made it so hard in the first place.

Main Theorem (see Theorem 6.1)**.** *Let X be a regular Noetherian and separated scheme over $\mathbb{Z}[\frac{1}{2}]$, of finite Krull dimension. Let $0 < d < n$ be*

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integers and let $\mathrm{Gr}_X(d,n)$ *be the Grassmannian of* d*-dimensional subbundles of the trivial* n*-dimensional vector bundle* $\mathcal{V}=\mathcal{O}_X^n$ *over* X*. (More generally, we treat any vector bundle* $\mathcal{V}$ *admitting a complete flag of subbundles.)*

Then the total Witt group of $\mathrm{Gr}_X(d,n)$ *is a free graded module over the total Witt group of* X *with an explicit basis indexed by so-called "even" Young diagrams. The basis element corresponding to an even Young diagram is essentially the push-forward of the unit along the inclusion of the corresponding Schubert variety. The cardinal of this basis equals* $2\cdot\dfrac{(d'+e')!}{d'!\cdot e'!}$ *where* $d'=\left[\frac{d}{2}\right]$ *and* $e'=\left[\frac{n-d}{2}\right]$*.*

Before explaining the statement in more detail, recall that the Grothendieck group, or the Chow group, of $\mathrm{Gr}_X(d,n)$ would also be free over that of X but with a basis indexed by *all* Young diagrams. We shall explain below why only some Young diagrams "make it to the Witt group".

The *total* Witt group refers to the sum of the Witt groups $\mathrm{W}^i(X,L)$,

$$\mathrm{W}^{\mathrm{tot}}(X)=\bigoplus_{\substack{i\in\mathbb{Z}/4,\\ [L]\in\mathrm{Pic}(X)/2}}\mathrm{W}^i(X,L)$$

for all possible shifts $i\in\mathbb{Z}/4$ and all possible twists $[L]\in\mathrm{Pic}(X)/2$ in the duality. Details about this total Witt group, including the dependency on choosing L in its class $[L]\in\mathrm{Pic}(X)/2$, can be found in Section 3. For this introduction, let us keep things simple: The total Witt group of X wraps up all Witt groups of X, for all possible shifts i and all twists L.

For $X=\mathrm{Spec}(F)$, the spectrum of a field, the total Witt group boils down to the classical Witt group $\mathrm{W}(F)$ but even in that case the above Theorem is new and the total Witt group of $\mathrm{Gr}_F(d,n)$ involves non-trivial shifted and twisted Witt groups. The result has a very round form when stated for total Witt groups, but Knebusch's classical unshifted Witt groups $\mathrm{W}^0(\mathrm{Gr}_X(d,n),L)$ can be isolated, as well as the unshifted and untwisted Witt group $\mathrm{W}(\mathrm{Gr}_X(d,n))=\mathrm{W}^0(\mathrm{Gr}_X(d,n),\mathcal{O})$. Indeed, the announced basis consists of homogeneous elements and we describe below how to read their explicit shifts i and twists L directly on the corresponding Young diagram. For instance, it is worth noting that there are no new interesting antisymmetric forms in the Witt groups of $\mathrm{Gr}_X(d,n)$, that is, except for those extended from X; see Corollary 7.2.

To describe our basis explicitly, we need to introduce *even* Young diagrams. We first consider ordinary Young diagrams sitting in the upper left corner of a rectangle with d rows and e columns, which we call the *frame* of the diagram; see Figure 1.

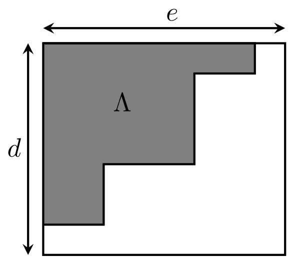


FIGURE 1. Young diagram Λ in $(d \times e)$-frame

We say that such a framed Young diagram Λ is *even* if all the segments of the boundary of Λ which are strictly inside the frame have even length. That is, we allow Λ to have odd-length segments on its boundary *only where it touches the outside frame*; see Figure 2 for examples. (In Figures 14, 15 and 16 we further give all even diagram in $(d\times e)$-frame for $(d,e) = (4,4)$, $(4,5)$ and $(5,5)$, respectively.)

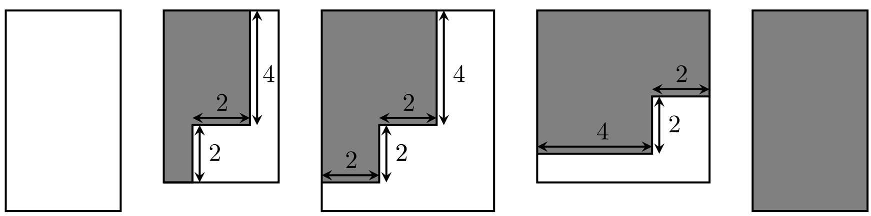


FIGURE 2. Five examples of even Young diagrams

We shall see that basis elements of $\mathrm{W}^{\mathrm{tot}}\big(\mathrm{Gr}_X(d,n)\big)$ are in bijective correspondence with even Young diagrams in $(d{\times}e)$-frame, for $e := n-d$. Moreover, as explained in Section 4, the Witt class $\phi_{d,e}(\Lambda)$ corresponding to such a diagram Λ lives in the Witt group $\mathrm{W}^i\big(\mathrm{Gr}_X(d,n), L\big)$ for the shift $i = |\Lambda| \in \mathbb{Z}/4$ equal to the *area* of the diagram and for the twist $[L] \in \mathrm{Pic}(\mathrm{Gr}_X(d,n))/2$ equal to the class of $t(\Lambda)\cdot\Delta$ where Δ is the determinant of the tautological bundle and where the integer $t(\Lambda)$ is half the *perimeter* of the diagram Λ (see Figure 9 in Section 4). More generally, when $\mathcal{V}$ is not free but admits a complete flag of subbundles, the twist of $\phi_{d,e}(\Lambda)$ also involves a multiple of the determinant of $\mathcal{V}$, in the direct summand of $\mathrm{Pic}(\mathrm{Gr}_X(d,n))/2$ coming from $\mathrm{Pic}(X)/2$.

Let us put our result in perspective. In its modern form, see for instance Laksov [15], the computation of the cohomology (or K-theory, or Chow groups) of a cellular variety uses essentially only localization long exact sequences and homotopy invariance, applied to the classical cellular decomposition of the Grassmannian. It took some time for Witt groups to reach the necessary cohomological maturity. Indeed, the localization long exact sequence could only be established by defining first "higher" or "shifted" Witt groups, as was done in [3] and [4] by the first author, using the framework of triangulated categories. Then, homotopy invariance was proved by Gille [11]. To be more precise, one actually also needs some form of *dévissage*, that allows us to compare the theory with supports on a closed subset Z with the theory of Z itself. This piece of theory was a hard nut to crack because dualities came in the way, but they are now well understood. (We return to these dualities below.)

This construction of the cohomological machines accounts for most of the delay that Witt groups accumulated in comparison to sister theories. However, this is not the end of the story. What makes the computation of the Witt groups of $\mathrm{Gr}_X(d,n)$ harder than that of classical theories is the following very interesting phenomenon. The classical computation proceeds by induction, using the closed embedding of a smaller Grassmannian $\mathrm{Gr}_X(d,n-1)$ inside $\mathrm{Gr}_X(d,n)$, whose open complement $U = \mathrm{Gr}_X(d,n) \smallsetminus \mathrm{Gr}_X(d,n-1)$ is an affine bundle over another smaller Grassmannian, $\mathrm{Gr}_X(d-1,n)$. (See more in Section 5.) Now the true miracle in the classical proof is that the restriction homomorphism, from the big Grassmannian $\mathrm{Gr}_X(d,n)$ to the open U, is *split surjective*. This holds more precisely for any *oriented* cohomology theory in the sense of Levine-Morel [16] or Panin [18]. In other words, there are no real localization *long* exact sequences involved in the classical proof, no real connecting homomorphisms, just split short exact sequences.

The interesting point is that this miracle ceases to happen for Witt groups: For a general $(i,[L]) \in \mathbb{Z}/4 \times \mathrm{Pic}(\mathrm{Gr}_X(d,n))/2$, the restriction homomorphism

$$\mathrm{W}^i(\mathrm{Gr}_X(d,n),L) \longrightarrow \mathrm{W}^i(U,L)$$

does not admit a section. Worse, it is not even surjective! That is, the connecting homomorphism in the localization long exact sequence is not zero in general. This makes the Witt group computation not a mere technical adaptation of the classical methods but a completely different story: One needs to *compute* a connecting homomorphism in geometric enough terms, in order to follow what happens on the given varieties. This very last pocket of Witt resistance has been cleared in the companion article [6] and Grassmannians are the first known examples where this phenomenon can be described explicitly.

Let us now comment on the organization of the paper. Sections 1 and 2 contain preparatory material on Grassmann varieties, desingularizations of Schubert varieties and even Young diagrams.

Section 3 formalizes the use of total Witt groups, since the group $\mathrm{W}^*(X, L)$ does not truly depend only on the class $[L] \in \mathrm{Pic}(X)/2$ but really on the representative L in this class. This is an old problem, rooting back to Knebusch [14], to which we proposed a solution in [7]. We recall this formalism in Section 3 with emphasis on the case of Grassmannians. Until then, this Introduction should therefore be read *cum grano salis*.

Our generators of $\mathrm{W}^{\mathrm{tot}}\big(\mathrm{Gr}_X(d,n)\big)$ are defined in Section 4 as push-forwards of the unit forms of certain desingularized Schubert varieties. The reader should observe that pushing the unit form is not always possible, due to the presence of line bundles in the definition of the push-forward. Indeed, for a proper morphism $f : \bar{Y} \to Y$ of constant relative dimension $\dim(f)$, between regular Noetherian schemes $\bar{Y}$ and Y (think $\dim f = \dim \bar{Y} - \dim Y$), the push-forward along f is defined between the following Witt groups:

$$\mathrm{W}^{i+\dim f}(\bar{Y}, \omega_f \otimes f^*L) \xrightarrow{f_*} \mathrm{W}^i(Y, L)\,, \tag{1}$$

where the special line bundle ω_f on the left is the *relative line bundle*. So, if unfortunately $\mathcal{O}_{\bar{Y}}$ is not (up to squares) of the form $\omega_f \otimes f^*L$ for any line bundle L over Y, then one simply *cannot* push-forward the unit form of $\bar{Y}$, which lives in $\mathrm{W}^0(\bar{Y}, \mathcal{O}_{\bar{Y}})$. This is why we start Section 4 by discussing the “parity” of the relevant canonical bundles ω_f. Although somewhat heavy, these computations are elementary and are all based on a repeated application of the computation of the relative canonical bundle of a Grassmann bundle (Proposition 1.5). The condition for a Young diagram to be even implies the existence of such a push-forward for the unit of the desingularized Schubert cell into the Grassmannian. Actually, we could push-forward the unit form for more Schubert cells but these additional generators would be redundant. The even Young diagrams are chosen so that the corresponding forms are also linearly independent.

Then, in Section 5, we recall the classical relative cellular structure of the Grassmann varieties. In Section 6, we compute how our candidate-generators behave under the morphisms in the long exact sequence, especially under the connecting homomorphism, which is most of the time not zero (Corollary 7.3). The proof of the main theorem (Theorem 6.1) then follows by induction on the rank of the vector bundle $\mathcal{V}$. The last section contains corollaries and examples.

This article is written in the language of “functors of points”, which means that we describe schemes in terms of their points (which are here flags) and morphisms of schemes as how they act on those points. This method is completely rigorous in this case. The original source is [1] and we also refer the reader to [9, § I.1] and [13, Part 2] for general considerations on this subject. This language is customary when dealing with flag varieties; see for instance [15] in which it is used for the computation of Chow groups of Grassmann varieties.

1. Combinatorics of Grassmann and flag varieties

We recall elementary facts about Grassmann varieties and desingularizations of Schubert cells. We also provide the necessary material about canonical bundles to treat the push-forward homomorphisms for Witt groups in Section 4.

1.1. Definition. A *subbundle* $\mathcal{P} \lhd \mathcal{V}$ of a vector bundle $\mathcal{V}$ over a scheme X is an $\mathcal{O}_X$-submodule which is locally a direct summand, i.e. $\mathcal{P}$ and $\mathcal{V}/\mathcal{P}$ are vector bundles.

1.2. Definition. Let $\mathcal{V}$ be a vector bundle of rank $n > 0$ over a scheme X and let d be an integer $0 \leqslant d \leqslant n$. We denote by $\mathrm{Gr}_X(d, \mathcal{V})$ the Grassmann bundle over X parameterizing the subbundles of rank d of $\mathcal{V}$. In the language of functors of points, it means that for any morphism $f : \mathrm{Spec}(R) \to X$, the set $\mathrm{Gr}_X(d, \mathcal{V})(R)$ consists of the R-submodules $P \lhd \mathcal{V}(R) = f^*(\mathcal{V})$ which are direct summands of rank d.

The scheme $\mathrm{Gr}_X(d, \mathcal{V})$ comes equipped with a smooth structural morphism $\pi : \mathrm{Gr}_X(d, \mathcal{V}) \to X$ and a tautological bundle $\mathcal{T}_d = \mathcal{T}_d^{\mathrm{Gr}_X(d,\mathcal{V})}$ of rank d.

1.3. Proposition. *The scheme $\mathrm{Gr}_X(d, \mathcal{V})$ is smooth over X of relative dimension $d(n-d)$. For $0 < d < n$, the Picard group of $\mathrm{Gr}_X(d, \mathcal{V})$ is given by*

$$\begin{array}{ccc} \mathrm{Pic}(X) \oplus \mathbb{Z} & \cong & \mathrm{Pic}(\mathrm{Gr}_X(d, \mathcal{V})) \\ (\ell, m) & \mapsto & \pi^*(\ell) \cdot [\det(\mathcal{T}_d)]^m. \end{array}$$

In case $d = 0$ or $d = n$, the morphism $\pi : \mathrm{Gr}_X(d, \mathcal{V}) \to X$ is the identity.

Proof. The Picard group of a regular scheme coincides with its Chow group CH^1, which is computed in [15] for Grassmannians; see Theorem 16 for the case where X is a field, and §13 to work over a regular base X. Using the Plücker embedding, one checks that the generator in loc. cit. is indeed $[\det(\mathcal{T}_d)]$. □

Let Δ_d denote the class of $\det(\mathcal{T}_d)$ in $\mathrm{Pic}\big(\mathrm{Gr}_X(d, \mathcal{V})\big)/2$.

1.4. Corollary. *If $0 < d < n$, we have a natural identification*

$$\mathrm{Pic}\big(\mathrm{Gr}_X(d, \mathcal{V})\big)/2 \;\cong\; \mathrm{Pic}(X)/2 \;\oplus\; \mathbb{Z}/2 \cdot \Delta_d.$$

1.5. Proposition. *The class of the relative canonical bundle* $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}$ *of the projection* $\pi : \mathrm{Gr}_X(d,\mathcal{V}) \to X$ *is* $[\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}] = [\det \mathcal{V}]^{-d} \cdot \Delta_d^n$ *in* $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))$. *In particular, if* $\mathcal{V} = \mathcal{O}_X^n$ *is trivial,* $[\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}] = \Delta_d^n$.

Proof. The morphism π is smooth, so $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}$ is the determinant (highest exterior power) of the relative cotangent bundle of π. This cotangent bundle is the tautological bundle tensored by the dual of the tautological quotient bundle (see [10, Appendix B.5.8]). Taking the determinant, we get the result. □

We now extend the previous results from Grassmannians to some flag varieties.

1.6. Definition. Let $k \geqslant 1$ and $(\underline{d}, \underline{e})$ be a pair of k-tuples of non-negative integers $\underline{d} = (d_1, \ldots, d_k)$ and $\underline{e} = (e_1, \ldots, e_k)$ satisfying

$$0 < d_1 < \cdots < d_k \qquad \text{and} \qquad e_1 + d_1 \leqslant \cdots \leqslant e_k + d_k\,. \tag{2}$$

(The second condition holds, in particular, if we have $e_1 \leqslant \cdots \leqslant e_k$.) Consider a flag

$$\mathcal{V}_{d_1+e_1} \lhd \cdots \lhd \mathcal{V}_{d_i+e_i} \lhd \cdots \lhd \mathcal{V}_{d_k+e_k} \tag{3}$$

of vector bundles over X, where $\lhd$ indicates subbundles in the strong sense of Definition 1.1 and where the rank is given by the index: $\mathrm{rk}_X(\mathcal{V}_r) = r$.

We associate to this data the scheme $\mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)$ over X, which parameterizes the flags of vector bundles $\mathcal{P}_{d_1} \lhd \mathcal{P}_{d_2} \lhd \cdots \lhd \mathcal{P}_{d_k}$ such that $\mathrm{rk}\,\mathcal{P}_{d_j} = d_j$ and $\mathcal{P}_{d_j} \lhd \mathcal{V}_{d_j+e_j}$. As a functor of points, this gives for any morphism $f : Y \to X$,

(4)
$$\mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)(Y) := \left\{ \begin{array}{ccccccc} 0 \lhd & \mathcal{P}_{d_1} & \lhd & \mathcal{P}_{d_2} & \lhd \cdots \lhd & \mathcal{P}_{d_k} \\ & \triangle\, e_1 & & \triangle\, e_2 & & \triangle\, e_k \\ 0 \lhd & f^*\mathcal{V}_{d_1+e_1} & \lhd & f^*\mathcal{V}_{d_2+e_2} & \lhd \cdots \lhd & f^*\mathcal{V}_{d_k+e_k} \end{array} \right\},$$

where all $\mathcal{P}_{d_i}$ are vector bundles over Y of rank d_i such that all inclusions are subbundles in the sense of Definition 1.1. The integers along inclusions indicate codimensions. Following general practice, we shall drop the mention of f^* in the sequel. Moreover, to avoid cumbersome notation, unless the original flag (3) varies, we drop the mention of $\mathcal{V}_\bullet$ from the notation: $\mathcal{F}l_X(\underline{d}, \underline{e}) = \mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)$.

1.7. Example. For $k = 1$, the scheme $\mathcal{F}l_X(\underline{d}, \underline{e})$ is simply $\mathrm{Gr}_X(d_1, \mathcal{V}_{d_1+e_1})$.

1.8. Remark. For any choice J of k' indices among $\{1, \ldots, k\}$, one can consider the pair of k'-tuples $(\underline{d}', \underline{e}')$ obtained from $(\underline{d}, \underline{e})$ by keeping d_i and e_i only for indices $i \in J$. There is a natural morphism $\mathcal{F}l_X(\underline{d}, \underline{e}) \to \mathcal{F}l_X(\underline{d}', \underline{e}')$ over X, obtained by dropping the $\mathcal{P}_{d_j}$ for those indices j which are not in the chosen J.

Furthermore, for any vector bundle $\mathcal{V}$ such that $\mathcal{V}_{d_k+e_k} \lhd \mathcal{V}$, there is a natural morphism $f_{\underline{d},\underline{e},\mathcal{V}}$ of schemes over X as follows:

$$
(5) \qquad \begin{array}{rccccc} f_{\underline{d},\underline{e},\mathcal{V}}: & \mathcal{F}l_X(\underline{d},\underline{e}) & \longrightarrow & \mathrm{Gr}_X(d,\mathcal{V}_{d_k+e_k}) & \hookrightarrow & \mathrm{Gr}_X(d,\mathcal{V}) \\ & (\mathcal{P}_{d_1},\ldots,\mathcal{P}_{d_k}) & \longmapsto & \mathcal{P}_{d_k} & \longmapsto & \mathcal{P}_{d_k}\,, \end{array}
$$

where the first morphism is as above and the second is a closed immersion.

1.9. Definition. The scheme $\mathcal{F}l_X(\underline{d},\underline{e})$ is equipped with *tautological bundles* $\mathcal{T}_{d_i}$, $1 \leqslant i \leqslant k$, of rank d_i, whose determinant classes are denoted by $\Delta_{d_i} := \det(\mathcal{T}_{d_i})$. The stalk of $\mathcal{T}_{d_i}$ at a point $(\mathcal{P}_{d_1},\ldots,\mathcal{P}_{d_k})$ is $\mathcal{P}_{d_i}$. In ambiguous cases, the full notation for $\mathcal{T}_{d_i}$ would be $\mathcal{T}_{d_i}^{\mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet)}$.

1.10. Remark. If $e_i = 0$, then the vector bundles $\mathcal{T}_{d_i} = \mathcal{V}_{d_i}$ and $\Delta_{d_i} = [\det \mathcal{V}_{d_i}]$ are both extended from X.

1.11. Lemma. *Let $k \geqslant 2$ and let $(\underline{d},\underline{e})$ be a pair of k-tuples satisfying* (2)*. Let $\mathcal{V}_\bullet$ be a flag as in* (3)*. Define the $(k-1)$-tuples $\underline{d}_{|_{k-1}}$ and $\underline{e}_{|_{k-1}}$ as the restrictions of $\underline{d}$ and $\underline{e}$ to the first $k-1$ entries. Consider the scheme*

$$
Y := \mathcal{F}l_X(\,\underline{d}_{|_{k-1}}\,,\,\underline{e}_{|_{k-1}}\,,\,\mathcal{V}_\bullet\,)\,,
$$

which only "uses" the first $k-1$ bundles $\mathcal{V}_{d_1+e_1} \lhd \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}}$. Consider the pull-back to Y of the remaining bundle, still denoted $\mathcal{V}_{d_k+e_k}$. Observe that $\mathcal{T}^Y_{d_{k-1}} \lhd \mathcal{V}_{d_k+e_k}$ and consider the quotient bundle

$$
\tilde{\mathcal{V}} := \mathcal{V}_{d_k+e_k}/\mathcal{T}^Y_{d_{k-1}}
$$

over Y. It has rank $d_k - d_{k-1} + e_k$. We then have a canonical isomorphism of schemes over Y (hence over X):

$$
(6) \qquad \mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet) \quad \cong \quad \mathrm{Gr}_Y\left(d_k - d_{k-1}, \tilde{\mathcal{V}}\right).
$$

Under this identification, we have $\mathcal{T}_{d_i}^{\mathcal{F}l(\underline{d},\underline{e},\mathcal{V}_\bullet)} = \mathcal{T}^Y_{d_i}$ for all $1 \leqslant i \leqslant k-1$ and

$$
(7) \qquad \mathcal{T}_{d_k}^{\mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet)}/\mathcal{T}_{d_{k-1}}^{\mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet)} = \mathcal{T}_{d_k-d_{k-1}}^{\mathrm{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})}\,.
$$

Proof. This simply amounts to the bijective correspondence between a flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_{d_k}$ satisfying $\mathcal{P}_{d_i} \lhd \mathcal{V}_{d_i+e_i}$ for all $1 \leqslant i \leqslant k$ and the following data:

(a) the beginning of this flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d_{k-1}}$ satisfying $\mathcal{P}_{d_i} \lhd \mathcal{V}_{d_i+e_i}$ for all $1 \leqslant i \leqslant k-1$,

(b) the bundle $\mathcal{P}_{d_k}$ such that $\mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_{d_k} \lhd \mathcal{V}_{d_k+e_k}$,

and to observe that (b) is equivalent to a subbundle $\tilde{\mathcal{P}} \lhd \mathcal{V}_{d_k+e_k}/\mathcal{P}_{d_{k-1}}$ of rank $d_k - d_{k-1}$, where $\tilde{\mathcal{P}} := \mathcal{P}_{d_k}/\mathcal{P}_{d_{k-1}}$. Details are left to the reader. □

1.12. Convention. When using k-tuples $\underline{d} = (d_1, \ldots, d_k)$, it will unify several formulas to simply define $d_0 = 0$.

1.13. Proposition. *Let $\underline{d}$ and $\underline{e}$ be two k-tuples as in (2) and $\mathcal{V}_\bullet$ be a flag as in (3). Then $\mathcal{F}l_X(\underline{d},\underline{e})$ is smooth over X of relative dimension $\sum_{i=1}^k (d_i - d_{i-1})\, e_i$. The Picard group of $\mathcal{F}l_X(\underline{d},\underline{e})$ is generated by $\operatorname{Pic}(X)$ and the "new" classes Δ_{d_i} :*

(8) $$\operatorname{Pic}(\mathcal{F}l_X(\underline{d},\underline{e})) \cong \operatorname{Pic}(X) \;\oplus \bigoplus_{\substack{1\leqslant i \leqslant k,\\ s.t.\ e_i\neq 0}} \mathbb{Z}\,\Delta_{d_i}\,.$$

The class of the relative canonical bundle $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}$ is given by the formula

(9)
$$[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}] = \prod_{i=1}^{k} [\det \mathcal{V}_{d_i+e_i}]^{-d_i+d_{i-1}} \cdot \prod_{i=1}^{k-1} \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_{d_k}^{d_k-d_{k-1}+e_k}\,,$$

where $\Delta_{d_i} = [\det \mathcal{V}_{d_i}]$ if $e_i = 0$ by Remark 1.10 and where we use Convention 1.12. In particular, for $k = 1$, we have $[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}] = [\det \mathcal{V}_{d_1+e_1}]^{-d_1} \cdot \Delta_{d_1}^{d_1+e_1}$.

Proof. By induction on k. The case $k = 1$ is that of a Grassmannian over X (Example 1.7) so the result follows from Propositions 1.3 and 1.5.

Let now $k \geqslant 2$. Consider $Y = \mathcal{F}l_X(\underline{d}|_{k-1}, \underline{e}|_{k-1}, \mathcal{V}_\bullet)$ and the bundle $\tilde{\mathcal{V}} = \mathcal{V}_{d_k+e_k}/\mathcal{T}_{d_{k-1}}$ over Y, as in Lemma 1.11. Recall that $\operatorname{rk}_Y(\tilde{\mathcal{V}}) = d_k - d_{k-1} + e_k$, which is always strictly positive ($d_k > d_{k-1}$) and which is bigger than or equal to $d_k - d_{k-1}$ with equality if and only if $e_k = 0$. Equation (6) and Propositions 1.3 and 1.5 immediately give smoothness, the formula for the relative dimension and that for the Picard group (8). Finally, to prove (9), observe that

$$\begin{aligned}
[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/Y}] &= [\omega_{\operatorname{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})\,/\,Y}] \\
&= [\det \tilde{\mathcal{V}}]^{-d_k+d_{k-1}} \cdot \Big(\Delta_{d_k-d_{k-1}}^{\operatorname{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})}\Big)^{\operatorname{rk}(\tilde{\mathcal{V}})} \\
&= [\det \tilde{\mathcal{V}}]^{-d_k+d_{k-1}} \cdot \Big(\Delta_{d_k}^{\mathcal{F}l_X(\underline{d},\underline{e})}\Big)^{\operatorname{rk}(\tilde{\mathcal{V}})} \cdot \Big(\Delta_{d_{k-1}}^{\mathcal{F}l_X(\underline{d},\underline{e})}\Big)^{-\operatorname{rk}(\tilde{\mathcal{V}})} \\
&= [\det \mathcal{V}_{d_k+e_k}]^{-d_k+d_{k-1}} \cdot \Delta_{d_{k-1}}^{-e_k} \cdot \Delta_{d_k}^{d_k-d_{k-1}+e_k}\,.
\end{aligned}$$

The first equality uses (6), the second comes from Proposition 1.5 and the third from (7). The last equality is a direct computation (in which we drop the mention of $\mathcal{F}l_X(\underline{d},\underline{e})$ for readability). By induction hypothesis, we get $[\omega_{Y/X}]$ from Equation (9) for $k-1$, that is, for the flag variety Y. Since $[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}] = [\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/Y}] \cdot [\omega_{Y/X}]$ over $\mathcal{F}l_X(\underline{d},\underline{e})$, we get (9) for k by adding the above. □

1.14. Corollary. *Let $\underline{d}$ and $\underline{e}$ be two k-tuples as in (2) and $\mathcal{V}_\bullet$ be a flag as in (3). Let $\mathcal{V}$ be a vector bundle of rank $d+e$ such that $\mathcal{V}_{d_k+e_k} \lhd \mathcal{V}$. The class in $\operatorname{Pic}(\mathcal{F}l_X(\underline{d},\underline{e}))$ of the relative canonical bundle for the morphism $f_{\underline{d},\underline{e},\mathcal{V}} : \mathcal{F}l_X(\underline{d},\underline{e}) \to \operatorname{Gr}_X(d,\mathcal{V})$ of (5) is given by*

$$
\begin{aligned}
[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/\operatorname{Gr}_X(d,\mathcal{V})}] &= \prod_{i=1}^{k} [\det \mathcal{V}_{d_i+e_i}]^{-d_i+d_{i-1}} \cdot [\det \mathcal{V}]^{d_k} \cdot \\
&\cdot \prod_{i=1}^{k-1} \quad \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_{d_k}^{-d_{k-1}+e_k-e},
\end{aligned} \tag{10}
$$

where $\Delta_{d_i} = [\det \mathcal{V}_{d_i}]$ if $e_i = 0$ by Remark 1.10 and where we use Convention 1.12. For $k = 1$, this reads $[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/\operatorname{Gr}_X(d,\mathcal{V})}] = [\det(\mathcal{V}/\mathcal{V}_{d_1+e_1})]^{d_1} \cdot \Delta_{d_1}^{e_1-e}$.

Proof. Subtract $(f_{\underline{d},\underline{e},\mathcal{V}})^*[\omega_{\operatorname{Gr}_X(d,\mathcal{V})/X}] = [\det \mathcal{V}]^{-d_k} \cdot \Delta_{d_k}^{d_k+e}$ (Proposition 1.5) from the bundle $[\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}]$ given in (9). □

1.15. Remark. When $\mathcal{V}_\bullet = \mathcal{O}_X^\bullet$, all the formulas are simpler, since all the $[\det \mathcal{V}_i]$ are trivial. This applies in particular when $X = \operatorname{Spec}(R)$ for a local ring R.

2. Even Young diagrams

We introduce *even* Young diagrams that will parameterize the basis of the total Witt group of the Grassmann variety, to be constructed in Section 4.

2.1. Definition. Let $d, e \geqslant 1$. We shall call *Young diagram in $(d\times e)$-frame*, or simply *(d,e)-diagram*, any d-tuple $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_d)$ of integers such that:

$$e \geqslant \Lambda_1 \geqslant \Lambda_2 \geqslant \ldots \geqslant \Lambda_d \geqslant 0.$$

See Figure 1 in the Introduction. The *area* of Λ is $|\Lambda| = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_d$. These (d,e)-diagrams are just ordinary Young diagrams displayed in the upper left corner of a rectangle with d rows and e columns, possibly leaving empty rows below and empty columns to the right of the Young diagram. So, an ordinary Young diagram with ρ rows and γ columns defines a (d,e)-diagram for any $d \geqslant \rho$ and $e \geqslant \gamma$.

2.2. Notation. The empty diagram $(0,\ldots,0) \in \mathbb{N}^d$ is denoted by $\square\!\!\!/$ and the full $(d\times e)$-rectangle $(e,\ldots,e) \in \mathbb{N}^d$ by $[d \times e]$.

2.3. Definition. Let $d, e \geqslant 1$ and let Λ be a Young diagram in $(d\times e)$-frame. The decreasing sequence $\Lambda_1 \geqslant \Lambda_2 \geqslant \ldots \geqslant \Lambda_d$ can be written in a unique way as a series of equalities and strict inequalities:

(11)
$$\underbrace{\Lambda_1 = \cdots = \Lambda_{d_1}}_{d_1 \text{ terms}} > \underbrace{\Lambda_{d_1+1} = \cdots = \Lambda_{d_2}}_{d_2 - d_1 \text{ terms}} > \cdots > \underbrace{\Lambda_{d_{k-1}+1} = \cdots = \Lambda_{d_k}}_{d_k - d_{k-1} \text{ terms}} = \Lambda_d\,.$$

Note that $d_k = d$. The integers $k \geqslant 1$ and $0 < d_1 < \ldots < d_k$ depend on Λ. If we need to stress this, we shall write $k = k(\Lambda)$ and $d_i = d_i(\Lambda)$ for $1 \leqslant i \leqslant k(\Lambda)$.

For fixed d and e, there is a bijection (pictured in Figure 3) between the Young diagrams Λ in $(d{\times}e)$-frame and pairs of k-tuples of integers

(12)
$$\begin{array}{lll} \underline{d} = (d_1, \ldots, d_k) & \text{such that} & 0 < d_1 < \cdots < d_k = d, \\ \underline{e} = (e_1, \ldots, e_k) & \text{such that} & 0 \leqslant e_1 < \cdots < e_k \leqslant e, \end{array}$$

with $1 \leqslant k \leqslant d$. The integers $k = k(\Lambda)$ and $d_i = d_i(\Lambda)$ are the ones above and we set $e_i = e_i(\Lambda) := e - \Lambda_{d_i}$ for all $i = 1, \ldots, k$. The converse construction is obvious.

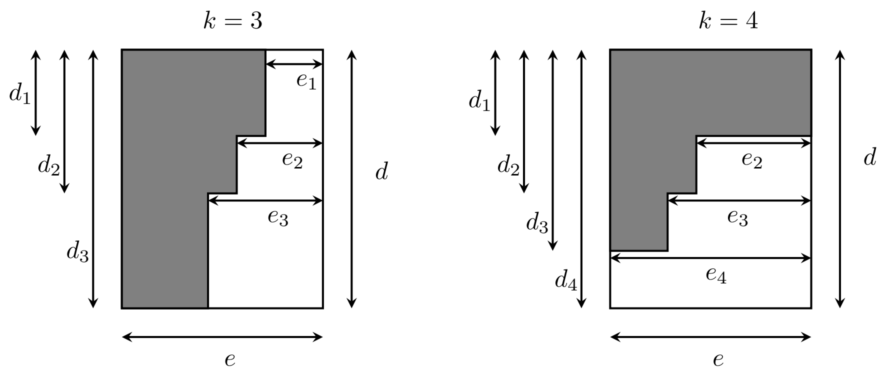


FIGURE 3. Two examples of the two k-tuples $(d_1, \ldots, d_k)$ and $(e_1, \ldots, e_k)$ corresponding to a Young diagram Λ in $(d \times e)$-frame.

2.4. Definition. Let $d, e \geqslant 1$. Fix a complete flag of vector bundles over X

(13)
$$0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_i \lhd \cdots \lhd \mathcal{V}_{d+e} =: \mathcal{V}\,.$$

Note that we baptize $\mathcal{V}$ the bundle of dimension $d + e$, to lighten notation.

Let Λ be a Young diagram in $(d{\times}e)$-frame. By Definition 2.3, this amounts to a pair $(\underline{d}, \underline{e})$ of k-tuples of integers satisfying (12), and hence satisfying (2). We can now apply Definition 1.6 to $\underline{d}$ and $\underline{e}$ and the flag (3) taken from (13)

above:
(14)

$$\mathcal{F}l_X(d,e,\mathcal{V}_\bullet;\Lambda) := \mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet) = \left\{ \begin{array}{ccccccc} 0 \lhd & \mathcal{P}_{d_1} & \lhd & \mathcal{P}_{d_2} \lhd & \cdots & \lhd & \mathcal{P}_{d_k} \\ & \vartriangle e_1 & & \vartriangle e_2 & & & \vartriangle e_k \\ 0 \lhd & \mathcal{V}_{d_1+e_1} & \lhd & \mathcal{V}_{d_2+e_2} \lhd & \cdots & \lhd & \mathcal{V}_{d_k+e_k} \end{array} \right\}.$$

As usual, instead of $\mathcal{F}l_X(d,e,\mathcal{V}_\bullet;\Lambda)$, we might simply write $\mathcal{F}l(\Lambda)$ or anything "in between" depending on what is obvious from the context.

As in (5), there is a natural morphism f_Λ from $\mathcal{F}l_X(d,e;\Lambda)$ to $\mathrm{Gr}(d,\mathcal{V})$,

(15)
$$f_\Lambda = f_{d,e;\Lambda} := f_{\underline{d},\underline{e},\mathcal{V}} : \begin{array}{ccc} \mathcal{F}l_X(d,e;\Lambda) & \longrightarrow & \mathrm{Gr}(d,\mathcal{V}) \\ (\mathcal{P}_{d_1},\ldots,\mathcal{P}_{d_k}) & \longmapsto & \mathcal{P}_{d_k}\,. \end{array}$$

When $X = \mathrm{Spec}(F)$ is a field, one can understand the image of f_Λ as the subset of those subspaces $\mathcal{P}_d \lhd \mathcal{V}$ whose intersection with each $\mathcal{V}_{d_i+e_i}$ is of dimension at least d_i. This is the classical Schubert cell associated to the diagram Λ. It is pretty clear that f_Λ is a birational morphism. The advantage of $\mathcal{F}l_X(d,e;\Lambda)$ over the Schubert cell is that $\mathcal{F}l_X(d,e;\Lambda)$ is not singular by Proposition 1.13.

2.5. Example. Following up on Example 1.7, when $\Lambda = \varnothing$ is the empty diagram, that is for $k=1$ and $e_1 = e$, we have $\mathcal{F}l_X(\varnothing) = \mathrm{Gr}_X(d,\mathcal{V})$ and $f_\varnothing$ is the identity. At the other end, for $\Lambda = [d\times e]$ the whole $(d\times e)$-rectangle, that is for $k=1$ and $e_1 = 0$, we have $\mathcal{F}l_X(d,e;\Lambda) = \mathrm{Gr}_X(d,\mathcal{V}_d) = X$ and f_Λ is a closed immersion.

2.6. Definition. Let Λ be a Young diagram in $(d\times e)$-frame. We define $\rho(\Lambda) \in \{0,\ldots,d\}$ to be the number of non-zero rows of Λ. Complementarily, we define $\zeta(\Lambda) = d - \rho(\Lambda)$ to be the number of zero rows at the end of Λ, that is

$$\begin{array}{llll} \rho(\Lambda) = d & \text{and} & \zeta(\Lambda) = 0 & \text{if } \Lambda_d > 0\,, \\ \rho(\Lambda) = d_{k-1} & \text{and} & \zeta(\Lambda) = d - d_{k-1} & \text{if } \Lambda_d = 0\,. \end{array}$$

For the empty diagram, we have $\rho(\varnothing) = 0$ and $\zeta(\varnothing) = d$.

We are going to use a certain class of (d,e)-diagrams, that we call the *even* (d,e)-diagrams. Defining them by a picture is very easy. The condition to be even is that any segment of the (d,e)-diagram which does not belong to the outer $(d\times e)$-frame must have even length; see Figure 2. The formal definition is the following.

2.7. Definition. Let Λ be a Young diagram in $(d\times e)$-frame and let $\underline{d}$ and $\underline{e}$ be the associated k-tuples as in Definition 2.3. We say that Λ is *even* if all the following conditions are satisfied:

(i) $d_{i+1} - d_i$ is even for all $i = 1, \ldots, k-2$ (for $k \geqslant 3$, otherwise no condition),
(ii) $e_{i+1} - e_i$ is even, for all $i = 1, \ldots, k-1$ (for $k \geqslant 2$),
(iii) when $0 < e_1 < e$ we also require d_1 to be even, and
(iv) when $0 < e_k < e$ we also require $d_k - d_{k-1}$ to be even.

2.8. Example. For any $d, e \geqslant 1$, both the empty diagram $\varnothing$ and the full-rectangle $[d \times e]$ are even (d, e)-diagrams (see Notation 2.2). Indeed, in both cases, $k = 1$ and $\underline{d} = (d)$, whereas $\underline{e}(\varnothing) = (e)$ and $\underline{e}([d \times e]) = (0)$; so there is no condition to check.

When $d = 1$ or $e = 1$, these are the only even Young diagram in $(d \times e)$-frame.

For more examples, the reader can find all even Young diagrams in the cases $(d, e) = (4, 4)$, $(4, 5)$ and $(5, 5)$ in Figures 14, 15 and 16, at the end of the paper.

2.9. Remark. Definition 2.7 depends on d and e as well as on the Young diagram Λ. For an even (d, e)-diagram Λ to remain even in a bigger frame, we might have one or two more conditions to check, namely (iii) or (iv) in Definition 2.7, in the case where Λ was touching the right border or the bottom border of its $(d \times e)$-frame.

2.10. Remark. For each even (d, e)-diagram we will construct an element in one of the Witt groups of the Grassmannian $\mathrm{Gr}_X(d, \mathcal{V})$. The proof that these Witt classes actually form a total basis will proceed by induction on $d + e = \mathrm{rk}(\mathcal{V})$, using the long exact sequence of localization associated to a natural "cellular" decomposition of the Grassmannians. In that proof, we shall need the description in terms of Young diagrams of the various Witt group homomorphisms appearing in that long exact sequence. As we shall see, these are the ones of the next proposition. This explains why the following constructions are relevant here.

2.11. Definition. Let Λ' be an even $(d, e-1)$-diagram with $d \geq 1$, $e \geq 2$ and such that $\zeta(\Lambda')$ is even. We define the (d, e)-diagram $\bar{\iota}(\Lambda')$ as

$$\bar{\iota}(\Lambda') = (\Lambda'_1 + 1, \ldots, \Lambda'_d + 1). \tag{16}$$

Let Λ be an even (d, e)-diagram with $d \geq 2$, $e \geq 1$ and such that $\Lambda_d = 0$. We define the $(d-1, e)$-diagram $\bar{\upsilon}(\Lambda)$ as

$$\bar{\upsilon}(\Lambda) = \Lambda_{|_{d-1,e}}. \tag{17}$$

Let Λ'' be an even $(d-1,e)$-diagram with $d,e \geq 2$ and such that Λ''_{d-1} is odd. We define the $(d,e-1)$-diagram $\bar{\partial}(\Lambda'')$ as

$$\bar{\partial}(\Lambda'') = (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0). \tag{18}$$

See Figures 4, 5 and 6.

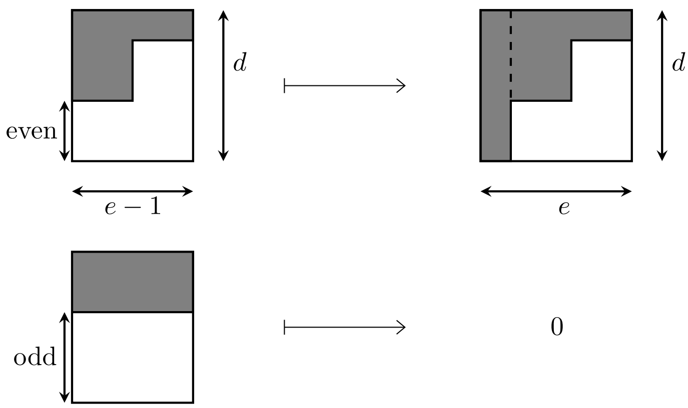


FIGURE 4. Morphism $\bar{\iota}$ on various $(d,e-1)$-diagrams Λ'.

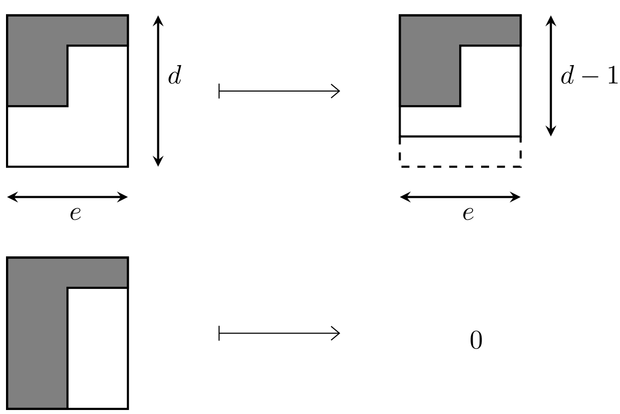


FIGURE 5. Morphism $\bar{\upsilon}$ on various (d,e)-diagrams Λ.

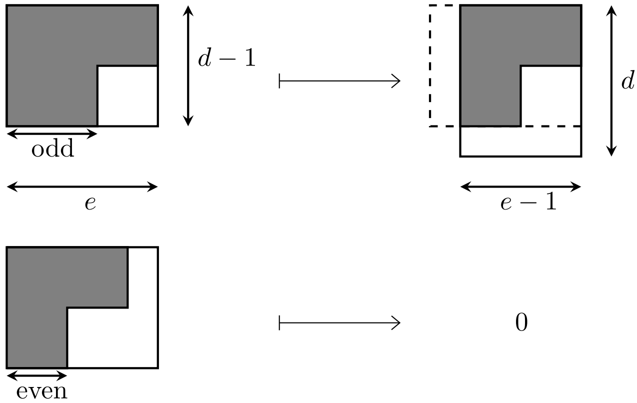


FIGURE 6. Morphism $\bar{\partial}$ on various $(d-1,e)$-diagrams Λ''.

2.12. Proposition. *(a) When $d \geq 1$ and $e \geq 2$, the map $\bar{\iota}$ defines a bijection*

$$\left\{\begin{array}{c}\textit{even Young } (d,e-1)\textit{-diagrams}\\ \Lambda' \textit{ such that } \zeta(\Lambda') \textit{ is even}\end{array}\right\} \overset{\simeq}{\longleftrightarrow} \left\{\begin{array}{c}\textit{even Young } (d,e)\textit{-diagrams}\\ \Lambda \textit{ such that } \Lambda_d > 0\end{array}\right\}$$

$$\begin{array}{ccc}\Lambda' & \longmapsto & \bar{\iota}(\Lambda') = (\Lambda'_1+1,\ldots,\Lambda'_d+1)\\ (\Lambda_1-1,\ldots,\Lambda_d-1) & \longleftarrow & \Lambda.\end{array}$$

(b) When $d \geq 2$ and $e \geq 2$, the map $\bar{\upsilon}$ defines a bijection

$$\left\{\begin{array}{c}\textit{even Young } (d,e)\textit{-diagrams}\\ \Lambda \textit{ such that } \Lambda_d = 0\end{array}\right\} \overset{\simeq}{\longleftrightarrow} \left\{\begin{array}{c}\textit{even Young } (d-1,e)\textit{-diagrams}\\ \Lambda'' \textit{ such that } \Lambda''_{d-1} \textit{ is even}\end{array}\right\}$$

$$\begin{array}{ccc}\Lambda & \longmapsto & \bar{\upsilon}(\Lambda) = \Lambda_{|_{d-1,e}}\\ (\Lambda''_1,\ldots,\Lambda''_{d-1},0) & \longleftarrow & \Lambda''.\end{array}$$

(c) When $d,e \geq 2$, the map $\bar{\partial}$ defines a bijection

$$\left\{\begin{array}{c}\textit{even Young } (d-1,e)\textit{-diagrams}\\ \Lambda'' \textit{ such that } \Lambda''_{d-1} \textit{ is odd}\end{array}\right\} \overset{\simeq}{\longleftrightarrow} \left\{\begin{array}{c}\textit{even Young } (d,e-1)\textit{-diagrams}\\ \Lambda' \textit{ such that } \zeta(\Lambda') \textit{ is odd}\end{array}\right\}$$

$$\begin{array}{ccc}\Lambda'' & \longmapsto & \bar{\partial}(\Lambda'') = (\Lambda''_1-1,\ldots,\Lambda''_{d-1}-1,0)\\ (1+\Lambda'_1,\ldots,1+\Lambda'_{d-1}) & \longleftarrow & \Lambda'.\end{array}$$

Proof. The proof essentially consists in checking that the announced constructions are well defined and that they preserve even diagrams. Checking that they are mutually inverse constructions is straightforward. The notation $\Lambda_{|_{d-1,e}}$ is the obvious one: we view a diagram with empty last row in a smaller frame. All this is most easily performed and followed on pictures. For instance, the maps from left to right are pictured in the upper parts of Figures 4, 5 and 6. □

3. Total bases and lax-similitude

3.1. Convention. From now on, X denotes a *regular* Noetherian $\mathbb{Z}[\frac{1}{2}]$-scheme of finite Krull dimension.

For precise statements and proofs of our results, it is convenient to use the language of total bases and lax similitude, as developed in [7]. Here is a brief list of the relevant facts.

We restrict to the subcategory $\mathcal{S}_X$ of X-schemes $\pi_Y : Y \to X$ that are smooth over X and that satisfy the following assumptions on Picard groups and global sections [7, Definition 4.1]:

(I) The map $\pi_Y^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective.

(II) The abelian group $\mathrm{Pic}_X(Y) := \mathrm{Pic}(Y)/\pi_Y^*\big(\mathrm{Pic}(X)\big)$ has no 2-torsion.

(III) The map $\pi_Y^* : \mathbb{G}_{\mathrm{m}}(X) \to \mathbb{G}_{\mathrm{m}}(Y)/\mathbb{G}_{\mathrm{m}}(Y)^2$ is surjective.

This ensures that the notions considered below are well behaved.

3.2. Remark. All schemes considered in the computations of the remaining Sections 4 to 6 are in the category $\mathcal{S}_X$. Indeed, most are flag varieties constructed iteratively from X as towers of Grassmann bundles (see Lemma 1.11), so the Picard group assumptions follow from Proposition 1.13. Similarly, Property (III) follows from [12, Theorem 2.3.1], i.e. in all our examples, $\mathbb{G}_{\mathrm{m}}(X) \to \mathbb{G}_{\mathrm{m}}(Y)$ is already surjective. The remaining schemes are vector bundles over these flag varieties, so each of them has the same Picard group and invertible global sections as its base.

3.3. Definition ([7, Definition 2.3])**.** Let L_1 and L_2 be line bundles over a scheme Y. An alignment $A : L_1 \rightsquigarrow L_2$ is a pair $A = (M, \psi)$ consisting of a line bundle M over Y together with an isomorphism $\psi : M^{\otimes 2} \otimes L_1 \xrightarrow{\sim} L_2$. Of course, such an alignment exists if and only if $[L_1] = [L_2]$ in $\mathrm{Pic}(Y)/2$. It induces an isomorphism on Witt groups

$$A^{\circlearrowleft} : \mathrm{W}^*(Y, L_1) \xrightarrow{\sim} \mathrm{W}^*(Y, L_2)$$

defined as the composition of multiplication by the form $M \xrightarrow{\sim} M^\vee \otimes M^{\otimes 2}$ (square periodicity) and of the identification of the dualities with values respectively in $M^{\otimes 2} \otimes L_1$ and in L_2 using ψ.

When Y is a scheme in $\mathcal{S}_X$, we also use a relative notion. An X-*alignment* from L_1 to L_2 is an alignment $A : \pi_Y^* K \otimes L_1 \rightsquigarrow L_2$ for some line bundle K over X. We denote this by $A : L_1 \underset{K}{\rightsquigarrow} L_2$.

3.4. Definition (see [7, Definition 2.5])**.** Two Witt classes $w_1 \in \mathrm{W}^j(Y, L_1)$ and $w_2 \in \mathrm{W}^j(Y, L_2)$ are *lax-similar* if there exists an alignment A such that $w_2 = A^{\circlearrowleft}(w_1)$. This is an equivalence relation written $w_1 \leftrightsquigarrow w_2$. Note also that $w_1 \leftrightsquigarrow w_2$ forces $[L_1] = [L_2]$ in $\mathrm{Pic}(Y)/2$.

3.5. Definition (see [7, Definition 3.4])**.** Let $f : \bar{Y} \to Y$, let L be a line bundle on Y and let $\bar{L}$ be a line bundle on $\bar{Y}$. If an alignment $\bar{A} : f^* L \rightsquigarrow \bar{L}$ exists, we define a *lax pull-back*

$$\bar{A}^{\circlearrowleft} \circ f^* \;:\quad \mathrm{W}^*_Z(Y, L) \longrightarrow \mathrm{W}^*_{f^{-1}Z}(\bar{Y}, f^* L) \xrightarrow{\sim} \mathrm{W}^*_{f^{-1}Z}(\bar{Y}, \bar{L})\,.$$

3.6. Remark. It is easy to see that two lax pull-backs (along the same morphism) of lax-similar elements are lax-similar.

3.7. Definition (see [7, Definition 3.5])**.** Similarly, if an alignment $\bar{A}$: $\bar{L} \rightsquigarrow \omega_f \otimes f^* L$ exists, we define a *lax push-forward*

$$f_* \circ \bar{A}^{\circlearrowleft} \;:\quad \mathrm{W}^{*+d}_{\bar{Z}}(\bar{Y}, \bar{L}) \xrightarrow{\sim} \mathrm{W}^{*+d}_{\bar{Z}}(\bar{Y}, \omega_f \otimes f^* L) \longrightarrow \mathrm{W}^*_{f\bar{Z}}(Y, L)\,.$$

The freedom in the use of lax push-forwards is summarized in the following:

3.8. Theorem. *Let $f : \bar{Y} \to Y$ be a morphism of schemes in $\mathcal{S}_X$ and let $\bar{L}$ be a line bundle over $\bar{Y}$.*

(a) *A lax push-forward starting from* $\mathrm{W}^*(\bar{Y}, \bar{L})$ *exists if and only if*

$$[\bar{L}] \in \mathrm{Im}\big([\omega_f] \otimes f^* : \mathrm{Pic}(Y)/2 \to \mathrm{Pic}(\bar{Y})/2\big) \tag{19}$$

or equivalently replacing $\mathrm{Pic}(-)/2$ *by* $\mathrm{Pic}_X(-)/2$.

(b) *Assuming* (19) *holds, the lax push-forward can be chosen to land in* $\mathrm{W}^*(Y, L)$ *for a line bundle L on Y if and only if*

$$[\omega_f \otimes f^* L] = [\bar{L}] \in \mathrm{Pic}(\bar{Y})/2. \tag{20}$$

(c) *Given two line bundles L_1 and L_2 on Y both satisfying* (20)*, lax push-forwards from* $\mathrm{W}^*(\bar{Y}, \bar{L})$ *to* $\mathrm{W}^*(Y, L_1)$ *and to* $\mathrm{W}^*(Y, L_2)$ *are lax-similar on Y (i.e. there exists an alignment $A^{\circlearrowleft}$:* $\mathrm{W}^*(Y, L_1) \to \mathrm{W}^*(Y, L_2)$ *turning one push-forward into the other) if and only if*

$$[L_1] = [L_2] \in \mathrm{Pic}(Y)/2 \tag{21}$$

or equivalently replacing $\mathrm{Pic}(-)/2$ *by* $\mathrm{Pic}_X(-)/2$*. This condition is automatically satisfied if $f^* : \mathrm{Pic}_X(Y)/2 \to \mathrm{Pic}_X(\bar{Y})/2$ is injective, so in that case, there is no need to specify the specific target of the lax push-forward if one is only interested in lax-similitude classes of the images.*

Proof. This is detailed in § 4 of [7]. The first two parts are straightforward from (1). Use [7, Lemma 4.3 (d)] to replace $\mathrm{Pic}(-)/2$ by $\mathrm{Pic}_X(-)/2$ in (a). To replace $\mathrm{Pic}(-)/2$ by $\mathrm{Pic}_X(-)/2$ in the last statement, use [7, Lemma 4.3 (c)], in which case [7, Proposition 4.7] gives that the images are lax-similar. □

3.9. Remark. Remark 3.6 and Theorem 3.8 mean that as long as one is only interested in elements up to lax-similarity, there is no need to be specific about where lax pull-backs and push-forwards start and land, as long as they exist. One only needs to keep track of classes of line bundles in $\mathrm{Pic}_X(-)/2$. See also [7, Remark 2.10] for connecting homomorphisms.

Let Y be a scheme in $\mathcal{S}_X$ and let Z be a closed subset of Y. Let $\mathcal{I}$ be a set. Given a family of line bundles $(L_i)_{i\in\mathcal{I}}$ and a class $p \in \mathrm{Pic}_X(Y)/2$, let $\mathcal{I}_p$ denote the subset of those $i \in \mathcal{I}$ such that $[L_i] = p$. Let $(w_i)_{i\in\mathcal{I}}$ be a family of Witt classes, for various shifts and twists: $w_i \in \mathrm{W}^{j_i}(Y, L_i)$.

3.10. Definition. Let L be a line bundle on Y and let $k \in \mathbb{Z}$ be an integer. Let $\mathcal{J} \subset \mathcal{I}_{[L]}$ be a finite subset. Given a family of X-alignments $(A_i : L_i \underset{K_i}{\rightsquigarrow} L)_{i\in\mathcal{J}}$ and a family of coefficients $(\lambda_i)_{i\in\mathcal{J}}$ with $\lambda_i \in \mathrm{W}^{k-j_i}(X, K_i)$, we can form the *linear combination*

$$\sum_{i\in\mathcal{J}} \lambda_i \underset{A_i}{\cdot} w_i := \sum_{i\in\mathcal{J}} A_i{}^{\circlearrowleft}\big(\pi_Y^*(\lambda_i)\cdot w_i\big),$$

which is an element of $\mathrm{W}^k(Y, L)$. See details in [7, § 6].

3.11. Definition. Let $P \subseteq \mathrm{Pic}_X(Y)/2$; typically P is the whole $\mathrm{Pic}_X(Y)/2$. The family $(w_i)_{i\in\mathcal{I}}$ is called a *total basis* of the P-part of the Witt groups of Y over X with support in Z, if $[L_i] \in P$ for all i and if for every line bundle L with $[L] \in P$, we have the following two properties:

(a) *Total generation*: Any element in $\mathrm{W}^k(Y, L)$ can be obtained as a linear combination of a finite subfamily $(w_i)_{i\in\mathcal{J}}$, i.e. alignments and coefficients as in Definition 3.10 can be found yielding the element as the linear combination.

(b) *Total independence*: Any linear combination of the w_i yielding the zero element in $\mathrm{W}^k(Y, L)$ has zero coefficients.

A total basis yields the following result, expressed in classical terms.

3.12. Theorem ([7, Proposition 6.9])**.** *For every line bundle L with $[L] \in P$, every $k \in \mathbb{Z}$ and* for every choice*, for those $i \in \mathcal{I}_{[L]}$, of a line bundle K_i over X and a K_i-alignment $C_i : L_i \underset{K_i}{\rightsquigarrow} L$, the following map is an isomorphism*

$$\begin{array}{rcl} \theta = \theta(C_\bullet) : \bigoplus_{i\in\mathcal{I}_{[L]}} \mathrm{W}^{k-j_i}(X, K_i) & \xrightarrow{\sim} & \mathrm{W}^k_Z(Y, L) \\ (x_i)_{i\in\mathcal{I}_{[L]}} & \longmapsto & \sum x_i \cdot_{C_i} w_i\,. \end{array} \tag{22}$$

3.13. Remark. In the same spirit, if one replaces elements of a total basis by lax-similar ones (Definition 3.5), they still form a total basis; see [7, Cor. 6.14].

Finally, here is a way to keep track of total bases along localization sequences. This will be our main tool to construct inductively total bases for Grassmann varieties, together with homotopy invariance and dévissage, under which total bases are naturally preserved (see [7, Corollaries 6.13 and 6.16] for precise statements).

Let U be the open complement of a closed subset $Z \subset Y$, and let $\upsilon : U \hookrightarrow Y$ be the corresponding open embedding. Assume $U \in \mathcal{S}_X$. Let $\mathrm{e} : \mathrm{W}^*_Z(Y, L) \to \mathrm{W}^*(Y, L)$ be the extension of the support map. Recall from [3] that there is a long exact sequence of localization

(23)

$$\cdots \longrightarrow \mathrm{W}^i_Z(Y, L) \xrightarrow{\mathrm{e}} \mathrm{W}^i(Y, L) \xrightarrow{\upsilon^*} \mathrm{W}^i(U, \upsilon^* L) \xrightarrow{\partial} \mathrm{W}^{i+1}_Z(Y, L) \to \cdots .$$

3.14. Theorem (see [7, Theorem 7.1])**.** *Let P be a subset of $\mathrm{Pic}_X(Y)/2$. Assume that the restriction $\upsilon^*_{|P} : P \to \mathrm{Pic}_X(U)/2$ is injective and let $P_U = \upsilon^*(P) \subset \mathrm{Pic}_X(U)/2$.*

Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be sets and let $(w'_i)_{i\in\mathcal{I}}$ and $(w_j)_{j\in\mathcal{J}}$ be Witt classes on Y, let $(v_i)_{i\in\mathcal{I}}$ and $(v'_k)_{k\in\mathcal{K}}$ be Witt classes on Y with support in Z and let $(u_k)_{k\in\mathcal{K}}$ and $(u'_j)_{j\in\mathcal{J}}$ be Witt classes on U, whose line bundles are restricted from Y. Suppose the following conditions hold (see Figure 7):

- *(a) for every $i \in \mathcal{I}$, we have lax-similitude $\mathrm{e}(v_i) \leftrightsquigarrow w'_i$;*
- *(b) for every $j \in \mathcal{J}$, we have lax-similitude $\upsilon^*(w_j) \leftrightsquigarrow u'_j$;*
- *(c) for every $k \in \mathcal{K}$, we have lax-similitude $\partial(u_k) \leftrightsquigarrow v'_k$.*

Then, the following properties are satisfied:

- *(1) for every $i \in \mathcal{I}$, we have $\upsilon^*(w'_i) = 0$;*
- *(2) for every $j \in \mathcal{J}$, we have $\partial(u'_j) = 0$;*
- *(3) for every $k \in \mathcal{K}$, we have $\mathrm{e}(v'_k) = 0$.*
- *(4) If, furthermore,*
  - *(i) the $(v_i)_{i\in\mathcal{I}}$ and $(v'_k)_{k\in\mathcal{K}}$ form together a total basis of the P-part of the Witt groups of Y with support in Z, over X,*
  - *(ii) the $(u_k)_{k\in\mathcal{K}}$ and $(u'_j)_{j\in\mathcal{J}}$ form together a total basis of the P_U-part of the Witt groups of U, over X,*

  then the $(w'_i)_{i\in\mathcal{I}}$ and $(w_j)_{j\in\mathcal{J}}$ form together a total basis of the P-part of the Witt groups of Y, over X.

Finally, we will need the following fact about push-forwards along blow-up:

3.15. Proposition. *Let X be a quasi-compact and quasi-separated scheme (e.g. an affine or a Noetherian scheme) and let $Z \hookrightarrow X$ be a regular immersion*

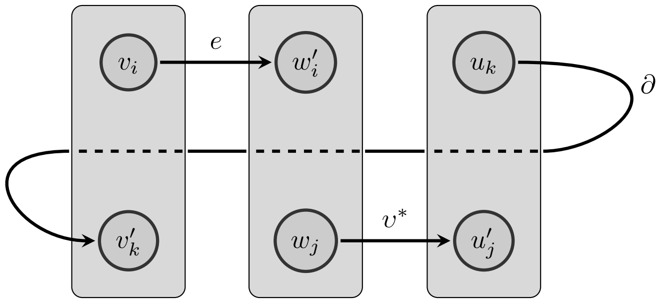


FIGURE 7. Families mapping to each other up to lax-similitude in Theorem 3.14. No arrow means mapped to zero.

of pure codimension d. Let $\pi : B \to X$ be the blow-up of X along Z. Then,

(a) *there is a natural isomorphism $R\pi_*(\mathcal{O}_B) \cong \mathcal{O}_X$ in the derived category of X;*

(b) *assume further that X is regular and that $\omega_{B/X}$ is a square, which happens exactly when d is odd by* [6, Proposition A.11(iii)]. *Then a lax push-forward (in the sense of Definition 3.7) $\mathrm{W}^0(B, \mathcal{O}_B) \to \mathrm{W}^0(X, \mathcal{O}_X)$ maps the unit class 1_B to an element lax-similar to the unit class 1_X.*

Proof. Part (a) can be found in SGA 6, see [2, Lemme VII.3.5, p. 441] or the more recent account in [19, Lemme 2.3 (a)].

Part (b) follows from (a) and holds at the level of symmetric forms already, before taking Witt classes. Indeed, when $d = 1$, we have $B = X$ and there is nothing to prove. When $d \geqslant 3$, then line bundles over X, and homomorphisms between them, are determined by their restriction to the open complement $U = X \smallsetminus Z$ of Z since Z is of codimension at least 2. The result follows by the base-change formula for push-forwards [8, Theorem 5.5] and by Theorem 3.8, since $\pi_{|_{\pi^{-1}(U)}} : \pi^{-1}(U) \to U$ is an isomorphism. □

4. Construction of the total basis

For this section, let Λ be a Young diagram in $(d \times e)$-frame and recall the k-tuples $\underline{d}$ and $\underline{e}$ associated to Λ in Definition 2.3.

4.1. Remark. Our goal is to construct classes in the total Witt group of $\mathrm{Gr}_X(d, \mathcal{V})$ by lax-pushing-forward the unit form $1 \in \mathrm{W}(\mathcal{F}l_X(\Lambda)) =$

$\mathrm{W}^0(\mathcal{F}l_X(\Lambda),\mathcal{O})$ along the morphism $f_\Lambda : \mathcal{F}l_X(\Lambda) \to \mathrm{Gr}_X(d,\mathcal{V})$ of (15). As recalled in Theorem 3.8(a), this lax push-forward only exists conditionally, namely only when the class of the relative canonical bundle $\omega_{\mathcal{F}l_X(\Lambda)/\,\mathrm{Gr}_X(d,\mathcal{V})}$ in $\mathrm{Pic}_X(\mathcal{F}l_X(\Lambda))/2$ belongs to the image of

$$(f_\Lambda)^* \ : \ \mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V}))/2 \longrightarrow \mathrm{Pic}_X(\mathcal{F}l_X(\Lambda))/2\,.$$

This is true if and only if the following conditions are satisfied:

(a) $d_i - d_{i-1} + e_{i+1} - e_i$ is even for every $i = 2,\dots,k-1$ (for $k \geqslant 3$);
(b) when $0 < e_1 < e$ and $k \geq 2$, require, moreover, $d_1 + e_2 - e_1$ even.

We shall be more precise in Proposition 4.8 below, but the reader can verify our claim using (10) in Corollary 1.14. For this, note that $\Delta_{d_1} = [\det \mathcal{V}_{d_1}]$ comes from X when $e_1 = 0$ and that Δ_d always comes from $\mathrm{Gr}_X(d,\mathcal{V})$ since $(f_\Lambda)^*\big(\Delta_d^{\mathrm{Gr}_X(d,\mathcal{V})}\big) = \Delta_d^{\mathcal{F}l_X(\Lambda)}$, as can be checked on the tautological bundles already.

Conditions (a) and (b) hold, in particular, when Λ is *even* in the sense of Definition 2.7. Indeed, for such Λ not only the sum $d_i - d_{i-1} + e_{i+1} - e_i$ is even but actually both terms $d_i - d_{i-1}$ and $e_{i+1} - e_i$ are. Compare Figure 8.

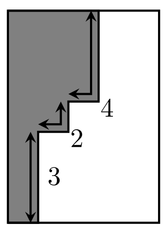


FIGURE 8. Framed Young diagram satisfying Conditions (a) and (b) of Remark 4.1 but which is not even (at all).

When (a) and (b) hold (e.g. for Λ even), there exists a line bundle L on $\mathrm{Gr}_X(d,\mathcal{V})$ such that $[\omega_{f_\Lambda}]\cdot f_\Lambda^*[L] = 1$ in $\mathrm{Pic}(\mathcal{F}l_X(\Lambda))/2$. Therefore, there is a lax push-forward 3.7 along $f = f_\Lambda$, i.e. a homomorphism:

$$\mathrm{W}^0\left(\mathcal{F}l(\Lambda),\mathcal{O}\right) \simeq \mathrm{W}^0\left(\mathcal{F}l(\Lambda),\omega_{f_\Lambda}\otimes f_\Lambda^* L\right) \xrightarrow{(f_\Lambda)_*} \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V}),L\right).$$

(Use that $\dim(f_\Lambda) = \dim \mathcal{F}l_X(\Lambda) - \dim \mathrm{Gr}_X(d,\mathcal{V}) = -|\Lambda|$ by Propositions 1.3 and 1.13.) Consequently, we can produce a Witt class over $\mathrm{Gr}_X(d,\mathcal{V})$, by pushing the unit form in the first group. This is what we are going to do below for Λ even, making the class of L in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$ more explicit in terms of the diagram Λ. Once this class of L in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$ or in $\mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V}))/2$ is fixed, the choices involved in the lax push-forward of

Definition 3.7 will be irrelevant. A nice fact is that this class in the Picard group can be read very easily on the diagram, as we now explain.

4.2. Remark. The perimeter of a Young diagram Λ is an even integer. Indeed, from the lower-left corner of Λ to its upper-right corner, there are two paths which follow the boundary (the upper path and the lower path) and they have the same length, namely the lattice distance between these two corners.

4.3. Definition. Let Λ be a Young diagram. We define $t(\Lambda) \in \mathbb{Z}/2$ to be the class of half the perimeter of Λ. From the above remark, $t(\Lambda)$ is also the class of the (lattice) distance from the lower-left corner of Λ to its upper-right corner. That is :

$$t(\Lambda) = [\Lambda_1 + \rho(\Lambda)] \in \mathbb{Z}/2$$

where $\rho(\Lambda)$ is the number of non-zero rows of Λ (Definition 2.6). Note that this Definition does not depend on an ambient frame.

4.4. Remark. On an even Young diagram Λ in $(d \times e)$-frame, there is another way to read $t(\Lambda) \in \mathbb{Z}/2$ on the diagram. Add the (parity of) the length of the segments where Λ touches the right and the bottom of the frame; see Figure 9. This is justified and generalized in Proposition 4.5.

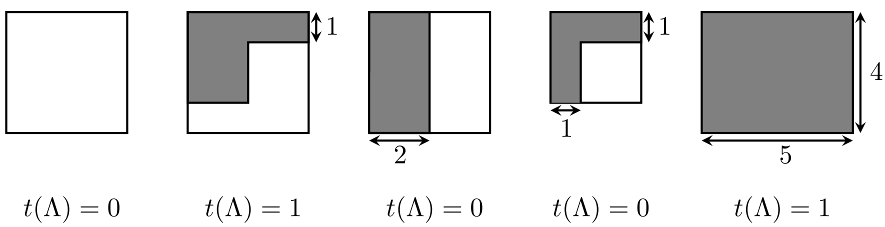


FIGURE 9. Class $t(\Lambda) \in \mathbb{Z}/2$, for different Λ.

4.5. Proposition. *Let Λ be an even Young diagram in $(d{\times}e)$-frame and let $\underline{d}, \underline{e}$ be the associated k-tuples (Definition 2.3). Then $t(\Lambda) = [d_i + (e - e_j)] \in \mathbb{Z}/2$ for any $i, j \in \{1, \ldots, k\}$ such that $e_i < e$ (only $i = k$ should be avoided when $e_k = e$).*

Proof. Measure the half-perimeter of Λ as the length of the lower boundary of Λ, from the lower-left corner of Λ to its upper-right corner (see Remark 4.2). Since Λ is even, all segments on this lower half-perimeter which are not on the outside frame have even length. So, the only two segments to contribute to the lower half-perimeter with possible odd length, are on the outside $(d{\times}e)$-frame, i.e. :

- the vertical segment most to the right, which has length d_1, and

- the lowest horizontal segment, which has length $e-e_k$ when Λ touches the lower part of the $(d\times e)$-frame (otherwise $e_k = e$ and this length is even).

In any case, this shows that $t(\Lambda) = [d_1+(e-e_k)] \in \mathbb{Z}/2$, that is, the announced formula for $i = 1$ and $j = k$. The other formulas follow from this one since by Definition 2.7 the successive differences $d_i - d_{i-1}$ and $e_{j+1} - e_j$ are even for all $i = 2,\dots,k-1$, for all $j = 1,\dots,k-1$, and also for $i = k$ when $e_k < e$. □

4.6. Definition. Let Λ be an even Young diagram in $(d\times e)$-frame. We define *the twist* $T(\Lambda)$ *of* Λ as the following class in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2 = \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2\cdot\Delta_d$ (see Corollary 1.4 and recall that Δ_d is the determinant of the tautological bundle):

$$T(\Lambda) = T(\Lambda,d,e) := [\det\mathcal{V}]^{\rho(\Lambda)}\cdot\Delta_d^{t(\Lambda)}\,,$$

where we recall that $t(\Lambda)$ is the half-perimeter of Λ modulo 2 (Definition 4.3) and that $\rho(\Lambda)$ is the number of non-zero rows of Λ (Definition 2.6).

4.7. Remark. The important part of $T(\Lambda)$ is of course $\Delta_d^{t(\Lambda)}$, which is not coming from the base X. Also, when $\mathcal{V}$ is trivial, the other term disappears anyway; this holds, in particular, with $X = \mathrm{Spec}(R)$ for a local ring R (e.g. a field).

4.8. Proposition. *Let Λ be an even Young diagram in $(d{\times}e)$-frame. Then*

$$[\omega_{f_\Lambda}]\cdot f_\Lambda^*\big(T(\Lambda)\big) = 1 \quad \textit{in} \quad \mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2\,.$$

Proof. Suppose first that $k = k(\Lambda)$ is at least 2. Remove from (10) all even exponents coming from the fact that Λ is even and use Proposition 4.5 for $i = k-1$ and $j = k$. This gives in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$:

$$[\omega_{f_\Lambda}] = [\det\mathcal{V}_{d_1+e_1}]^{d_1}\cdot[\det\mathcal{V}_{d+e_k}]^{d+d_{k-1}}\cdot[\det\mathcal{V}]^d\cdot\Delta_{d_1}^{d_1}\cdot\Delta_d^{t(\Lambda)}\,.$$

Now observe that $[\det\mathcal{V}_{d_1+e_1}]^{d_1}\cdot\Delta_{d_1}^{d_1} = 1$ in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$. Indeed, either $e_1 > 0$, hence d_1 is even, or $e_1 = 0$, hence $\Delta_{d_1} = [\det\mathcal{V}_{d_1}]$ by Remark 1.10. So, we can simplify the above equation in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$:

$$[\omega_{f_\Lambda}] = [\det\mathcal{V}_{d+e_k}]^{d+d_{k-1}}\cdot[\det\mathcal{V}]^d\cdot\Delta_d^{t(\Lambda)}\,.$$

Now, if $e_k < e$, then $d - d_{k-1} = d_k - d_{k-1}$ is even and $\rho(\Lambda) = d$; on the other hand, if $e_k = e$, then $\rho(\Lambda) = d_{k-1}$. In both cases, the above expression becomes $[\det\mathcal{V}]^{\rho(\Lambda)}\cdot\Delta_d^{t(\Lambda)}$, which is $T(\Lambda)$ by Definition 4.6.

Similarly, the case $k = 1$ is an easy consequence of Corollary 1.14. □

Let Λ be an even Young diagram in $(d{\times}e)$-frame. Choose L_Λ a line bundle of class $T(\Lambda)$ in $\mathrm{Pic}\big(\mathrm{Gr}_X(d,\mathcal{V})\big)/2$. (For simplicity, choose $L_{\varnothing} = \mathcal{O}$.) By the equality of Proposition 4.8, there exists an isomorphism $\psi_\Lambda : M_\Lambda^{\otimes 2} \xrightarrow{\sim} \omega_{f_\Lambda}\otimes f_\Lambda^* L_\Lambda$ over $\mathcal{F}l_X(\Lambda)$, i.e. an alignment $A_\Lambda = (M_\Lambda,\psi_\Lambda) : \mathcal{O}_{\mathcal{F}l_X(\Lambda)} \rightsquigarrow \omega_{f_\Lambda}\otimes$

$f_\Lambda^* L_\Lambda$, which induces a lax push-forward along the morphism $f_\Lambda : \mathcal{F}l_X(\Lambda) \longrightarrow \mathrm{Gr}_X(d,\mathcal{V})$, as in Definition 3.7:

$$(f_\Lambda)_* \circ A_\Lambda{}^{\circlearrowleft} : \mathrm{W}^0\left(\mathcal{F}l_X(\Lambda),\mathcal{O}\right) \longrightarrow \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V}), L_\Lambda\right). \tag{24}$$

4.9. Definition. For any even Young diagram Λ in $(d{\times}e)$-frame, we define

$$\phi_{d,e}(\Lambda) \in \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V})\,,\, L_\Lambda\right) \tag{25}$$

as the image of the unit form $1 \in \mathrm{W}^0(\mathcal{F}l_X(\Lambda),\mathcal{O})$ by the lax push-forward constructed in (24).

4.10. Remark. The definition of $\phi_{d,e}(\Lambda)$ involves the choice of L_Λ in the class $T(\Lambda)$ and the choices of M_Λ and of the isomorphism $\psi_\Lambda : M_\Lambda^{\otimes 2} \xrightarrow{\sim} \omega_{f_\Lambda} \otimes f_\Lambda^* L_\Lambda$. By Theorem 3.8(c), the lax-similitude class of $\phi_{d,e}(\Lambda)$ is independent of these choices. By Remark 3.13, our main Theorem 6.1 (that they form a total basis) will hold regardless of these choices. For these reasons, and also to lighten notation, we do not incorporate these choices in the notation $\phi_{d,e}(\Lambda)$.

4.11. Example. Let $\Lambda = \varnothing$ be the empty Young diagram in $(d{\times}e)$-frame, which is even, as we know. Then $T(\Lambda)$ is trivial and $f_{\varnothing} : \mathcal{F}l_X(d,e;\varnothing) \to \mathrm{Gr}_X(d,\mathcal{V})$ is the identity, so $\omega_{f_\Lambda} = \mathcal{O}$ and $\phi_{d,e}(\varnothing) = 1 \in \mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}),\mathcal{O})$ is the unit form on the Grassmannian if $L_\Lambda = M_\Lambda = \mathcal{O}$ and the isomorphism $M_\Lambda^{\otimes 2} \simeq \omega_{f_\Lambda} \otimes f_\Lambda^* L_\Lambda$ is the trivial one. (Otherwise, $\phi_{d,e}(\varnothing)$ is only lax-similar to the unit form.)

4.12. Remark. By Proposition 3.15, the lax-similitude class of our Witt classes $\phi_{d,e}(\Lambda)$ does not really depend on the chosen desingularization $\mathcal{F}l(\Lambda)$ of the Schubert subvariety corresponding to the Young diagram Λ. Indeed the unit will remain the unit when pushed-forward between two such desingularizations, if one is obtained from the other by blow-up. (The condition on the relative bundle being even in Proposition 3.15 is automatically satisfied if both desingularizations have even relative bundle with respect to the Grassmannian.)

5. Cellular decomposition

We describe the usual relative cellular decomposition of Grassmannians. Fix $d, e \geqslant 2$ for the whole section.

5.1. Notation. Fix a complete flag $\mathcal{V}_\bullet$ of vector bundles on X,

$$0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_{d+e} = \mathcal{V}\,,$$

as in (13). We set $\mathcal{V}^1 = \mathcal{V}_{d+e-1}$ to be the chosen codimension one subbundle of $\mathcal{V}$. We have an obvious closed immersion $\mathrm{Gr}_X(d,\mathcal{V}^1) \hookrightarrow \mathrm{Gr}_X(d,\mathcal{V})$, of codimension d, whose open complement is denoted by $U_X(d,\mathcal{V}_\bullet)$.

5.2. Notation. Let $\mathcal{P}_d \lhd \mathcal{V}$ be a subbundle of rank d. We write $\mathcal{P}_d \dot{\not\lhd} \mathcal{V}^1$ to express that $\mathcal{P}_d$ is not a subbundle of $\mathcal{V}^1$ but moreover satisfies the equivalent conditions:

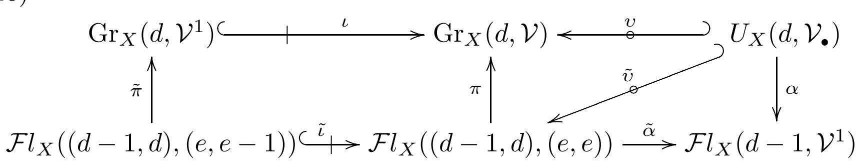
(a) The natural map from $\mathcal{P}_d/(\mathcal{P}_d \cap \mathcal{V}^1) = (\mathcal{P}_d + \mathcal{V}^1)/\mathcal{V}^1$ into $\mathcal{V}/\mathcal{V}^1$ is an isomorphism.

(b) $\mathcal{P}_d \cap \mathcal{V}^1$ is a subbundle of $\mathcal{P}_d$ (in the strong sense of Definition 1.1).

Over a field, this amounts to $\mathcal{P}_d \not\subset \mathcal{V}^1$ but this is not sufficient in general.

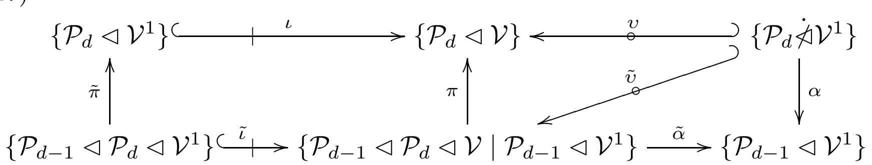
5.3. Definition. Using the notation of Section 1, we have a commutative diagram,

(26)



which looks as follows on points:

(27)



Here ι, $\tilde{\iota}$, π, $\tilde{\pi}$ and $\tilde{\alpha}$ are the obvious morphisms. The morphism $\tilde{\upsilon}$ maps $\mathcal{P}_d$ to the flag $\mathcal{P}_{d-1} \lhd \mathcal{P}_d$ with $\mathcal{P}_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ (see Notation 5.2). Finally α is defined as $\tilde{\alpha} \circ \tilde{\upsilon}$.

5.4. Proposition. *In Diagram* (26)*, the scheme* $\mathcal{F}l_X((d-1,d),(e,e))$ *is the blow-up of* $\mathrm{Gr}_X(d,\mathcal{V})$ *along* $\mathrm{Gr}_X(d,\mathcal{V}^1)$ *with exceptional fiber* $\mathcal{F}l_X((d-1,d),(e,e-1))$*.*

Proof. This is probably folklore to algebraic geometers. By compatibility of blow-ups with pull-backs, we can reduce to the case where X is affine (even $X = \mathrm{Spec}(\mathbb{Z})$) and suppose that $\mathcal{V}$ is free and that $\mathcal{V}^1 = \ker(\mathcal{V} \twoheadrightarrow \mathcal{O})$ is the kernel of a (split) epimorphism to $\mathcal{O}$. We omit X in the notation for the rest of the proof.

Let us check that $B := \mathcal{F}l((d-1,d),(e,e))$ has the universal property of the blow-up (see [17, § 8.1.2, Corollary 1.16]), i.e. it is final among schemes over $\mathrm{Gr}(d,\mathcal{V})$ in which the preimage of $Z := \mathrm{Gr}(d,\mathcal{V}^1)$ is an effective Cartier divisor (i.e. a codimension one closed subscheme locally given by a principal ideal). Let us first check that B indeed has this property. Note that the left-hand square of (26) is Cartesian. Moreover, we have an identification $B = \mathcal{F}l((d-1,d),(e,e)) = \mathbb{P}_Y(\mathcal{V}/\mathcal{T}_{d-1})$ where $Y = \mathrm{Gr}(d-1,\mathcal{V}^1)$ as in Lemma 1.11. Under this identification, the inverse image $\mathcal{F}l((d-1,d),(e,e-1))$ of Z becomes $\mathbb{P}_Y(\mathcal{V}^1/\mathcal{T}_{d-1})$. So this inverse image is locally $\mathbb{P}^{e-1}_Y \subset \mathbb{P}^e_Y$ hence an effective Cartier divisor, as wanted.

Suppose now that $f : W \to \mathrm{Gr}_X(d,\mathcal{V})$ is a morphism for which $f^{-1}(\mathrm{Gr}_X(d,\mathcal{V}^1))$ is an effective Cartier divisor. Let us consider W as a functor of points and show that there exists a unique morphism $g : W \to B$ such that $\tilde{\pi} \circ g = f$. Consider on $\mathrm{Gr}(d,\mathcal{V})$, the morphism $s : \mathcal{T}_d \to \mathcal{O}$ obtained by composing the inclusion $\mathcal{T}_d \rightarrowtail \mathcal{V}$ and the projection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}^1 = \mathcal{O}$. By definition, Z is the zero locus of this morphism, i.e. it is defined by the ideal $\mathrm{im}(s) \subset \mathcal{O}_{\mathrm{Gr}(d,\mathcal{V})}$. By assumption on $f : W \to \mathrm{Gr}_X(d,\mathcal{V})$, the ideal $\mathrm{im}(f^*(s)) = f^{-1}(\mathrm{im}(s)) \subset \mathcal{O}_W$ is invertible. Hence, over each point $\mathrm{Spec}(R) \to W$ of W, $\ker(s_{|_R})$ is a codimension one subbundle of $\mathcal{T}_d$, which is contained in $\mathcal{V}^1$ by construction. This defines the wanted morphism $g : W \to B$ sending each $\mathrm{Spec}(R) \to W$ to $\ker(s_{|_R}) \lhd \mathcal{P}_d$. Uniqueness is easy. If $g : W \to B$ satisfies $g \circ \tilde{\pi} = f$, then g maps a point $w : \mathrm{Spec}(R) \to W$ to $\mathcal{P}_{d-1} \lhd \mathcal{P}_d$ with $\mathcal{P}_d = f(w)$ forced. On the other hand $\mathcal{P}_{d-1} \subset \mathcal{V}^1$ forces $\mathcal{P}_{d-1}$ to be in $\ker(s_{|_R})$, hence to be equal to it by dimension counting. ☐

5.5. Definition. Let $B_X(d,\mathcal{V}_\bullet) = \mathcal{F}l_X((d-1,d),(e,e))$ be the blow-up of $\mathrm{Gr}_X(d,\mathcal{V})$ along $\mathrm{Gr}_X(d,\mathcal{V}^1)$ and let $E_X(d,\mathcal{V}_\bullet) = \mathcal{F}l_X((d-1,d),(e,e-1))$ be the exceptional fiber. By (26), $\mathrm{Gr}_X(d,\mathcal{V})$ now has a decomposition as in [6, Hypothesis 1.2], namely there exists an auxiliary morphism $\tilde{\alpha} : B_X(d,\mathcal{V}) \to Y := \mathrm{Gr}_X(d-1,\mathcal{V}^1)$ from the blow-up to another scheme Y, such that $\alpha := \tilde{\alpha} \circ \tilde{v}$ is an $\mathbb{A}^*$-bundle:

$$
\begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}^1) & \xrightarrow{\ \iota\ } & \mathrm{Gr}_X(d,\mathcal{V}) & \xleftarrow{\ v\ } & U_X(d,\mathcal{V}_\bullet) \\
\uparrow{\scriptstyle \tilde{\pi}} & & \uparrow{\scriptstyle \pi} & \swarrow{\scriptstyle \tilde{v}} & \downarrow{\scriptstyle \alpha} \\
E_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \tilde{\iota}\ } & B_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \tilde{\alpha}\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1)\,.
\end{array}
\tag{28}
$$

Indeed, α is an $\mathbb{A}^e$-bundle because of the canonical isomorphism between $U_X(d,\mathcal{V}_\bullet)$ and $\mathbb{P}_Y(\mathcal{V}/\mathcal{T}^Y_{d-1}) \smallsetminus \mathbb{P}_Y(\mathcal{V}^1/\mathcal{T}^Y_{d-1})$, under which α corresponds to the structure morphism to Y.

5.6. Remark. We compute the relevant Picard groups and canonical bundles, via the methods of Section 1. Let us start with Picard groups, using (8).

Since $\mathrm{Pic}(X)$ is a direct summand of the Picard group of all schemes in (26), we focus on the relative Picard groups $\mathrm{Pic}_X(-) := \mathrm{Pic}(-)/\mathrm{Pic}(X)$. Then "$\mathrm{Pic}_X(-)$ of (26)" equals

(29)

$$\begin{array}{ccccc}
\mathbb{Z}\Delta_d & \xleftarrow{\ \iota^*=1\ } & \mathbb{Z}\Delta_d & \xrightarrow{\ \upsilon^*=1\ } & \mathbb{Z}\upsilon^*(\Delta_d) = \mathbb{Z}\alpha^*(\Delta_{d-1}) \\
{\scriptstyle \tilde{\pi}^*=\binom{0}{1}}\big\downarrow & & {\scriptstyle \pi^*=\binom{0}{1}}\big\downarrow & \nearrow{\scriptstyle \tilde{\upsilon}^*=(1\ 1)} & \big\uparrow{\scriptstyle \alpha^*=1} \\
\mathbb{Z}\Delta_{d-1}\oplus\mathbb{Z}\Delta_d & \xleftarrow[\ \tilde{\iota}^*=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\ ]{} & \mathbb{Z}\Delta_{d-1}\oplus\mathbb{Z}\Delta_d & \xleftarrow[\ \tilde{\alpha}^*=\binom{1}{0}\ ]{} & \mathbb{Z}\Delta_{d-1}.
\end{array}$$

(In the case $X = \mathrm{Spec}(R)$ for a local ring R, the Picard groups are exactly as above.) Here we used that the closed subscheme $\mathrm{Gr}_X(d, \mathcal{V}^1)$ is of codimension $d \geqslant 2$ in $\mathrm{Gr}_X(d, \mathcal{V})$ to see that $\upsilon^* : \mathrm{Pic}(\mathrm{Gr}_X(d, \mathcal{V})) \cong \mathrm{Pic}(U_X(d, \mathcal{V}_\bullet))$. We also used that $e \geqslant 2$, otherwise $e - 1 = 0$ and $\Delta_{d-1} \in \mathrm{Pic}(X)$ by Remark 1.10. (When $d = 1$, respectively $e = 1$, we loose all components $\mathbb{Z}\Delta_{d-1}$, respectively all components $\mathbb{Z}\Delta_d$ in the left column, in the previous diagram.) Alternatively, "$\mathrm{Pic}_X((26)) = (29)$" follows from the computation of the Picard groups provided in [6, Proposition A.6]. Finally, the maps into the upper right corner of Diagram (29) are deduced from

$$\upsilon^*(\Delta_d) \cdot \alpha^*(\Delta_{d-1})^{-1} = [\mathcal{V}/\mathcal{V}^1]\,, \tag{30}$$

which itself follows from Condition (a) in Notation 5.2.

We shall use push-forwards along some morphisms of (26). The classes of the relevant relative canonical bundles in the respective (plain) Picard groups are:

$$[\omega_\iota] = [\mathcal{V}/\mathcal{V}^1]^d \cdot \Delta_d^{-1}, \tag{31}$$

$$[\omega_\pi] = [\mathcal{V}/\mathcal{V}^1]^{d-1} \cdot \Delta_{d-1}^{d-1} \cdot \Delta_d^{1-d}, \tag{32}$$

$$[\omega_{\tilde{\iota}}] = [\mathcal{V}/\mathcal{V}^1] \cdot \Delta_{d-1} \cdot \Delta_d^{-1}, \tag{33}$$

$$[\omega_{\tilde{\pi}}] = \Delta_{d-1}^{d} \cdot \Delta_d^{1-d}\,. \tag{34}$$

Indeed, Corollary 1.14 gives (31), (32) and $[\omega_{\iota\tilde{\pi}}] = [\omega_{\pi\tilde{\iota}}] = [\mathcal{V}/\mathcal{V}^1]^d \cdot \Delta_{d-1}^d \cdot \Delta_d^{-d}$, out of which the other two follow by multiplicativity of ω_-. Again, when the fixed complete flag $\mathcal{V}_\bullet = \mathcal{O}_X^\bullet$ is trivial, the "noise" $\mathcal{V}/\mathcal{V}^1$ vanishes.

We end this section with two geometric lemmas which will be useful in the proof of the main theorem.

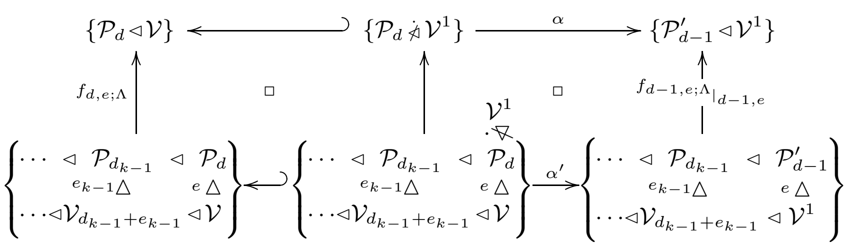
5.7. Lemma. *Let $d, e \geqslant 2$ and let Λ be an even Young (d, e)-diagram with empty last row (i.e. $\Lambda_k = 0$, i.e. $\zeta(\Lambda) > 0$). Hence $\Lambda_{|_{d-1,e}}$ is an even $(d-1, e)$-diagram. Then the base-changes to $U_X(d, \mathcal{V}_\bullet)$ of the morphisms $f_{d,e;\Lambda}$*

and $f_{d-1,e;\Lambda_{|d-1,e}}$ *coincide, that is, we have two Cartesian squares:*

$$\begin{array}{ccccc} \mathrm{Gr}_X(d,\mathcal{V}) & \xleftarrow{\ \upsilon\ } & U_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \alpha\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1) \\ \uparrow{\scriptstyle f_{d,e;\Lambda}} & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda_{|d-1,e}}} \\ \mathcal{F}l_X(d,e;\Lambda) & \hookleftarrow & U' & \xrightarrow{\ \alpha'\ } & \mathcal{F}l_X(d-1,e;\Lambda_{|d-1,e})\,. \end{array} \tag{35}$$

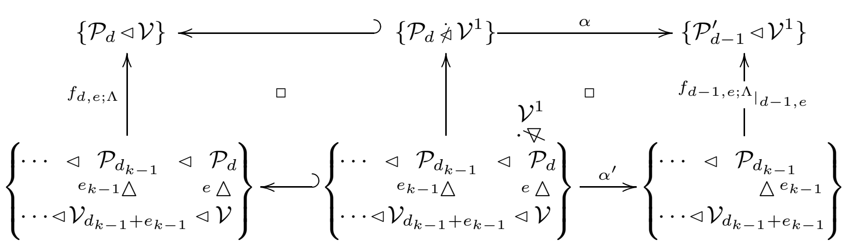
Proof. Let us check this on points. Let $\underline{d}$ and $\underline{e}$ be the k-tuples associated to Λ as usual (Definition 2.3). We need to distinguish two cases, namely $d_k - d_{k-1} > 1$ and $d_k - d_{k-1} = 1$.

When $d_k > d_{k-1} + 1$, that is, when there is more than one zero line at the end of Λ (i.e. $\zeta(\Lambda) > 1$), we then have $k(\Lambda_{|d-1,e}) = k(\Lambda) = k$ and the k-tuples $\underline{d}(\Lambda_{|d-1,e})$ and $\underline{e}(\Lambda_{|d-1,e})$ are almost the same as $\underline{d}$ and $\underline{e}$ except for the last entry of $\underline{d}(\Lambda_{|d-1,e})$ which becomes $d-1$. Diagram (35) then looks as follows on points (as usual the $\mathcal{P}_i$ and $\mathcal{P}'_j$ are "variables" whereas the $\mathcal{V}_i$ belong to the fixed complete flag):



where the morphisms α send $\mathcal{P}_d$ to $\mathcal{P}'_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ and similarly for α' .

On the other hand, when $d_k = d_{k-1} + 1$, that is, when Λ has only one zero line (i.e. $\zeta(\Lambda) = 1$), then we have $k(\Lambda_{|d-1,e}) = k(\Lambda) - 1 = k - 1$ and the $(k-1)$-tuples $\underline{d}(\Lambda_{|d-1,e})$ and $\underline{e}(\Lambda_{|d-1,e})$ are respectively $\underline{d}$ and $\underline{e}$ truncated from their last entry. Diagram (35) then looks as follows on points:



where α still sends $\mathcal{P}_d$ to $\mathcal{P}'_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ and where $f_{d-1,e;\Lambda_{|d-1,e}}$ sends a flag to $\mathcal{P}_{d_{k-1}}$. Note that in this case, α' drops the last subspace $\mathcal{P}_d$ in the flag.

In both cases, it is easy to check that the two squares are Cartesian. $\square$

5.8. Lemma. *Let $d, e \geqslant 2$ and let Λ'' be an even $(d-1,e)$-diagram such that Λ''_{d-1} is odd. Hence we can consider the even $(d,e-1)$-diagram $\Lambda' = (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0)$. Then, there exists a commutative diagram*

$$
(36)\qquad
\begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}^1) & \xleftarrow{\ \tilde\pi\ } & E_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \tilde\alpha\,\tilde\iota\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1) \\
{\scriptstyle f_{d,e-1;\Lambda'}}\uparrow & & {\scriptstyle f'}\uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda''}} \\
\mathcal{F}l_X(d,e-1;\Lambda') & \xleftarrow[\ \pi'\ ]{} & F' & \longrightarrow & \mathcal{F}l_X(d-1,e;\Lambda'')
\end{array}
$$

where $E_X(d,\mathcal{V}_\bullet)$ is the exceptional fiber of Diagram (26) and where the right-hand square is Cartesian. Moreover, either π' is an isomorphism or the scheme F' (with the morphism π') identifies with the blow-up of $\mathcal{F}l_X(d,e-1;\Lambda')$ along a closed regular subscheme of odd codimension.

Proof. Let $k = k(\Lambda'')$, $\underline{d} = \underline{d}(\Lambda'')$ and $\underline{e} = \underline{e}(\Lambda'')$ as usual (Definition 2.3). We need to distinguish two cases, namely $\Lambda''_{d-1} > 1$ and $\Lambda''_{d-1} = 1$.

Suppose first that $\Lambda''_{d-1} > 1$. Then $k(\Lambda') = k+1$ and $\underline{d}(\Lambda')$ and $\underline{e}(\Lambda')$ are just $\underline{d}(\Lambda'')$ and $\underline{e}(\Lambda'')$ with one more entry at the end, namely d and $e-1$ respectively. We can describe the pull-back in Diagram (36) as follows (on points):

$$
\begin{array}{ccc}
\{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} & \longrightarrow & \{\mathcal{P}_{d-1} \lhd \mathcal{V}^1\} \\
{\scriptstyle f'}\uparrow & \square & \uparrow \\
\left\{\begin{array}{ccccccc}
\mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\
\triangle & & \triangle & & \triangle \\
\mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d-1+e_k} & \lhd & \mathcal{V}^1
\end{array}\right\}
& \longrightarrow &
\left\{\begin{array}{ccc}
\mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} \\
\triangle & & \triangle \\
\mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d-1+e_k}
\end{array}\right\}
\end{array}
$$

This pull-back F' is $\mathcal{F}l_X(d,e-1;\Lambda')$. So, take $\pi' = \mathrm{id}$. The morphism f' composed with $\tilde\pi$ sends a flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d-1} \lhd \mathcal{P}_d$ to $\mathcal{P}_d$ and so does $f_{d,e-1;\Lambda'}$.

Suppose now that $\Lambda''_{d-1} = 1$. Then $k(\Lambda') = k$ and $\underline{e}(\Lambda') = \underline{e}(\Lambda'')$, whereas $\underline{d}(\Lambda')$ is obtained from $\underline{d}(\Lambda'')$ by replacing its last entry by d. We can describe the pull-back in Diagram (36) as follows:

$$
\begin{array}{ccc}
\{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} & \longrightarrow & \{\mathcal{P}_{d-1} \lhd \mathcal{V}^1\} \\
{\scriptstyle f'}\uparrow & \square & \uparrow \\
\left\{\begin{array}{ccccccccc}
\mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\
\triangle & & \triangle & & \triangle & & \triangle \\
\mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}_{d+e-2} & \lhd & \mathcal{V}^1
\end{array}\right\}
& \longrightarrow &
\left\{\begin{array}{ccc}
\mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} \\
\triangle & & \triangle \\
\mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d+e-2}
\end{array}\right\}
\end{array}
$$

where f' is the obvious morphism. The left-hand square of (36) is defined by:

$$\begin{array}{ccc}
\{\mathcal{P}_d \lhd \mathcal{V}^1\} & \xleftarrow{\quad\tilde{\pi}\quad} & \{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} \\
\uparrow & & \uparrow{\scriptstyle f'} \\
\left\{\begin{array}{ccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_d \\ \vartriangle & & \vartriangle & & \vartriangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}^1 \end{array}\right\} & \xleftarrow{\pi'} & \left\{\begin{array}{ccccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\ \vartriangle & & \vartriangle & & \vartriangle & & \vartriangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}_{d+e-2} & \lhd & \mathcal{V}^1 \end{array}\right\}
\end{array}$$

The morphism $\pi' : F' \to \mathcal{F}l_X(d, e-1; \Lambda')$ simply drops $\mathcal{P}_{d-1}$ in this case.

Let $\mathcal{V}^2 := \mathcal{V}_{d+e-2}$ and $Y := \mathcal{F}l_X((d_1, \ldots, d_{k-1}), (e_1, \ldots, e_{k-1}), \mathcal{V}_\bullet)$. We have $\mathcal{F}l_X(d, e-1; \Lambda') = \mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ by Lemma 1.11. As in Definition 5.5, we consider the blow-up $B_Y(d - d_{k-1}\,,\, \mathcal{V}^2/\mathcal{T}_{d_k-1} \lhd \mathcal{V}^1/\mathcal{T}_{d_k-1})$ of $\mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ along the closed regular immersion of $\mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^2/\mathcal{T}_{d_{k-1}})$. By Proposition 5.4, this blow-up coincides with F' and the morphism π of (27) here becomes the above morphism π'. In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}l_X(d, e-1; \Lambda') & \xleftarrow{\qquad\pi'\qquad} & F' \\
\| & & \| \\
\mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}}) & \xleftarrow{\text{“}\pi\text{”}} & B_Y(d - d_{k-1},\ \mathcal{V}^2/\mathcal{T}_{d_k-1} \lhd \mathcal{V}^1/\mathcal{T}_{d_{k-1}})\,.
\end{array}$$

The closed immersion $\mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^2/\mathcal{T}_{d_{k-1}}) \hookrightarrow \mathrm{Gr}_Y(d - d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ is of odd codimension equal to $d - d_{k-1}$. □

6. Main result

We are now ready to state and prove the main result of the paper.

6.1. Theorem. *Let $d, e \geqslant 1$. Let X be a regular scheme over $\mathbb{Z}[\frac{1}{2}]$. Let $\mathcal{V}$ be a vector bundle of rank $d+e$ which admits a complete flag* (13) *of subbundles (for instance, $\mathcal{V}$ free). Then, the elements $\phi_{d,e}(\Lambda)$ of Definition* 4.9*, for Λ even, form a total basis of the Witt groups of* $\mathrm{Gr}_X(d, \mathcal{V})$ *over X.*

Theorem 6.1 is unchanged when the $\phi_{d,e}(\Lambda)$ are changed to lax-similar elements, so it holds independently of all choices in the construction of $\phi_{d,e}(\Lambda)$, as mentioned in Remark 4.10. Furthermore, lax pull-backs, push-forwards and connecting homomorphisms are compatible with lax-similarity by Remark 3.9. We therefore use the following convention:

6.2. Convention. All pull-backs, push-forwards and connecting homomorphisms in the rest of Section 6 are *lax*.

This precisely means that they are classical pull-backs, push-forwards and connecting homomorphisms, preceded or followed by alignment morphisms in order to start and end in Witt groups with specific line bundles, chosen for the various statements to make sense. These alignments exist and the freedom in their choices has no influence on the lax-similarity classes considered, by the results of [7], as summarized in Section 3.

6.3. Remark. The case $d = 1$ or $e = 1$ is that of the projective bundle $\mathbb{P}(\mathcal{V})$. In that case, Walter [20] has proved such a result (see [20, Proposition 8.1] when $\mathcal{V}$ has a complete flag). One can check that his generator ξ is lax-similar in the sense of Definition 3.4 to our $\phi(\Lambda)$ when $\Lambda = [d \times e]$ is the only non-empty even diagram in $(d \times e)$-frame. The same result was also proved in the case $\mathcal{V} = \mathcal{O}_X^{d+e}$ in [5], using methods closer to the present geometric philosophy. We could therefore assume the starting point of the induction (i.e. the case $d = 1$ or $e = 1$). However our proof of the induction step covers this “fringe” case as well.

Proof of Theorem 6.1. The proof occupies the rest of Section 6. We proceed by induction on $\mathrm{rk}(\mathcal{V}) = d + e$. The initial case $d = e = 1$ will follow from the same proof, except that we shall not use any induction hypothesis in that case. In other words, from now on, *when* $d + e \geq 2$, *we can assume the result for vector bundles* $\mathcal{V}$ *of rank less than* $d + e$. We fix a complete flag

$$0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_{d+e} = \mathcal{V}\,.$$

We set as before $\mathcal{V}^1 := \mathcal{V}_{d+e-1}$.

The idea of the proof is to use Theorem 3.14 in the localization setting of Section 5, i.e. for the regular closed immersion $\iota : Z = \mathrm{Gr}_X(d, \mathcal{V}^1) \hookrightarrow \mathrm{Gr}_X(d, \mathcal{V})$ with open complement $U = U_X(d, \mathcal{V}_\bullet)$. For every line bundle L on $\mathrm{Gr}_X(d, \mathcal{V})$, the localization long exact sequence (23) is as follows:

(37)

$$\begin{array}{l} \cdots \longrightarrow \mathrm{W}^i_Z(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\mathrm{e}} \mathrm{W}^i(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\upsilon^*} \mathrm{W}^i(U_X(d,\mathcal{V}_\bullet),\upsilon^*L) \\ \quad \xrightarrow{\partial} \mathrm{W}^{i+1}_Z(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\mathrm{e}} \mathrm{W}^{i+1}(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\upsilon^*} \mathrm{W}^{i+1}(U_X(d,\mathcal{V}_\bullet),\upsilon^*L) \longrightarrow \cdots \end{array}$$

where $\mathrm{e} : \mathrm{W}^i_Z(\mathrm{Gr}_X(d,\mathcal{V}),L) \to \mathrm{W}^i(\mathrm{Gr}_X(d,\mathcal{V}),L)$ is extension of support.

6.4. Remark. The reader will notice along the proof that when $d = 1$ (respectively, $e = 1$), the situation is somewhat particular. This is because the map υ^* (respectively, ι^*) on relative Picard groups modulo 2 is no longer injective. So, some adjustments need to be made. In a first reading, it might be a good idea to skip these special cases to more easily follow the structure of the proof.

Before applying Theorem 3.14, we first need to use dévissage to obtain a total basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V})$ with support in $\mathrm{Gr}_X(d, \mathcal{V}^1)$, from a total basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V}^1)$. Let $(\iota_Z)_*$ be the push-forward along ι from the Witt groups of Z to the Witt groups with support in Z. We therefore have $\mathrm{e} \circ (\iota_Z)_* = \iota_*$, where e is the extension of support from Z to the whole $\mathrm{Gr}_X(d, \mathcal{V})$.

6.5. Lemma. *Assume $e \geq 2$ (and recall Convention 6.2). The elements*

$$(\iota_Z)_*\big(\phi_{d,e-1}(\Lambda'')\big) \qquad \text{with } \Lambda'' \text{ even } (d, e-1)\text{-diagram}$$

form a total basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V})$ with support in $Z = \mathrm{Gr}_X(d, \mathcal{V}^1)$.

Proof. Since $e \geq 2$ the map $\iota^* : \mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))/2 \to \mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}^1))/2$ is an isomorphism (see Remark 5.6). We can therefore apply [7, Corollary 6.16] with $P = \mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))$, using that the $\phi_{d,e-1}(\Lambda'')$ form a total basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V}^1)$ by the induction assumption. □

When $e = 1$, the situation is slightly different. Since $\mathrm{Gr}_X(d, \mathcal{V}^1) = X$, the map ι^* is no longer injective on Picard groups modulo 2, and there are two essentially different lax push-forwards. Indeed, in that case $\mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))/2 = \mathbb{Z}/2 \cdot \Delta_d$ and $\mathrm{Pic}_X(X)/2 = 1$. Observe the convenient $\mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))/2 = \{\Delta_d^d, \Delta_d^{d+1}\}$.

6.6. Definition. Assume $e = 1$ and therefore $\mathrm{Gr}_X(d, \mathcal{V}^1) = X$. Let L_d (respectively, L_{d+1}) be a line bundle on $\mathrm{Gr}_X(d, \mathcal{V})$ such that $[L_d] = \Delta_d^d \in \mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))/2$ (respectively, $[L_{d+1}] = \Delta_d^{d+1}$). Let δ_d (respectively, δ_{d+1}) be a lax push-forward of the unit form 1_X to $\mathrm{W}^*_X(\mathrm{Gr}_X(d, \mathcal{V}), L_d)$ (respectively, to $\mathrm{W}^*_X(\mathrm{Gr}_X(d, \mathcal{V}), L_{d+1})$) along $\iota : X \hookrightarrow \mathrm{Gr}_X(d, \mathcal{V})$.

6.7. Lemma (Analogue of Lemma 6.5 in a special case)**.** *Assume $e = 1$. Then*

- *(a) The element δ_d (respectively, δ_{d+1}) forms a total basis of the $\{\Delta_d^d\}$-part (respectively, the $\{\Delta_d^{d+1}\}$-part) of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V})$ with support in $X = \mathrm{Gr}_X(d, \mathcal{V}^1)$.*
- *(b) Together, δ_d and δ_{d+1} form a total basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V})$ with support in $X = \mathrm{Gr}_X(d, \mathcal{V}^1)$.*

Proof. Part (a) follows from [7, Corollary 6.15], applied to the subset $P = \{\Delta_d^d\}$ (respectively, $P = \{\Delta_d^{d+1}\}$). Part (b) follows from (a) since the union of total bases of disjoint parts P_1 and P_2 forms a total basis of the part $P = P_1 \cup P_2$; see [7, Lemma 6.10]. Here we use $\mathrm{Pic}_X(\mathrm{Gr}_X(d, \mathcal{V}))/2 = \{\Delta_d^d\} \sqcup \{\Delta_d^{d+1}\}$. □

We need to track our classes $\phi_{d,e}(\Lambda)$ along the morphisms involved in the long exact sequence of localization. Recall the constructions $\bar{\iota}$, $\bar{\upsilon}$ and $\bar{\partial}$ on even

Young diagrams (Definition 2.11) and the lax-similarity equivalence relation $\leftrightsquigarrow$ (Definition 3.4).

6.8. Proposition. *With the above notation (and with Convention 6.2), we have*

(a) *Let Λ' be an even Young diagram in $(d\times(e-1))$-frame, with $d\geq 1$, $e\geq 2$. The push-forward ι_* satisfies (see Figure 4):*

$$\iota_*(\phi_{d,e-1}(\Lambda')) \leftrightsquigarrow \phi_{d,e}(\bar\iota(\Lambda')) \qquad \text{if } \zeta(\Lambda') \text{ is even.}$$

(b) *Let Λ be an even Young diagram in $(d\times e)$-frame, with $d\geq 2$ and $e\geq 1$. The restriction morphism υ^* satisfies (see Figure 5):*

$$\upsilon^*\big(\phi_{d,e}(\Lambda)\big) \leftrightsquigarrow \alpha^*\big(\phi_{d-1,e}(\bar\upsilon(\Lambda))\big) \qquad \text{if } \Lambda_d = 0.$$

(c) *Let Λ'' be an even Young diagram in $((d-1)\times e)$-frame, with $d,e\geq 2$. The connecting homomorphism ∂ satisfies (see Figure 6):*

$$\partial\big(\alpha^*(\phi_{d-1,e}(\Lambda''))\big) \leftrightsquigarrow (\iota_Z)_*\big(\phi_{d,e-1}(\bar\partial(\Lambda''))\big) \qquad \text{if } \Lambda''_{d-1} \text{ is odd.}$$

Proof. *(a)* Let Λ' be an even $(d,e-1)$-diagram such that $\zeta(\Lambda')$ is even. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Definition 2.3). By assumption we can consider the even (d,e)-diagram $\bar\iota(\Lambda') := (\Lambda'_1+1,\ldots,\Lambda'_d+1)$. Observe that it has the same associated k-tuples (with respect to the $(d\times e)$-frame). From (14), it is then easy to see that $\mathcal{F}l_X(d,e-1;\Lambda') = \mathcal{F}l_X(d,e;\bar\iota(\Lambda'))$ and that the diagram

$$\begin{array}{ccc}
\mathrm{Gr}_X(d,\mathcal{V}^1) & \xrightarrow{\ \iota\ } & \mathrm{Gr}_X(d,\mathcal{V}) \\
{\scriptstyle f_{d,e-1;\Lambda'}}\uparrow & & \uparrow{\scriptstyle f_{d,e,\bar\iota(\Lambda')}} \\
\mathcal{F}l_X(d,e-1;\Lambda') & = & \mathcal{F}l_X(d,e;\bar\iota(\Lambda'))
\end{array}$$

commutes. Thus, (a) follows by composition of push-forwards and by Theorem 3.8.

(b) Let Λ be an even (d,e)-diagram such that $\Lambda_d=0$. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Definition 2.3). By assumption we can consider the even $(d-1,e)$-diagram $\bar\upsilon(\Lambda) := \Lambda_{|_{d-1,e}}$. We then have two Cartesian squares:

$$\begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}) & \xleftarrow{\ \upsilon\ } & U_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \alpha\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1) \\
{\scriptstyle f_{d,e;\Lambda}}\uparrow & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\bar\upsilon(\Lambda)}} \\
\mathcal{F}l_X(d,e;\Lambda) & \hookleftarrow & U' & \longrightarrow & \mathcal{F}l_X(d-1,e;\bar\upsilon(\Lambda))
\end{array}$$

by Lemma 5.7. The equality in (b) then follows by base change on the above two Cartesian squares (see [8, Theorem 5.5], the horizontal morphisms are smooth, so flat, and the vertical maps are proper, including $U'\to U_X(d,\mathcal{V}_\bullet)$).

Using Remark 3.6 and Theorem 3.8, both sides of the desired equation are lax-similar to the push-forward of the same $1_{U'} \in \mathrm{W}^0(U')$ along $U' \to U_X(d, \mathcal{V}_\bullet)$.

(c) Let Λ'' be an even $(d-1,e)$-diagram such that Λ''_{d-1} is odd. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Definition 2.3). By assumption, we can consider the even $(d, e-1)$-diagram

$$\bar{\partial}(\Lambda'') := (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0)\,.$$

The key tool here is [6, Theorem 2.6] which allows us to compute ∂ geometrically, as the pull-back to the exceptional fiber along $\tilde{\alpha} \circ \tilde{\iota} : E_X(d, \mathcal{V}_\bullet) \to \mathrm{Gr}_X(d{-}1, \mathcal{V}^1)$ followed by the push-forward along $\tilde{\pi} : E_X(d, \mathcal{V}_\bullet) \to \mathrm{Gr}_X(d, \mathcal{V}^1)$, as long as the twist of the Witt class under consideration satisfies the assumptions of [6, Theorem 2.6].

Here, the twist $\Delta_{d-1}^{t(\Lambda'')}$ of $\phi_{d-1,e}(\Lambda'')$ is given by $t(\Lambda'') = [d{+}1] \in \mathbb{Z}/2$. Using Diagram (29) and Equation (32), one can easily check that this is the twist for which [6, Theorem 2.6] applies. By Lemma 5.8, we have the right-hand Cartesian square in the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Gr}_X(d, \mathcal{V}^1) & \xleftarrow{\ \tilde{\pi}\ } & E_X(d, \mathcal{V}_\bullet) & \xrightarrow{\ \tilde{\alpha}\tilde{\iota}\ } & \mathrm{Gr}_X(d-1, \mathcal{V}^1) \\
{\scriptstyle f_{d,e-1;\bar{\partial}(\Lambda'')}} \uparrow & & \uparrow & \square & \uparrow {\scriptstyle f_{d-1,e;\Lambda''}} \\
\mathcal{F}l_X(d, e-1; \bar{\partial}(\Lambda'')) & \xleftarrow{\ \pi'\ } & F' & \longrightarrow & \mathcal{F}l_X(d-1, e; \Lambda'')\,.
\end{array}$$

We can now compute $\partial(\alpha^*(\phi_{d-1,e}(\Lambda'')))$ by starting with the unit form 1 in the lower right corner, pushing-forward along $f_{d-1,e;\Lambda''}$, pulling back along $\tilde{\alpha}\tilde{\iota}$ and then pushing forward along $\tilde{\pi}$. By base-change on the right-hand square, this class $\partial(\alpha^*(\phi_{d-1,e}(\Lambda'')))$ is also the (lax) push-forward of the unit form on F' along π' and then along $f_{d,e-1;\bar{\partial}(\Lambda'')}$. The key point is to check that π'_* preserves the unit form, up to lax-similitude. By Lemma 5.8, we know that π' is either an isomorphism or a blow-up along a closed regular immersion of odd codimension. In both cases $\pi'_*(1) \leftrightsquigarrow 1$ by Proposition 3.15 and we get the result. □

6.9. Proposition (Analogue of Proposition 6.8 in special cases)**.** *When* $e = 1$, *the push-forward* $\iota_* : \mathrm{W}^0(X, \mathcal{O}_X) \to \mathrm{W}^d(\mathrm{Gr}_X(d, \mathcal{V}), L_{d+1})$ *satisfies*

$$\iota_*(1_X) = \mathrm{e}(\delta_{d+1}) \leftrightsquigarrow \phi_{d,1}([d \times 1]). \tag{38}$$

When $d = 1$, *the restriction morphism* υ^* *satisfies*

$$\upsilon^*(\phi_{1,e}(\not\square)) \leftrightsquigarrow \alpha^*(1_X). \tag{39}$$

When $e = 1$ *and* $d \geq 2$, *the connecting homomorphism of the localization sequence with twist* L_d *satisfies*

$$\partial\big(\alpha^*(\phi_{d-1,1}([(d-1) \times e]))\big) \leftrightsquigarrow \delta_d\,. \tag{40}$$

When $e = d = 1$, the connecting homomorphism of the localization sequence with twist $L_d = L_1$ satisfies

$$\partial\big(\alpha^*(1_X)\big) \leftrightsquigarrow \delta_1\,. \tag{41}$$

Proof. The proof is entirely analogous to the one of Proposition 6.8, so we leave the details to the reader. □

We now apply localization, using Theorem 3.14, in which a part P of the relative Picard group modulo 2 needs to be specified. Again, there is a general case, when $d \geq 2$, and a particular case where $d = 1$.

Case $d, e \geq 2$. We choose P to be the whole $\mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V}))/2$. Note that we indeed have $\upsilon^* : \mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V})/2 \to \mathrm{Pic}_X(U_X(d,\mathcal{V}_\bullet))/2$ injective by (29). By induction and by Lemma 6.5, the $\phi_{d,e-1}(\Lambda')$, Λ' even, form a total basis of the Witt groups of $\mathrm{Gr}_X(d,\mathcal{V})$ with support in $\mathrm{Gr}_X(d,\mathcal{V}^1)$. By induction, we also know that the $\phi_{d-1,e}(\Lambda'')$, Λ'' even, form a total basis of the Witt groups of $\mathrm{Gr}_X(d-1,\mathcal{V}^1)$. Homotopy invariance and [7, Corollary 6.13] ensure that the $\alpha^*(\phi_{d-1,e}(\Lambda''))$ form a total basis of the Witt groups of $U_X(d,\mathcal{V}_\bullet)$. The correspondence of Propositions 6.8 and 2.12 then show that assumptions (a), (b) and (c) of Theorem 3.14 are satisfied when choosing the following collections of Witt classes:

$$\mathcal{I} = \left\{ \begin{matrix}\text{even Young } (d,e-1)\text{-diagrams}\\ \Lambda' \text{ such that } \zeta(\Lambda') \text{ is even}\end{matrix} \right\} \xrightarrow[\bar\iota]{\simeq} \left\{ \begin{matrix}\text{even Young } (d,e)\text{-diagrams}\\ \Lambda \text{ such that } \Lambda_d > 0\end{matrix} \right\},$$

$$v_{\Lambda'} = (\iota_Z)_*(\phi_{d,e-1}(\Lambda')) \quad\text{and}\quad w'_{\Lambda'} = \phi_{d,e}(\bar\iota(\Lambda')) \leftrightsquigarrow \mathrm{e}(v_{\Lambda'}) \quad \text{for all } \Lambda' \in \mathcal{I},$$

$$\mathcal{J} = \left\{ \begin{matrix}\text{even Young } (d,e)\text{-diagrams}\\ \Lambda \text{ such that } \Lambda_d = 0\end{matrix} \right\} \xrightarrow[\bar\upsilon]{\simeq} \left\{ \begin{matrix}\text{even Young } (d-1,e)\text{-diagrams}\\ \Lambda'' \text{ such that } \Lambda''_{d-1} \text{ is even}\end{matrix} \right\},$$

$$w_\Lambda = \phi_{d,e}(\Lambda) \quad\text{and}\quad u'_\Lambda = \alpha^*(\phi_{d-1,e}(\bar\upsilon(\Lambda))) \leftrightsquigarrow \upsilon^*(w_\Lambda) \quad \text{for all } \Lambda \in \mathcal{J},$$

$$\mathcal{K} = \left\{ \begin{matrix}\text{even Young } (d-1,e)\text{-diagrams}\\ \Lambda'' \text{ such that } \Lambda''_{d-1} \text{ is odd}\end{matrix} \right\} \xrightarrow[\bar\partial]{\simeq} \left\{ \begin{matrix}\text{even Young } (d,e-1)\text{-diagrams}\\ \Lambda' \text{ such that } \zeta(\Lambda') \text{ is odd}\end{matrix} \right\},$$

$$u_{\Lambda''} = \alpha^*(\phi_{d-1,e}(\Lambda'')) \text{ and } v'_{\Lambda''} = (\iota_Z)_*(\phi_{d,e-1}(\bar\partial(\Lambda''))) \leftrightsquigarrow \partial(u_{\Lambda''}) \quad \text{for all } \Lambda'' \in \mathcal{K}.$$

We therefore conclude by part (4) of Theorem 3.14 that we have a total basis $\{\phi_{d,e}(\Lambda)\}_\Lambda$ indexed by Λ in $\{\Lambda \text{ even with } \Lambda_d = 0\} \sqcup \{\Lambda \text{ even with } \Lambda_d > 0\}$, i.e. by all even Young diagrams in $(d\times e)$-frame. This is the result.

Case $d \geq 2$, $e = 1$. The proof is the same as in the case $d, e \geq 2$, except that Proposition 6.9 is used instead of Proposition 6.8. We take (see Figure 10)

$$\begin{aligned}
&\mathcal{I} = \{\star\}, &&v_\star = \delta_{d+1}, &&w'_\star = \phi_{d,1}([d \times 1]) \leftrightsquigarrow \mathrm{e}(v_\star),\\
&\mathcal{J} = \{\boxslash\}, &&w_{\boxslash} = \phi_{d,1}(\boxslash), &&u'_{\boxslash} = \alpha^*(\phi_{d-1,1}(\boxslash)) \leftrightsquigarrow \upsilon^*(w_{\boxslash}),\\
&\mathcal{K} = \{\diamond\}, &&u_\diamond = \alpha^*(\phi_{d-1,1}(\boxslash)), &&v'_\diamond = \delta_d \leftrightsquigarrow \partial(u_{\boxslash}).
\end{aligned}$$

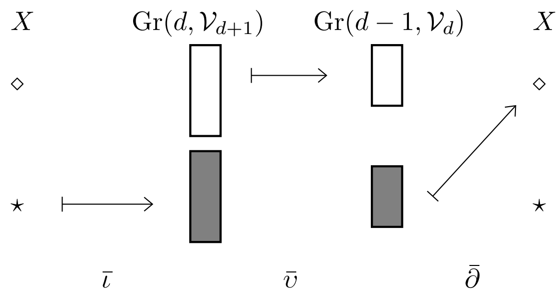


FIGURE 10. Image of the diagrams by the morphisms $\bar{\iota}$, $\bar{\upsilon}$ and $\bar{\partial}$ when $d \geq 2$ and $e = 1$. No arrow means the corresponding generator is mapped to zero.

Case $d = 1$, $e \geq 2$. In that case, $\mathrm{Gr}_X(d-1, \mathcal{V}^1) = \mathrm{Gr}_X(0, \mathcal{V}^1) = X$, so we can no longer choose P to be the whole relative Picard group modulo 2 because injectivity of υ^* would not hold (see Remark 5.6). We therefore first choose $P = P_1 := \{\Delta_d^{d+1}\} = \{1\}$ and take (see the P_1-part of Figure 11)

$$\begin{aligned}
&\mathcal{I} = \{[1 \times (e-1)]\}, &&v_{[1\times(e-1)]} = (\iota_Z)_*(\phi_{1,e-1}([1 \times (e-1)])),\\
& &&w'_{[1\times(e-1)]} = \phi_{1,e}([1 \times e]) \leftrightsquigarrow \mathrm{e}(v_{[1\times(e-1)]}),\\
&\mathcal{J} = \{\boxslash\}, &&w_{\boxslash} = \phi_{1,e}(\boxslash),\\
& &&u'_{\boxslash} = \alpha^*(1_X) \leftrightsquigarrow \upsilon^*(w_{\boxslash}),\\
&\mathcal{K} = \emptyset.
\end{aligned}$$

As in previous cases, by part (a) of Proposition 6.8, by equation (39) and by induction, using Theorem 3.14, we get that $\phi_{1,e}([1 \times e])$ and $\phi_{1,e}(\boxslash)$ form a basis of the $\{1\}$-part of the Witt groups of $\mathrm{Gr}_X(1, \mathcal{V})$.

We then choose $P = P_2 := \{\Delta_d\}$, and take (see the P_2-part of Figure 11)

$$\begin{aligned}
&\mathcal{I} = \mathcal{J} = \emptyset,\\
&\mathcal{K} = \{\star\}, \qquad u_\star = \alpha^*(1_X), \quad v'_\star = (\iota_Z)_*(\phi_{1,e}(\boxslash)) \leftrightsquigarrow \partial(u_\star).
\end{aligned}$$

By Theorem 3.14 using (39), we obtain that the empty set is a basis of the $\{\Delta_d\}$-part of the Witt groups of $\mathrm{Gr}_X(1, \mathcal{V})$.

Putting together both bases (one of which is empty), we obtain a basis of the Witt groups of $\mathrm{Gr}_X(d, \mathcal{V})$ by [7, Lemma 6.10], since $P_1 \sqcup P_2 = \mathrm{Pic}_X(\mathrm{Gr}_X(1, \mathcal{V}))$. This basis contains exactly the only two generators corresponding to even diagrams of size $(1, e)$, i.e. $\not\Box$ and $[1 \times e]$; see Figure 11.

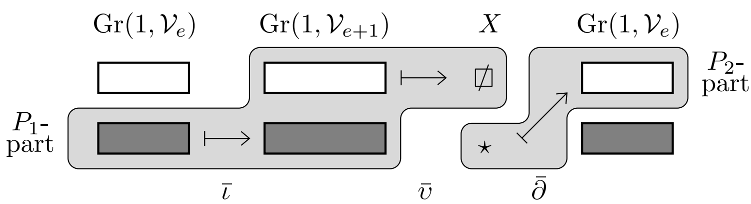


FIGURE 11. Image of the diagrams by the morphisms $\bar{\iota}$, $\bar{\upsilon}$ and $\bar{\partial}$ when $d = 1$ and $e \geq 2$. No arrow means the corresponding generator is mapped to zero.

Case $d = e = 1$. It works as in the case $d = 1$, $e \geq 2$ except that $(\iota_Z)_*(\phi_{1,e-1}(\not\Box))$ is replaced by δ_1 and that $(\iota_Z)_*(\phi_{1,e}([1 \times 1]))$ is replaced by δ_0; see Figure 12. Note that, as announced, this case does not use any induction assumption. This completes the induction and therefore the proof of the main Theorem 6.1. □

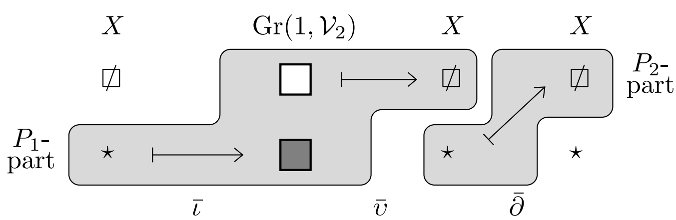


FIGURE 12. Image of the diagrams by the morphisms $\bar{\iota}$, $\bar{\upsilon}$ and $\bar{\partial}$ when $d = e = 1$. No arrow means the corresponding generator is mapped to zero.

7. Corollaries and examples

Here is how to deduce a more classical result on Witt groups that does not involve "total" concepts. Recall that π is the structural morphism of

$\mathrm{Gr}_X(d,\mathcal{V})$ and that $\mathcal{T}_d$ is its tautological bundle. Also recall that the generator $\phi_{d,e}(\Lambda)$ lives in $\mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V}), L_\Lambda\right)$ where $|\Lambda|$ is the area of Λ. Any line bundle on $\mathrm{Gr}_X(d,\mathcal{V})$ is, up to isomorphism, of the form $\pi^*K\otimes\det(\mathcal{T}_d)^{\otimes l}$, for some line bundle K on X and some $l\in\mathbb{Z}$. A diagram Λ is such that $[L_\Lambda]=[\pi^*K\otimes\det(\mathcal{T}_d)^{\otimes l}]\in\mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V}))/2$ if and only if $t(\Lambda)=l\in\mathbb{Z}/2$, where $t(\Lambda)$ is the half-perimeter of Λ.

7.1. Corollary. *Let K be a line bundle on X and let l be an integer. For each even (d,e)-diagram Λ such that $l=t(\Lambda)\in\mathbb{Z}/2$, choose a line bundle N_Λ over $\mathrm{Gr}_X(d,\mathcal{V})$ and an isomorphism $N_\Lambda^{\otimes 2}\otimes\pi^*(K\otimes\det(\mathcal{V})^{\otimes-\rho(\Lambda)})\otimes L_\Lambda\overset{\sim}{\to}\pi^*(K)\otimes\det(\mathcal{T}_d)^l$. Then the morphism of $\mathrm{W}^0(X,\mathcal{O}_X)$-modules*

$$\bigoplus_{\substack{\Lambda \text{ even s.t.}\\ t(\Lambda)=l\in\mathbb{Z}/2}}\mathrm{W}^{k-|\Lambda|}\left(X,K\otimes\det(\mathcal{V})^{\otimes-\rho(\Lambda)}\right)\overset{\sim}{\to}\mathrm{W}^k\left(\mathrm{Gr}_X(d,\mathcal{V}),\pi^*(K)\otimes\det(\mathcal{T}_d)^{\otimes l}\right)$$

sending (x_Λ) to $\sum x_\Lambda\cdot\phi_{d,e}(\Lambda)$ (notation of Definition 3.10) is an isomorphism.

Proof. Apply Theorem 3.12 to the total basis formed by the $\phi_{d,e}(\Lambda)$. □

7.2. Corollary. *Let d' (respectively, e') be the integral part of $d/2$ (respectively, $e/2$) and consider the binomial coefficient $\binom{a+b}{a}=\frac{(a+b)!}{a!b!}$. The cardinal of a total basis of the Witt groups of $\mathrm{Gr}_X(d,\mathcal{V})$ over X is $2\binom{d'+e'}{e'}$. If we assume moreover that $\mathrm{W}^i(X,K)=0$ for $i\not\equiv 0 \mod 4$ or $[K]\neq[\mathcal{O}_X]\in\mathrm{Pic}(X)/2$, for instance for X local (e.g. a field), then, as a module over $\mathrm{W}^0(X,\mathcal{O}_X)$,*

- *the classical Witt group of symmetric forms $\mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}),\mathcal{O})$ has rank $\binom{d'+e'}{e'}$,*
- *the classical Witt group of antisymmetric forms $\mathrm{W}^2(\mathrm{Gr}_X(d,\mathcal{V}),\mathcal{O})$ is zero,*
- *the Witt groups $\mathrm{W}^1(\mathrm{Gr}_X(d,\mathcal{V}),L)$ and $\mathrm{W}^3(\mathrm{Gr}_X(d,\mathcal{V}),L)$ are zero when $[L]=\Delta_d\in\mathrm{Pic}_X(\mathrm{Gr}_X(d,\mathcal{V}))/2$.*

Proof. Let "4-block" mean "2×2 square". Every even Young diagram Λ is either

(a) a union of 4-blocks in which case $\phi(\Lambda)$ counts in $\mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}),\mathcal{O})$,
(b) a single row plus 4-blocks (e even) in which case $\phi(\Lambda)$ counts in $\mathrm{W}^e(\mathrm{Gr}_X(d,\mathcal{V}),L_\Lambda)$,
(c) a single column plus 4-blocks (d even) in which case $\phi(\Lambda)$ counts in $\mathrm{W}^d(\mathrm{Gr}_X(d,\mathcal{V}),L_\Lambda)$,
(d) a single row and a single column plus 4-blocks (d and e odd) in which case $\phi(\Lambda)$ counts in $\mathrm{W}^{d+e-1}(\mathrm{Gr}_X(d,\mathcal{V}),\mathcal{O})$.

All possibilities (a)–(d) are exclusive and can be enumerated easily by counting the diagrams of 4-blocks, which amounts to counting the usual Young diagrams in $(d'\times e')$-frame. We get $\binom{d'+e'}{e'}$ elements in case (a), and the other results depend on the parities of d and e but are also very easy to figure out in each case. $\square$

7.3. Corollary. *The connecting homomorphism ∂ is zero for all line bundles L (and thus the long exact sequence* (37) *decomposes as split short exact sequences as for Chow groups) if and only if both d and e are even.*

Proof. Looking back at the proof of the main theorem, and at (18) (or Figure 6), we see that ∂ is zero if and only if there is no even $(d-1, e)$-diagram Λ'' such that Λ''_{d-1} is odd. This implies that e is even (otherwise $\Lambda'' = [(d-1) \times e]$ would be such an even diagram) and that $d-1$ is odd (otherwise $\Lambda'' = (1, \ldots, 1)$ would be such an even diagram). Conversely, assume e even and the existence of an even $(d-1, e)$-diagram Λ'' such that Λ''_{d-1} is odd. Then e_k is odd (since $e = \Lambda''_{d-1} + e_k$ is even), hence all e_i are odd since Λ'' is an even diagram. In particular, e_1 is odd, hence $e_1 > 0$ and therefore d_1 is even. This implies that $d-1 = d_k = (d_k - d_{k-1}) + \ldots + (d_2 - d_1) + d_1$ is even, i.e. d is odd, as was to be shown. $\square$

7.4. Notation. For $d, e \geqslant 1$, we write $\mathbb{G}_X(d,e)$ for the split Grassmannian

$$\mathbb{G}_X(d,e) = \mathrm{Gr}_X(d, \mathcal{O}_X^{d+e}) \,.$$

7.5. Example. Figure 13 shows how the different generators are mapped to each other, up to lax-similitude, by ι_*, υ^* and ∂ in the long exact sequence (37) for $\mathbb{G}(3,3)$. We use dévissage (respectively, homotopy invariance) to identify the generators of the Witt groups of $\mathbb{G}(3,2)$ (respectively, of $\mathbb{G}(2,3)$) with generators of the Witt groups of $\mathbb{G}(3,3)$ with support on $\mathbb{G}(3,2)$ (respectively, of the open complement of $\mathbb{G}(3,2)$).

7.6. Example. Figures 14, 15 and 16 give the even Young diagrams in $(d \times e)$-frame and the corresponding shifts in $\mathbb{Z}/4$ and twists Δ_d or $\mathcal{O}$ for the Grassmannians $\mathbb{G}(4,4)$, $\mathbb{G}(4,5)$ and $\mathbb{G}(5,5)$.

We conclude with a few comments.

7.7. Remark. As mentioned in Remark 6.3, we could have considered a larger set of elements $\phi_{d,e}(\Lambda)$ using Remark 4.1 instead of assuming Λ even. This larger set is also stable by applying ι_*, υ^* and ∂. Some of these extra elements are then easily seen to be zero from the exact sequence, but some of them are non-zero.

7.8. Remark. Part of the multiplicative structure on Witt groups can be computed at each induction step using the projection formula. Unfortunately, this is not enough for the whole computation. Despite the results for the Grothendieck and the Chow rings using the basis of Schubert cells, it is unclear to the authors what kind of Littlewood-Richardson type rule one should expect.

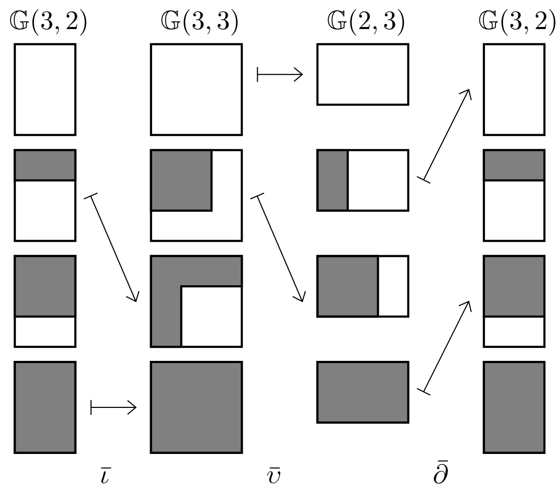


Figure 13. Maps $\bar{\iota}$, $\bar{\upsilon}$ and $\bar{\partial}$ on diagrams, corresponding to the maps ι_*, υ^* and ∂ on generators (no arrow means mapped to zero).

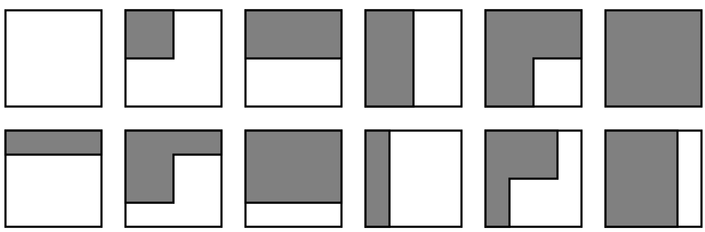


Figure 14. Diagrams of generators for $\mathbb{G}(4,4)$: first row in shift 0 and twist $\mathcal{O}$, second row in shift 0 and twist Δ_d.

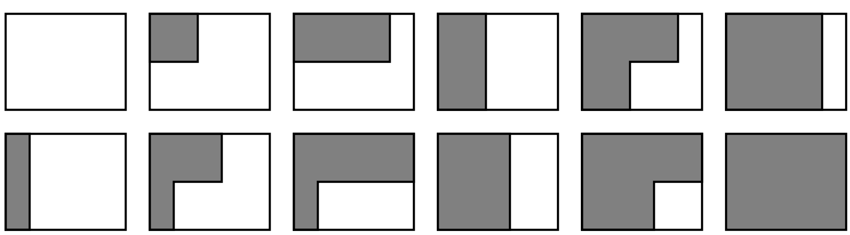


Figure 15. Diagrams of generators for $\mathbb{G}(4,5)$: first row in shift 0 and twist $\mathcal{O}$, second row in shift 0 and twist Δ_d.

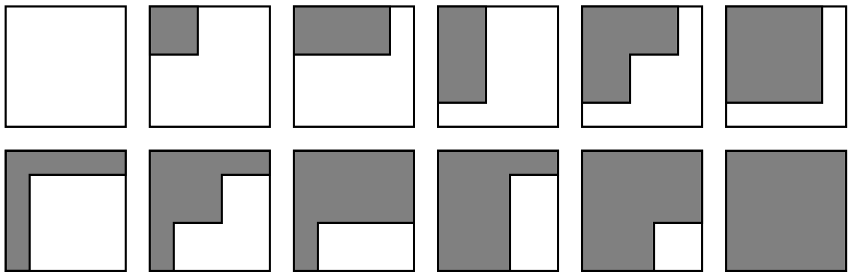


FIGURE 16. Diagrams of generators for $\mathbb{G}(5,5)$: first row in shift 0 and twist $\mathcal{O}$, second row in shift 1 and twist Δ_d.

Note however that all generators $\phi_{d,e}(\Lambda)$ are nilpotent except for $\Lambda = \boxslash$. This can be checked using homotopy invariance and a discussion on the supports or simply the fact that these Witt classes are generically trivial.

7.9. Remark. Although we do not need it here, it is possible to show that, for $\mathcal{V} = \mathcal{O}_X^{d+e}$, the isomorphism $\mathbb{G}(d,e) = \mathrm{Gr}(d,\mathcal{V}) \simeq \mathrm{Gr}(e,\mathcal{V}^\vee) = \mathbb{G}(e,d)$ sends $\phi_{d,e}(\Lambda)$ to $\phi_{e,d}(\Lambda^\vee)$, up to lax-similitude of course, where $\Lambda^\vee$ is the dual partition.

References

1. *Schémas en groupes I, II et III*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151, 152 and 153, Springer-Verlag, Berlin, 1962/1970.
2. *Théorie des intersections et théorème de Riemann-Roch*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Springer Lecture Notes in Mathematics, Vol. 225. MR0354655 (50:7133)
3. P. Balmer, *Triangular Witt groups Part I: The 12-term localization exact sequence*, K-Theory **4** (2000), no. 19, 311–363. MR1763933 (2002h:19002)
4. ———, *Triangular Witt groups Part II: From usual to derived*, Math. Z. **236** (2001), no. 2, 351–382. MR1815833 (2002h:19003)
5. ———, *Products of degenerate quadratic forms*, Compositio Mathematica **6** (2005), no. 141, 1374–1404. MR2188441 (2006k:11059)
6. P. Balmer and B. Calmès, *Geometric description of the connecting homomorphism for Witt groups*, Doc. Math. **14** (2009), 525–550. MR2565903 (2010j:19009)
7. ———, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl., to appear.
8. B. Calmès and J. Hornbostel, *Push-forwards for Witt groups of schemes*, Comment. Math. Helv. **86** (2011), no. 2, 437–468. MR2775136 (2012e:19007)
9. M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970. MR0302656 (46:1800)

10. W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 2, Springer-Verlag, Berlin, 1998. MR1644323 (99d:14003)
11. S. Gille, *Homotopy invariance of coherent Witt groups*, Math. Z. **244** (2003), no. 2, 211–233. MR1992537 (2004f:19009)
12. A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167.
13. N. A. Karpenko, *Cohomology of relative cellular spaces and of isotropic flag varieties*, Algebra i Analiz **12** (2000), no. 1, 3–69. MR1758562 (2001c:14076)
14. M. Knebusch, *Symmetric bilinear forms over algebraic varieties*, Queen's Papers in Pure and Appl. Math., No. 46, Queen's Univ., Kingston, Ont., 1977, pp. 103–283. MR0498378 (58:16506)
15. D. Laksov, *Algebraic Cycles on Grassmann Varieties*, Adv. in Math. **9** (1972), no. 3, 267–295. MR0318145 (47:6694)
16. M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR2286826 (2008a:14029)
17. Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. MR1917232 (2003g:14001)
18. I. Panin, *Riemann-Roch theorems for oriented cohomology*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II, vol. 131, Kluwer, Dordrecht, 2004, pp. 261–333. MR2061857 (2005g:14025)
19. R. W. Thomason, *Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire*, Invent. Math. **112** (1993), no. 1, 195–215. MR1207482 (93k:19005)
20. C. Walter, *Grothendieck-Witt groups of projective bundles*, preprint, 2003.

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