WITT MOTIVES, TRANSFERS AND DÉVISSAGE

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Abstract. We define *transfer morphisms* (also called *push-forwards* or *norm morphisms*) between coherent Witt groups of schemes along proper morphisms and establish the base change and projection formulae for those. We then use this to define the category of *Witt motives*. We also deduce a dévissage theorem. Finally, we obtain some results about Witt groups of cellular varieties.

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Introduction

Transfer maps exist for many cohomological theories over schemes, e.g. for K-theory, (higher) Chow groups and algebraic cobordism. They are undoubtedly a useful tool for understanding and computing those cohomology theories. This article is about the construction of transfer maps for Witt groups (as defined by Balmer [3] in the framework of triangulated categories). To illustrate the importance of

transfers, we prove a dévissage theorem for Witt groups of closed embeddings. This is then applied to obtain partial results about Witt groups of varieties with a given cellular decomposition. Besides the classical case of finite field extensions, some special cases of transfers for Witt groups have been treated in [9], [29] and [20].

Although similar to some extent to the case of K_0, the situation for Witt groups is more complicated. Witt groups of schemes (equipped with dualizing complexes) are the first example of a "non-orientable cohomology theory" for which a reasonable theory of transfer maps with respect to proper morphisms may be established. For *coherent* Witt groups, we prove (see Theorems 2.20, 2.17 and 2.16)

Theorem 0.1. *Let $f : X \to Y$ be a proper map of relative dimension d between smooth Noetherian schemes of finite Krull dimension over a regular connected base and L a line bundle on Y. Then we can construct a transfer map of degree $-d$*

$$f_* : W^{*+d}(X, f^*L \otimes_{\mathcal{O}_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$$

which satisfies the projection formula and the base change formula with respect to a flat morphism.

This is a consequence of a more general result (see Theorem 2.15). Observe the twists and shifts that show up. The construction of the transfer map is more tricky than one might expect as one has to carefully keep track of the dualities and isomorphisms between objects and their biduals involved as well as all kind of compatibilities with the triangulated structure and the monoidal structure. This is carried out in Section 1.

We show that the natural isomorphism from the identity to the bidual and various other isomorphisms can be constructed from an internal Hom adjoint to some tensor product. We also show that this and other constructions related to the adjointness of Lf^*, Rf_* and $f^!$ can be carried out in a compatible way, These verifications are rather long, but we see no way of avoiding them. We have presented them in a general framework. As long as possible - namely the entire section 1 - we stay in this general framework rather than appealing to known results or arguments related to varieties, dualizing complexes and Witt groups. This has at least two advantages. First, it emphasizes which of the results are formal and which depend on the special case of Witt groups and varieties. Second, Section 1 may be applied to other fields of mathematics, for instance to other "motivic" categories or to stable homotopy theory.

This general setting can be summarized as follows.

> Let us consider a collection of symmetric monoidal triangulated categories, each equipped with an internal Hom adjoint to the tensor product, and some exact monoidal functors f^* between them with reasonable composition properties. If these f^* have right adjoints f_* (as usual functors), which themselves have right adjoints $f^!$, then we can trivially define natural transfers (induced by f_*) between Witt groups associated to some dualities defined by the internal Homs.

A careful reader might not be satisfied with such a sentence, and this is why we have given full details and proofs in Section 1. Let us make a few remarks.

(1) When defining "duality preserving functors" that induce morphisms on Witt groups, and proving the desired formulas, the main problem is to show that some diagrams are commutative (see for example [7, Definition 2.6]). Most of the articles on the subject usually deal with concrete situations, where the objects are complexes in a derived category, and checking the

commutativity of those diagrams reduces to checking signs that appear, depending on the position in the complex. When the situation becomes too involved, this can generate "sign" mistakes. We completely avoid this problem here. We start with very few abstract triangulated functors (so of course, ultimately, "signs" in derived categories are hidden there), such as an internal tensor product and a functor called f^*. We assume that those functors have adjoints as usual functors (for example the internal Hom and f_*), and we *make* the adjoints triangulated in such a way that the required diagrams are commutative. A corollary of this is that whenever one of our references contains a sign mistake, this has no influence in our work.

(2) In our setting, each diagram could be obtained in some empirical way, by fiddling with the previous ones. However, since this would be difficult to follow for a reader, we have tried to obtain as many diagrams as possible in a systematical way, using *ad nauseam* an elementary lemma (namely, Lemma 1.7) and some sophisticated versions of it (Theorem 1.9, Lemmas 1.19 and 1.40 and Theorems 1.41 and 1.55). We do not claim any deep innovation in the notions introduced to do so: they are formal, and are just there to simplify and unify the exposition (see for example Definition 1.49).

Reading this first section might seem very unpleasant at first glance. Don't get discouraged: writing it and checking all the details one is tempted to believe anyway was even more unpleasant. Of course, you can trust us (in the spirit of [11, pp. 117-119]) and skip this section. If you don't, you will find a survey of its subsections at the beginning of it.

In Section 2, applying the abstract framework of Section 1 to the case of schemes by using results from the theory of duality (Grothendieck, Hartshorne, Deligne, Verdier *et al.*), we obtain our main results, in particular Theorem 0.1. This allows us in particular to construct the category of *Witt motives* (see Section 2.3) which are slightly more complicated but similar in spirit to Panin's K_0-motives (and Grothendieck-Manin's classical motives). We also explain the usual structures (graph functor, pseudoabelian completion, tensor product and duality) on this category of Witt motives.

In the third section, we use the transfers and the base change theorem to prove a dévissage theorem (Theorem 3.1) for Witt groups. As a corollary, for Z a closed subscheme of X we obtain a localization exact sequence (Corollary 3.4)

$$\cdots \xrightarrow{\partial} W^n(Z, f^!L) \xrightarrow{f_*} W^n(X, L) \xrightarrow{j^*} W^n(X - Z, j^*L) \xrightarrow{\partial} W^{n+1}(Z, f^!L) \to \cdots$$

which is useful for computations, of course.

The last section is devoted to Witt groups of smooth projective varieties equipped with a cellular decomposition.

In the appendix, we provide a choice of signs so that the derived functors used in Section 2 are compatible with the axioms required in Section 1. This choice of signs is compatible with the various sign conditions used by Balmer [3], [4], Gille [7] and Gille-Nenashev [10].

We believe the transfer morphisms constructed in this paper will have other applications than the dévissage theorem provided here. In particular, we have become interested in the problem when trying to compute the Witt groups of projective homogeneous varieties by adapting the motivic and equivariant methods of Panin in K-theory (see [22]) in which he reduces the problem to well-known computations of Grothendieck groups of the categories of representations of split algebraic groups. We consider this paper (especially Section 2.3) as part of this attempt, even though

much more has to be done to complete it (for example extending the results of this article to the equivariant setting).

This paper essentially extends the results of Section 4 and the appendix of the preprint [6], except the short section on dévissage (of which a sketch was provided in an update in 2005) and on cellular varieties. We thank Bruno Kahn for discussions about and around Grothendieck-Verdier duality, and Stefan Gille for useful comments on earlier drafts.

1. Duality formalism and adjunctions

In this section, we obtain formal consequences of adjunctions of the type $\otimes$-Hom, f^*-f_* and f_*-$f^!$ in tensor-triangulated categories that are useful for Balmer's theory of higher Witt groups. Some people would probably call this the formalism of the five functors in triangulated categories. Our philosophy is to exhibit a minimal axiomatic setting which can be verified without too much work in the examples of interest and from which everything can be deduced in a formal way. The example of a triangulated category to keep in mind for this article is the derived category $D^b_c(X)$ of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology on a separated Noetherian scheme X.

Here is a survey of the different subsections of this section.

- Section 1.1 explains the general notation.
- Section 1.2 recalls elementary properties of adjunctions and proves useful lemmas about it.
- Section 1.3 explains how to deal with bifunctors that are adjoint when we fix one variable (such as the tensor product and the internal Hom), and translates a few lemmas on adjunction in this context.
- Section 1.4 introduces suspended categories, suspended functors and suspended bifunctors.
- Section 1.5 explains how suspended functors can form an adjoint couple, and how one can carry a suspended structure from defined for one member of a couple to the other one; it includes the case of bifunctors.
- Section 1.6 introduces the main concept of duality on a category, duality preserving pairs and functors, in a uniform way for usual functors or for suspended functors, so that we can more or less forget about the suspension in the rest of the proofs (all methods to obtain new morphisms will then carry the extra suspension structure without any further problem).
- Sections 1.7 and 1.8 show that the dualities constructed with the internal Hom are indeed dualities in the sense of the previous section.
- Section 1.9 is the core of the abstract part. We obtain the morphisms of functors and the properties we are interested in by using the lemmas on adjunction. This includes in particular the definition of the push-forward (Theorem 1.75).
- Section 1.10 recalls properties related to the associativity of the tensor product and prepares what is needed for the proof of the projection formula.
- Section 1.11 recalls compatibilities with the unit of the monoidal structure, which are useful in applications to prove that some morphisms of functors are isomorphisms.
- Section 1.12 proves the projection formula 1.92.
- Section 1.13 shows that the push-forward respects composition.
- Section 1.14 proves the base change theorem 1.101.
- Section 1.15 restates the main theorems using $f^!$ of the unit objects which will be useful for applications.

We have presented some aspects in slightly greater generality and provided some more results than we actually need for our applications in section 2 for possible future applications (e.g., other motivic categories or the stable homotopy category).

1.1. **Notations and conventions.** The opposite category of a category $\mathcal{C}$ is denoted by $\mathcal{C}^o$.

When F and G are functors with same source and target, we denote a morphism of functors between them as $t : F \to G$. When $s : G \to H$ is another one, their composition is denoted by $s \circ t$. When $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2 : \mathcal{D} \to \mathcal{E}$, $f : F_1 \to F_2$ and $g : G_1 \to G_2$, we denote by gf the morphism of functors defined by $(fg)_A = G_2(f_A) \circ g_{F_1(A)} = g_{F_2(A)} \circ G_1(f_A)$ on any object A. When $F_1 = F_2 = F$ and $f = id_F$ (resp. $G_1 = G_2 = G$ and $g = id_G$), we usually use the notation gF (resp. Gf). With this convention, $gf = G_2 f \circ gF_1 = gF_2 \circ G_1 f$. When a commutative diagram is obtained by this equality or other properties immediate from the definition of a morphism of functors, we just put an $\boxed{mf}$ label on it and avoid further justification. To save space, it may happen that when labeling maps in diagrams, we drop the functors from the notation, and just keep the important part, that is the morphism of functors (thus FgH might be reduced to g). Many of the commutative diagrams in the article will be labeled by a symbol in a box (letters or numbers, such as in $\boxed{H}$ or $\boxed{3}$). When they are used in another commutative diagram, eventually after applying a functor to them, we just label them with the same symbol, so that the reader recognizes them, but without further comment.

1.2. **Useful properties of adjunctions.** This section is devoted to easy facts and theorems about adjunctions, that are repeatedly used throughout the first part of this article. All these facts are obvious, and we only prove the ones that are not completely classical. A good reference for the background on categories and adjunctions as discussed here is [14].

Definition 1.1. An adjoint couple (L, R) is the data consisting of two functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ and equivalently:

- a bijection $\mathrm{Hom}(LA, B) \simeq \mathrm{Hom}(A, RB)$, functorial in $A \in \mathcal{C}$ and $B \in \mathcal{D}$, or
- two morphism of functors $\eta : Id_{\mathcal{C}} \to RL$ and $\epsilon : LR \to Id_{\mathcal{D}}$, called respectively unit and counit, such that the resulting compositions $R \overset{\eta R}{\to} RLR \overset{R\epsilon}{\to} R$ and $L \overset{L\eta}{\to} LRL \overset{\epsilon L}{\to} L$ are identities.

In the couple, L is called the left adjoint and R the right adjoint. When we want to specify the unit and counit of the couple and the categories involved, we say (L, R, η, ϵ) is an adjoint couple from $\mathcal{C}$ to $\mathcal{D}$.

When the commutativity of a diagram follows by one of the above compositions giving the identity, we label it $\boxed{adj}$.

Remark 1.2. Adjunctions between functors that are contravariant can be considered in two different ways, by taking the opposite category of the source of L or R. This does *not* lead to the same notion, essentially because if (L, R) is an adjoint couple, then (R^o, L^o) is an adjoint couple (instead of (L^o, R^o)). For this reason, we only use covariant functors in adjoint couples.

Lemma 1.3. *Let (L, R, η, ϵ) and $(L', R', \eta', \epsilon')$ be two adjoint couples between the same categories $\mathcal{C}$ and $\mathcal{D}$, and let $l : L \to L'$ (resp. $r : R \to R'$) be an isomorphism. Then, there is a unique isomorphism $r : R \to R'$ (resp. $l : L \to L'$) such that $\eta' = rl \circ \eta$ and $\epsilon' = \epsilon \circ l^{-1} r^{-1}$.*

Proof: The morphism r is given by the composition $R'\epsilon \circ R' l^{-1} R \circ \eta' R$ and its inverse by the composition $R\epsilon' \circ RlR' \circ \eta R'$. $\square$

Corollary 1.4. *If a functor has a right (resp. left) adjoint, this adjoint is unique up to unique isomorphism.*

Lemma 1.5. *An equivalence of categories is an adjoint couple (F, G, a, b) for which the unit and counit are isomorphisms. In particular, (G, F, b, a) is also an adjoint couple.*

Lemma 1.6. *Let (L, R, η, ϵ) (resp. $(L', R', \eta', \epsilon')$) be an adjoint couple from $\mathcal{C}$ to $\mathcal{D}$ (resp. from $\mathcal{D}$ to $\mathcal{E}$). Then $(L'L, RR', R\eta' L \circ \eta, \epsilon' \circ L'\epsilon R')$ is an adjoint couple from $\mathcal{C}$ to $\mathcal{E}$.*

We now turn to a series of less standard results, nevertheless very easy.

Lemma 1.7. *Let H, H', J_1, K_1, J_2 and K_2 be functors with sources and targets as on the following diagram.*

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "H"] \arrow[d, shift left, "K_1"] & \mathcal{C}_2 \arrow[d, shift left, "K_2"] \\
\mathcal{C}'_1 \arrow[r, "H'"'] \arrow[u, shift left, "J_1"] & \mathcal{C}'_2 \arrow[u, shift left, "J_2"]
\end{tikzcd}$$

Assume $(J_i, K_i, \eta_i, \epsilon_i)$, $i = 1, 2$ are adjoint couples. Let $a : J_2H' \to HJ_1$ (resp. $b : H'K_1 \to K_2H$) be a morphism of functors. Then there exists a unique morphism of functors $b : H'K_1 \to K_2H$ (resp. $a : J_2H' \to HJ_1$) such that the diagrams

$$\begin{tikzcd}
J_2H'K_1 \arrow[r, "J_2b"] \arrow[d, "aK_1"'] \arrow[dr, phantom, "\boxed{H}"] & J_2K_2H \arrow[d, "\eta_2 H"] \\
HJ_1K_1 \arrow[r, "H\epsilon_1"'] & H
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
H' \arrow[r, "H'\eta_1"] \arrow[d, "\eta_2 H'"'] \arrow[dr, phantom, "\boxed{H'}"] & H'K_1J_1 \arrow[d, "bJ_1"] \\
K_2J_2H' \arrow[r, "K_2a"'] & K_2HJ_1
\end{tikzcd}$$

are commutative. Furthermore, given two morphisms of functors a and b, the commutativity of one diagram is equivalent to the commutativity of the other one.

Proof: We only prove that the existence of a implies the uniqueness and existence of b, the proof of the other case is similar. Assume that b exists and makes the diagrams commutative. The commutative diagram

$$\begin{tikzcd}
H'K_1 \arrow[r, "H'\eta_1K_1"] \arrow[d, "\eta_2H'K_1"'] \arrow[dr, phantom, "\boxed{H'}"] & H'K_1J_1K_1 \arrow[r, "H'K_1\epsilon_1"] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & H'K_1 \arrow[d, "b"] \\
K_2J_2H'K_1 \arrow[r, "K_2aK_1"'] & K_2HJ_1K_1 \arrow[r, "K_2H\epsilon_1"'] & K_2H
\end{tikzcd}$$

in which the upper horizontal composition is the identity of $H'K_1$ (by adjunction) shows that b has to be given by the composition

$$H'K_1 \xrightarrow{\eta_2H'K_1} K_2J_2H'K_1 \xrightarrow{K_2aK_1} K_2HJ_1K_1 \xrightarrow{K_2H\epsilon_1} K_2H\ .$$

This proves uniqueness. Now let b be given by the above composition. The commutative diagram

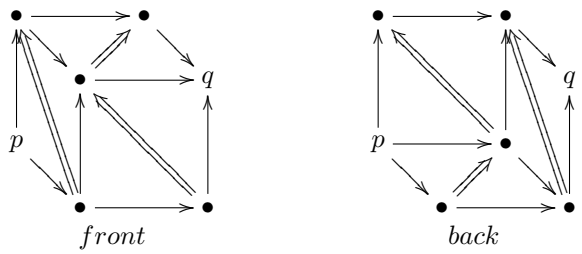
$$\begin{tikzcd}
J_2H'K_1 \arrow[r] \arrow[rrr, bend left, "J_2b"] \arrow[dr, equal] & J_2K_2J_2H'K_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{adj}"] & J_2K_2HJ_1K_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & J_2K_2H \arrow[d, "\epsilon_2H"] \\
& J_2H'K_1 \arrow[r, "aK_1"'] & HJ_1K_1 \arrow[r, "H\epsilon_1"'] & H
\end{tikzcd}$$

proves $\boxed{H}$ and the commutative diagram

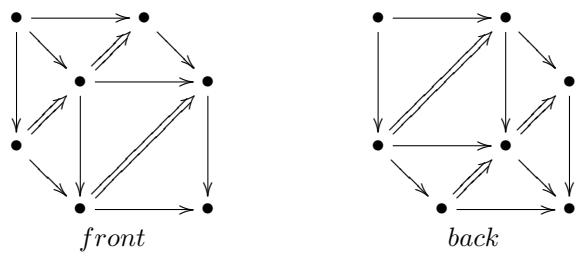
$$\begin{tikzcd}
H'K_1J_1 \arrow[r] \arrow[rrr, bend left, "bJ_1"] & K_2J_2H'K_1J_1 \arrow[r] & K_2HJ_1K_1J_1 \arrow[r] & K_2HJ_1 \\
H' \arrow[u, "H'\eta_1"] \arrow[r, "\eta_2H'"'] \arrow[ur, phantom, "\boxed{mf}"] & K_2J_2H' \arrow[u] \arrow[r, "K_2a"'] \arrow[ur, phantom, "\boxed{mf}"] & K_2HJ_1 \arrow[u] \arrow[ur, phantom, "\boxed{adj}"] \arrow[ur, equal] &
\end{tikzcd}$$

proves $\boxed{H'}$. The fact that the commutativity of one of the diagrams implies commutativity to the other is left to the reader. □

Lemma 1.8. *Let us consider a cube of functors and morphisms of functors*



that is commutative in the following sense: The morphism between the two outer compositions of functors from p to q given by the composition of the three morphisms of functors of the front is equal to the composition of the three morphism of functors of the back. Assume that the vertical maps have right adjoints. Then, by Lemma 1.7 applied to the vertical squares, we obtain the following cube (the top and bottom squares have not changed).



This cube is commutative (in the sense just defined).

Proof: This is straightforward, using the commutative diagrams of Lemma 1.7, and left to the reader. □

We now use Lemma 1.7 to prove a theorem, which doesn't contain a lot more than the lemma, but is stated in a convenient way for future reference in the applications we are interested in.

Theorem 1.9. *Let L, R, L', R', F_1, G_1, F_2, G_2 be functors whose sources and targets are specified by the diagram*

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, shift left, "L"] \arrow[d, shift left, "G_1"] & \mathcal{C}_2 \arrow[l, shift left, "R"] \arrow[d, shift left, "G_2"] \\
\mathcal{C}_1' \arrow[r, shift left, "L'"] \arrow[u, shift left, "F_1"] & \mathcal{C}_2'. \arrow[l, shift left, "R'"] \arrow[u, shift left, "F_2"]
\end{tikzcd}$$

We will study morphisms of functors f_L, f'_L, g_L, g'_L, f_R, f'_R, g_R and g'_R whose sources and targets will be as follows:

$$LF_1 \underset{f'_L}{\overset{f_L}{\rightleftarrows}} F_2L' \qquad L'G_1 \underset{g_L}{\overset{g'_L}{\rightleftarrows}} G_2L$$

$$F_1R' \underset{f_R}{\overset{f'_R}{\rightleftarrows}} RF_2 \qquad G_1R \underset{g'_R}{\overset{g_R}{\rightleftarrows}} R'G_2$$

Let us consider the following diagrams, in which the maps and their directions will be the obvious ones induced by the eight maps above and the adjunctions accordingly to the different cases discussed below.

$$\begin{tikzcd} F_2L'G_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{L}"] & LF_1G_1 \arrow[d, dash] \\ F_2G_2L \arrow[r, dash] & L \end{tikzcd} \qquad \begin{tikzcd} L' \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{L'}"] & G_2F_2L' \arrow[d, dash] \\ L'G_1F_1 \arrow[r, dash] & G_2LF_1 \end{tikzcd}$$

$$\begin{tikzcd} F_1R'G_2 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{R}"] & RF_2G_2 \arrow[d, dash] \\ F_1G_1R \arrow[r, dash] & R \end{tikzcd} \qquad \begin{tikzcd} R' \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{R'}"] & G_1F_1R' \arrow[d, dash] \\ R'G_2F_2 \arrow[r, dash] & G_1RF_2 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{G_1}"] & G_1RL \arrow[d, dash] \\ R'L'G_1 \arrow[r, dash] & R'G_2L \end{tikzcd} \qquad \begin{tikzcd} L'G_1R \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{G_2}"] & G_2LR \arrow[d] \\ L'R'G_2 \arrow[r] & G_2 \end{tikzcd}$$

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{F_1}"] & F_1R'L' \arrow[d, dash] \\ RLF_1 \arrow[r, dash] & RF_2L' \end{tikzcd} \qquad \begin{tikzcd} LF_1R' \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{F_2}"] & F_2L'R' \arrow[d] \\ LRF_2 \arrow[r] & F_2 \end{tikzcd}$$

Then

1. *Let (G_i, F_i), $i = 1, 2$ be adjoint couples. Let g_L (resp. f_L) be given, then there is a unique f_L (resp. g_L) such that $\boxed{L}$ and $\boxed{L'}$ are commutative. Let g_R (resp. f_R) be given, then there is a unique f_R (resp. g_R) such that $\boxed{R}$ and $\boxed{R'}$ are commutative.*

1'. *Let (F_i, G_i), $i = 1, 2$ be adjoint couples. Let g'_L (resp. f'_L) be given, then there is a unique f'_L (resp. g'_L) such that $\boxed{L}$ and $\boxed{L'}$ are commutative. Let g'_R (resp. f'_R) be given, then there is a unique f'_R (resp. g'_R) such that $\boxed{R}$ and $\boxed{R'}$ are commutative.*

2. *Let (L, R) and (L', R') be adjoint couples. Let f_L (resp. f'_R) be given, then there is a unique f'_R (resp. f_L) such that $\boxed{F_1}$ and $\boxed{F_2}$ are commutative. Let g'_L (resp. g_R) be given, then there is a unique g_R (resp. g'_L) such that $\boxed{G_1}$ and $\boxed{G_2}$ are commutative.*

3. *Assuming (G_i, F_i), $i = 1, 2$, (L, R) and (L', R') are adjoint couples, and g_L, $g'_L = g_L^{-1}$ are given (resp. f_R and $f'_R = f_R^{-1}$). By 1 and 2, we obtain f_L and g_R (resp. g_R and f_L). We then may construct f_R and f'_R (resp. g_L and g'_L) which are inverse to each other.*

3'. *Assuming (F_i, G_i), $i = 1, 2$, (L, R) and (L', R') are adjoint couples, f_L and $f'_L = f_L^{-1}$ are given (resp. g_R and $g'_R = g_R^{-1}$). By 1' and 2, we obtain g'_L and f'_R (resp. f'_R and g'_L). We then may construct g'_R and g_R (resp. f'_L and f_L) which are inverse to each other.*

Proof: Points 1, 1' and 2 are obvious translations of the previous lemma. We only prove Point 3, since 3' is dual to it. Let (L, R, η, ϵ), $(L', R', \eta', \epsilon')$ and $(G_i, F_i, \eta_i, \epsilon_i)$, $i = 1, 2$, be the adjoint couples. Using 1 and 2, we first obtain f_L and g_R, as well as the commutative diagrams $\boxed{L}$, $\boxed{L'}$ (both involving $g_L = (g'_L)^{-1}$), $\boxed{G_1}$ and $\boxed{G_2}$ (both involving $g'_L = (g_L)^{-1}$). The morphisms of functors f'_R and f_R are respectively defined by the compositions

$$F_1R' \xrightarrow{\eta F_1 R'} RLF_1R' \xrightarrow{Rf_L R'} RF_2L'R' \xrightarrow{RF_2\epsilon'} RF_2$$

and

$$RF_2 \xrightarrow{\eta_1 RF_2} F_1G_1RF_2 \xrightarrow{F_1 g_R F_2} F_1R'G_2F_2 \xrightarrow{F_1R'\epsilon_2} F_1R' \ .$$

We compute $f_R \circ f'_R$ as the upper right composition of the following commutative diagram

$$\begin{tikzcd}
F_1R' \arrow[r, "\eta"] \arrow[d, "\eta_1"'] \arrow[dr, phantom, "\boxed{mf}"] & RLF_1R' \arrow[r, "f_L"] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & RF_2L'R' \arrow[r, "\epsilon'"] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & RF_2 \arrow[d, "\eta_1"] \\
F_1G_1F_1R' \arrow[r] \arrow[dr, phantom, "\boxed{G_1}"] \arrow[ddr, bend right, "\eta'"'] & F_1G_1RLF_1R' \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & F_1G_1RF_2L'R' \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & F_1G_1RF_2 \arrow[d, "g_R"] \\
 & F_1R'G_2LF_1R' \arrow[r] \arrow[d, "(g'_L)^{-1}=g_L"] \arrow[dr, phantom, "\boxed{L'}"] & F_1R'G_2F_2L'R' \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & F_1R'G_2F_2 \arrow[d, "\epsilon_2"] \\
 & F_1R'L'G_1F_1R' \arrow[r, "\epsilon_1"'] & F_1R'L'R' \arrow[r, "\epsilon'"'] & F_1R'
\end{tikzcd}$$

The lower left composition in the above diagram is the identity because it appears as the upper right composition of the commutative diagram

$$\begin{tikzcd}
F_1R' \arrow[r, "\eta_1"] \arrow[dr, equal] & F_1G_1F_1R' \arrow[r, "\eta'"] \arrow[d, "adj"] \arrow[dr, phantom, "\boxed{mf}"] & F_1R'L'G_1F_1R' \arrow[d, "\epsilon_1"] \\
 & F_1R' \arrow[r] \arrow[dr, equal] \arrow[dr, phantom, "\boxed{adj}"] & F_1R'L'R' \arrow[d, "\epsilon'"] \\
 & & F_1R'
\end{tikzcd} \ .$$

The composition $f'_R \circ f_R = id$ is proved in a similar way, involving the diagrams $\boxed{L}$ and $\boxed{G_2}$. $\square$

The reader has certainly noticed that there is a statement $2'$ which we didn't spell out because we don't need it.

1.3. **Bifunctors and adjunctions.** We have to deal with couples of bifunctors that give adjoint couples of usual functors when one of the entries in the bifunctors is fixed. We need to formalize how these adjunctions are functorial in this entry. The standard example for that is the classical adjunction between tensor product and internal Hom.

Definition 1.10. Let $\mathcal{X}$ be a category. We denote by $\mathcal{M}or\mathcal{X}$ a category of morphisms of $\mathcal{X}$ constructed as follows. The objects of $\mathcal{M}or\mathcal{X}$ are morphisms $[f : A \to B]$ from $\mathcal{X}$ and a morphism from $[f_1 : A_1 \to B_1]$ to $[f_2 : A_2 \to B_2]$ is a

pair (a,b) of morphisms of $\mathcal{X}$ such that $a : A_2 \to A_1$, $b : B_1 \to B_2$ and

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] & B_1 \arrow[d, "b"] \\ A_2 \arrow[u, "a"] \arrow[r, "f_2"] & B_2 \end{tikzcd}$$

is commutative. Morphisms are composed in the obvious way.

Sending a morphism to its target (resp. to its source) defines a functor from $\mathcal{M}or\mathcal{X}$ to $\mathcal{X}$ (resp. $\mathcal{X}^o$) denoted by $P_{\mathcal{X}}$ (resp. $Q_{\mathcal{X}}$). For any $F : \mathcal{X} \to \mathcal{C}$ (resp. $F : \mathcal{X}^o \to \mathcal{C}$), we define $\bar{F} = FP_{\mathcal{X}}$ (resp. $\bar{F} = FQ_{\mathcal{X}}$).

Lemma 1.11. *Let $F_1, F_2 : \mathcal{X} \to \mathcal{C}$ (resp. $\mathcal{X}^o \to \mathcal{C}$). Sending a morphism of functors $\mu : F_1 \to F_2$ to $\bar{\mu} = \mu P_{\mathcal{X}}$ (resp. $\bar{\mu} = \mu Q_{\mathcal{X}}$) is a one-to-one correspondence between the morphisms of functors from F_1 to F_2 and the morphisms of functors from $\bar{F}_1$ to $\bar{F}_2$.*

Proof: Given $\bar{\mu}$, we can recover μ by defining μ_X, for an object X in $\mathcal{X}$ by the composition

$$F_1(X) = \bar{F}_1([id_X]) \xrightarrow{\bar{\mu}_{[id_X]}} \bar{F}_2([id_X]) = F_2(X)$$

We have to prove that μ satisfies the usual commutative diagrams of morphism of functors. Let $f : X \to Y$ be a morphism in $\mathcal{X}$. In $\mathcal{M}or\mathcal{X}$, we have the morphisms $(id, f) : [id_X] \to [f]$ and $(f, id) : [id_Y] \to [f]$. Since $\bar{\mu}$ is a morphism of functors, we have the commutative diagram

$$\begin{tikzcd} F_1(X) \arrow[r, equal] \arrow[d, "F_1(f)"'] & \bar{F}_1([id_X]) \arrow[r, "\bar{\mu}_{[id_X]}"] \arrow[d, "\bar{F}_1(id,f)"'] & \bar{F}_2([id_X]) \arrow[r, equal] \arrow[d, "\bar{F}_2(id,f)"] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, equal] \arrow[d, equal, "F_1(id)"'] & \bar{F}_1([f]) \arrow[r, "\bar{\mu}_{[f]}"] & \bar{F}_2([f]) \arrow[r, equal] & F_2(Y) \arrow[d, equal, "F_2(id)"] \\ F_1(Y) \arrow[r, equal] & \bar{F}_1([id_Y]) \arrow[r, "\bar{\mu}_{[id_Y]}"] \arrow[u, "\bar{F}_1(f,id)"] & \bar{F}_2([id_Y]) \arrow[r, equal] \arrow[u, "\bar{F}_2(f,id)"] & F_2(Y) \end{tikzcd}$$

whose outer part shows what we want. It is easy to see that this defines an inverse to $\mu \mapsto \bar{\mu}$. $\square$

Let $\mathcal{C}$, $\mathcal{C}'$ be two categories, and let $L : \mathcal{X} \times \mathcal{C}' \to \mathcal{C}$ and $R : \mathcal{X}^o \times \mathcal{C} \to \mathcal{C}'$ be bifunctors. We define $\tilde{L} : \mathcal{M}or\mathcal{X} \times \mathcal{C}' \to \mathcal{M}or\mathcal{X} \times \mathcal{C}$ (resp. $\tilde{R} : \mathcal{M}or\mathcal{X} \times \mathcal{C} \to \mathcal{M}or\mathcal{X} \times \mathcal{C}'$) via the projection to $\mathcal{M}or\mathcal{X}$ and the morphism $\mathcal{M}or\mathcal{X} \times \mathcal{C}' \xrightarrow{L \circ (P_{\mathcal{X}} \times id)} \mathcal{C}$ (resp. $\mathcal{M}or\mathcal{X} \times \mathcal{C} \xrightarrow{R \circ (Q_{\mathcal{X}} \times id)} \mathcal{C}'$). By a slight abuse of notation the latter morphism will be sometimes denoted by $\bar{L}$.

Definition 1.12. We say that (L, R) form an adjoint couple of bifunctors (abbreviated as ACB) from $\mathcal{C}'$ to $\mathcal{C}$ with parameter in $\mathcal{X}$ if $(\tilde{L}, \tilde{R})$ is an adjoint couple in the usual sense, with a unit and counit whose component with respect to $\mathcal{M}or\mathcal{X}$ is the identity. In particular, for any morphism $f : A \to B$ the unit and counit yield morphisms of functors

$$Id_{\mathcal{C}'} \to R(A, L(B, -)) \qquad\qquad L(B, R(A, -)) \to Id_{\mathcal{C}}.$$

We sometimes use the notation $(L(*, -), R(*, -))$, where the $*$ is the entry in $\mathcal{X}$ and write

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{X} \arrow[l, shift left, "R"] & \mathcal{C}' \end{tikzcd}$$

in diagrams.

Lemma 1.13. *If (L, R) is an ACB, then $(L(X, -), R(X, -))$ is an adjoint couple for every X in $\mathcal{X}$.*

Proof: Easy and left to the reader. □

Example 1.14. Let $\mathcal{C} = \mathcal{C}' = \mathcal{X}$ be the category of modules over a commutative ring. The tensor product and the internal Hom form an ACB, with the usual unit and counit.

Lemma 1.15. *Let $F : \mathcal{X}' \to \mathcal{X}$ be a functor, and (L, R) be an ACB with parameter in $\mathcal{X}$. Then $(L(F(*), -), R(F(*), -))$ is again an ACB in the obvious way.*

Lemma 1.16. *Let (L, R) and (L', R') be ACBs, and let $l : L \to L'$ (resp. $r : R \to R'$) be an isomorphism of bifunctors. We define $\tilde{l} : \tilde{L} \to \tilde{L}'$ (resp. $\tilde{r} : \tilde{R} \to \tilde{R}'$) as $(id_{p\mathcal{M}or\mathcal{X}}, \bar{l})$ (resp. $(id_{p\mathcal{M}or\mathcal{X}}, \bar{r})$). Then, there exists a unique isomorphism of bifunctors $r : R \to R'$ (resp. $l : L \to L'$) such that $\tilde{r} : \tilde{R} \to \tilde{R}'$ and $\tilde{l} : \tilde{L} \to \tilde{L}'$ induced by r and l satisfy $\eta' = \tilde{r}\tilde{l} \circ \eta$ and $\epsilon' = \epsilon \circ \tilde{l}^{-1}\tilde{r}^{-1}$, where η, η' and ϵ, ϵ' are the units and counits of the ACBs.*

Proof: Apply Lemma 1.3 to the functors with˜and lift the resulting isomorphism of functors using Lemma 1.11. □

Corollary 1.17. *A right (resp. left) adjoint in an ACB is unique up to unique isomorphism.*

Lemma 1.18. *Let (L, R) be an ACB from $\mathcal{C}'$ to $\mathcal{C}$ with parameter in $\mathcal{X}$ and let (F, G) be an adjoint couple from $\mathcal{D}$ to $\mathcal{C}'$, then $(L(*, F(-)), GR(*, -))$ is an ACB (with unit and counit given in an obvious and natural way).*

Proof: Left to the reader. □

We now give a version of Lemma 1.7 for ACBs.

Lemma 1.19. *Let (J_1, K_1) (resp. (J_2, K_2)) be an ACB from $\mathcal{C}'_1$ to $\mathcal{C}_1$ (resp. $\mathcal{C}'_2$ to $\mathcal{C}_2$) with parameter in $\mathcal{X}$, let $H : \mathcal{C}_1 \to \mathcal{C}_2$ and $H' : \mathcal{C}'_1 \to \mathcal{C}'_2$ be functors. Let $a : J_2(*, H'(-)) \to HJ_1(*, -)$ (resp. $b : H'K_1(*, -) \to K_2(*, H(-))$) be a morphism of bifunctors. Then there exists a unique morphism of bifunctors $b : H'K_1(*, -) \to K_2(*, H(-))$ (resp. $a : J_2(*, H'(-)) \to HJ_1(*, -)$) such that the diagrams*

$$\begin{tikzcd}
J_2(B, H'K_1(A, -)) \arrow[r, "J_2 b"] \arrow[d, "aK_1"'] \arrow[dr, phantom, "\boxed{H}"] & J_2(B, K_2(A, H(-))) \arrow[d, "\eta_2 H"] \\
HJ_1(B, K_1(A, -)) \arrow[r, "H\epsilon_1"'] & H
\end{tikzcd}$$

and

$$\begin{tikzcd}
H' \arrow[r, "H'\eta_1"] \arrow[d, "\eta_2 H'"'] \arrow[dr, phantom, "\boxed{H'}"] & H'K_1(A, J_1(B, -)) \arrow[d, "bJ_1"] \\
K_2(A, J_2(B, H'(-))) \arrow[r, "K_2 a"'] & K_2(A, HJ_1(B, -))
\end{tikzcd}$$

are commutative for every morphism $f : A \to B$ in $\mathcal{X}$. The maps η_i and ϵ_i are the components of the unit and counit of the ACBs to $\mathcal{C}'_i$ and to $\mathcal{C}_i$.

Proof: We apply Lemma 1.7 to the functors in the diagram

$$\begin{tikzcd} \mathcal{M}or\mathcal{X} \times \mathcal{C}_1 \arrow[r, "Id\times H"] \arrow[d, shift left, "\tilde{K}_1"] & \mathcal{M}or\mathcal{X} \times \mathcal{C}_2 \arrow[d, shift left, "\tilde{K}_2"] \\ \mathcal{M}or\mathcal{X} \times \mathcal{C}_1' \arrow[r, "Id\times H'"'] \arrow[u, shift left, "\tilde{J}_1"] & \mathcal{M}or\mathcal{X} \times \mathcal{C}_2' \arrow[u, shift left, "\tilde{J}_2"] \end{tikzcd}$$

and to the morphism of functors

$$\tilde{a} : \tilde{J}_2(Id \times H') \to (Id \times H)\tilde{J}_1$$

or

$$\tilde{b} : (Id \times H)\tilde{K}_1 \to \tilde{K}_2(Id \times H')$$

induced by a or b. Then we use Lemma 1.11 to lift the morphism of functor obtained (between functors with $\tilde{}$) to a morphism of (bi)functors. □

1.4. **Suspended and triangulated categories.** We recall here what we need about triangulated categories. The reason why we use the concept of a suspended category is because all the commutative diagrams that have to be satisfied when we deal with Witt groups are just related to the suspension, and not to the exactness of the functors involved. So when we need to prove the commutativity of those diagrams, we forget about the exactness of our functors, and just think of them as suspended functors, in the sense described below.

Definition 1.20. A suspended category is an additive category $\mathcal{C}$ together with an adjoint couple (T, T^{-1}) from $\mathcal{C}$ to $\mathcal{C}$ which is an equivalence of category (the unit and counit are isomorphisms).

Remark 1.21. We assume furthermore in all what follows that TT^{-1} and $T^{-1}T$ are the identity of $\mathcal{C}$ and that the unit and counit are also the identity. This assumption is not true in some suspended (triangulated) categories arising in stable homotopy theory. Nevertheless, it simplifies the exposition which is already sufficiently technical. When working in an example where this assumption does not hold, it is of course possible to make the modifications to get this even more general case.

Between suspended categories $(\mathcal{C}, T_{\mathcal{C}})$ and $(\mathcal{D}, T_{\mathcal{D}})$, we use suspended functors:

Definition 1.22. A suspended functor (F, f) from $\mathcal{C}$ to $\mathcal{D}$ is a functor F together with an isomorphism of functors $f : FT_{\mathcal{C}} \to T_{\mathcal{D}}F$. We sometimes forget about f in the notation.

Without the assumption in Remark 1.21, we would need another isomorphism $f' : FT^{-1} \to T^{-1}F$ and compatibility diagrams analogous to the ones in Lemma 1.7. Then, we would have to carry those compatibilities in our constructions. Again, this would not be a problem, just making things even more tedious.

Suspended functors can be composed in an obvious way, and (T, id_{T^2}) and (T^{-1}, id_{Id}) are suspended endofunctors of $\mathcal{C}$ that we call T and T^{-1} for short.

Definition 1.23. To a suspended functor F, one can associate "shifted" ones, composing F by T or T^{-1} several times on either sides.

Definition 1.24. The opposite suspended category $\mathcal{C}^o$ of a suspended category $\mathcal{C}$ is given the suspension $(T_{\mathcal{C}}^{-1})^o$.

With this convention, we can deal with contravariant suspended functors in two different ways (depending where we put the "op"), and this yields essentially the same thing, using the definition of shifted suspended functors.

Definition 1.25. A morphism of suspended functors $h : (F, f) \to (G, g)$ is a morphism of functors $h : F \to G$ such that the diagram

$$\begin{tikzcd} FT \arrow[r, "f"] \arrow[d, "hT"'] \arrow[dr, phantom, "\boxed{sus}"] & TF \arrow[d, "Th"] \\ GT \arrow[r, "g"'] & TG \end{tikzcd}$$

is commutative.

Lemma 1.26. *The composition of two morphisms of suspended functors yields a morphism of suspended functors.*

Proof: Straightforward. □

A triangulated category is a suspended category with the choice of some exact triangles, satisfying some axioms. This can be found in text books as [28](see also the nice introduction in [3, Section 1]). We include the enriched octahedron axiom in the list of required axioms as it is suitable to deal with Witt groups, as explained in *loc. cit.*

Definition 1.27. (see for example [10, § 1.1]) Let $(F, f) : \mathcal{C} \to \mathcal{D}$ be a covariant (resp. contravariant) suspended functor. We say that (F, f) is δ-exact ($\delta = \pm 1$) if for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

the triangle

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\delta f_A \circ Fw} TFA$$

respectively

$$FC \xrightarrow{Fv} FB \xrightarrow{Fu} FA \xrightarrow{\delta f_C \circ FT^{-1} w} TFC$$

is exact.

Remark 1.28. With this definition, T and T^{-1} are (-1)-exact functors, because of the second axiom of triangulated categories, and the composition of exact functors multiplies their signs. Thus, if F is δ-exact, then $T^i F T^j$ is $(-1)^{i+j}\delta$-exact.

To define morphisms between exact functors F and G, the signs δ_F and δ_G of the functors have to be taken into account, so that the morphism of functors induces a morphism between the triangles obtained by applying F or G to a triangle and making the sign modifications.

Definition 1.29. We say that $h : F \to G$ is a morphism of exact functors if the diagram $\boxed{sus}$ in Definition 1.25 is $\delta_F \delta_G$ commutative.

On the other hand, we have the following lemma.

Lemma 1.30. *Let $h : F \to G$ be an isomorphism of suspended functors such that $\boxed{sus}$ is ν-commutative. Assume F is δ-exact. Then G is $\delta\nu$-exact.*

Proof: For any triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

the triangle

$$GA \xrightarrow{Gu} GB \xrightarrow{Gv} FC \xrightarrow{\delta g_A \circ Gw} TGA$$

is easily shown to be isomorphic to

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\nu\delta f_A \circ Fw} TFA$$

□

We also need to deal with bifunctors from two suspended categories to another one. These are just suspended functors in each variable, with a compatibility condition. Examples are the internal Hom or the tensor product in triangulated categories.

Definition 1.31. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ be suspended categories. A suspended bifunctor from $\mathcal{C}_1 \times \mathcal{C}_2$ to $\mathcal{D}$ is a triple (B, b_1, b_2) where $B : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ is a functor and two morphisms of functors $b_1 : B(T(-), *) \to TB(-, *)$ and $b_2 : B(-, T(*)) \to TB(-, *)$, such that the diagram

$$\begin{tikzcd} B(TA, TC) \arrow[r, "b_{2,TA,C}"] \arrow[d, "b_{1,A,TC}"'] \arrow[dr, phantom, "-1"] & TB(TA, C) \arrow[d, "b_{1,A,C}"] \\ TB(A, TC) \arrow[r, "b_{2,A,C}"'] & T^2B(A, C) \end{tikzcd}$$

anti-commutes for every A and C.

Definition 1.32. A morphism of suspended bifunctors from a suspended bifunctor (B, b_1, b_2) to a suspended bifunctor (B', b'_1, b'_2) is a morphism of functors $f : B \to B'$ such that the two diagrams

$$\begin{tikzcd} B(TA, C) \arrow[r, "b_{1,A,C}"] \arrow[d, "f_{TA,C}"'] & TB(A, C) \arrow[d, "Tf_{A,C}"] & B(A, TC) \arrow[l, "b_{2,A,C}"'] \arrow[d, "f_{A,TC}"] \\ B'(TA, C) \arrow[r, "b'_{1,A,C}"] & TB'(A, C) & B'(A, TC) \arrow[l, "b'_{2,A,C}"'] \end{tikzcd}$$

are commutative for every A and C.

By composing with a usual suspended functor to $\mathcal{C}_1$ or $\mathcal{C}_2$ or from $\mathcal{D}$, we get other suspended bifunctors (the verification is easy). But, if we do that several times, using different functors, the order in which we do it *does* matter. For example, as with usual suspended functors, it is possible to define shifted versions by composing with the suspensions in each category. This can be useful. Unfortunately, according to the order in which we do this, we don't get the same isomorphism of functors, even though we get the same functors in the pair. One has to be careful about that.

1.5. **Suspended adjunctions.** As with usual functors, there is a notion of adjunction well suited for suspended functors.

Definition 1.33. A suspended adjoint couple (L, R) is a adjoint couple in the usual sense in which L and R are suspended functors and the unit and counit are morphisms of suspended functors.

Definition 1.34. When (L, R) is an adjoint couple of suspended functors, using Lemma 1.3 and Definition 1.23 we obtain shifted versions $(T^iLT^j, T^{-j}RT^{-i})$.

The following proposition seems to be well-known.

Proposition 1.35. *Let (L, R) be an adjoint couple from $\mathcal{C}$ to $\mathcal{D}$ (of usual functors) and let (L, l) be a suspended functor. Then*

1. *there is a unique isomorphism of functors $r : RT \to TR$ that turns (R, r) into a suspended functor and (L, R) into a suspended adjoint couple.*
2. *if furthermore $\mathcal{C}$ and $\mathcal{D}$ are triangulated and (L, l) is δ-exact, then (R, r) is also δ-exact (with the same δ).*

Proof: Point 1 is a direct corollary of Point 3 of Theorem 1.9, by taking $L = L'$, $R = R'$, $F_1 = T_{\mathcal{C}}$, $G_1 = T_{\mathcal{C}}^{-1}$, $F_2 = T_{\mathcal{D}}$, $G_2 = T_{\mathcal{D}}^{-1}$, $g_L = (g'_L)^{-1} = T^{-1}lT^{-1}$. This gives $f_R = r$. The commutative diagrams $\boxed{F_1}$ and $\boxed{F_2}$ exactly tell us that the unit and counit are suspended morphisms of functors with this choice of r. To prove Point 2, we have to show that the pair (R, r) is exact. Let

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

be an exact triangle. We want to prove that the triangle

$$RA \xrightarrow{u} RB \xrightarrow{v} RC \xrightarrow{r_A \circ Rw} TRA$$

is exact. We first complete $RA \xrightarrow{u} RB$ as an exact triangle

$$RA \xrightarrow{u} RB \xrightarrow{v'} C' \xrightarrow{w'} TRA$$

and we prove that this triangle is in fact isomorphic to the previous one. To do so, one completes the incomplete morphism of triangles

$$\begin{tikzcd}
LRA \arrow[r, "LRu"] \arrow[d] & LRB \arrow[r, "LRv"] \arrow[d] & LC' \arrow[r, "f_{RA} \circ LRw"] \arrow[d, dotted, "h"] & TLRA \arrow[d] \\
A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & TA
\end{tikzcd}.$$

Looking at the adjoint diagram, we see that $ad(h) : C' \to RC$ is an isomorphism by the five lemma for triangulated categories. □

Theorem 1.36. *Lemma 1.7, Lemma 1.8 and Theorem 1.9 holds when we replace every functor by a suspended functor, every adjoint couple by a suspended adjoint couple and every morphism of functor by a morphism of suspended functors.*

Proof: The same proofs hold, since they only rely on operations and properties of functors and morphism of functors, such as composition or commutative diagrams, that exist and behave the same way in the suspended case. □

We now need to deal with suspended bifunctors and adjunctions. When $\mathcal{X}$ is a suspended category, there is an induced suspension on $\mathcal{M}or\mathcal{X}$ which we always use except when we mention otherwise. When L is a suspended bifunctor, then $\tilde{L}$ is also one in the obvious way.

When (L, R) is an ACB in which (L, l_1, l_2) and (R, r_1, r_2) are suspended bifunctors, there are two ways to obtain an ACB $(L(*, T(-)), T^{-1}R(*, -))$. The first one is to apply Lemma 1.18 to (L, R) and (T, T^{-1}); the second one is to apply Lemma 1.15 to (L, R) and T, to get an ACB $(L(T(*), -), R(T(*), -))$ and then use this last couple and the isomorphisms

$$L(T(*), -) \xrightarrow{l_2} TL(*, -) \xrightarrow{l_1^{-1}} L(*, T(-))$$

and

$$R(T(*), -) \xrightarrow{T^{-1}r_1^{-1}(T, Id)} T^{-1}R(*, -)$$

in Lemma 1.16. Thus, we use the following definition.

Definition 1.37. Let (L, R) be an ACB from $\mathcal{C}'$ to $\mathcal{C}$ with parameter in $\mathcal{X}$, where $\mathcal{C}, \mathcal{C}'$ and $\mathcal{X}$ are suspended categories. Assume moreover that (L, l_1, l_2) and (R, r_1, r_2) are suspended bifunctors. We say that (L, R) is a suspended adjoint couple of bifunctors if

(1) $(\tilde{L}, \tilde{R})$ is a suspended adjoint couple (in the usual sense) when we take the suspensions (Id, T) on $\mathcal{M}or\mathcal{X} \times \mathcal{C}$ and $\mathcal{M}or\mathcal{X} \times \mathcal{C}'$ and

(2) the two ACBs $(L(*,T(-)),T^{-1}R(*,-))$ obtained as mentioned above coincide (*i.e.* their units and counits are the same).

Remark 1.38. Note that (1) ensures that we have suspended adjoint functors when we fix the parameter and (2) explains the compatibility of the suspensions (including the one for the parameter).

The following proposition is an analogue of Proposition 1.35 for suspended bifunctors.

Proposition 1.39. *Let (L,R) be an ACB such that (L,l_1,l_2) (resp. (R,r_1,r_2)) is a suspended bifunctor. Then*

(1) *there exist unique r_1,r_2 (resp. l_1,l_2) such that (L,R) is a suspended ACB,*
(2) *if $(L(A,-),l_1)$ (resp. $(L(*,A))$ is δ-exact for some object A, then so is $R(A,-)$ (resp. $R(*,A)$).*

Proof: In Point 1, the uniqueness is clear by Point 2 in Definition 1.37 for r_2 and by Lemma 1.3 for r_1. The existence is obtained by applying Point 1 of Proposition 1.35 to $\tilde{L}$ and $\tilde{R}$ (and Point 1 of Definition 1.37) to get r_2 from l_2 (again, one has to lift using Lemma 1.11) and then Lemma 1.16 to get r_1. The fact that the required diagram of Definition 1.31 anti-commutes is again a consequence of the uniqueness part of Lemma 1.16. Point 2 is proved in the same way as Point 2 in Proposition 1.35. □

We need a version of Lemma 1.19 for suspended bifunctors.

Lemma 1.40. *Lemma 1.19 holds when all the functors, bifunctors and adjunctions become suspended ones.*

Proof: The proof of Lemma 1.19 works since it only involves compositions and commutative diagrams that exist in the suspended case.
□

Finally, there is also a version of Theorem 1.9 for (suspended) bifunctors, which is our main tool for the applications. For this reason, we state it in full detail.

Theorem 1.41. *Let L, R, L', R', F_1, G_1, F_2, G_2 be (suspended) functors whose sources and targets are specified by the diagram (recall the notation of Definition 1.12),*

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, shift left, "L"] \arrow[d, shift left, "G_1"] \arrow[d, phantom, "\mathcal{X}"] & \mathcal{C}_2 \arrow[l, shift left, "R"] \arrow[d, shift left, "G_2"] \arrow[d, phantom, "\mathcal{X}"] \\
\mathcal{C}_1' \arrow[r, shift left, "L'"] \arrow[u, shift left, "F_1"] & \mathcal{C}_2' \arrow[l, shift left, "R'"] \arrow[u, shift left, "F_2"]
\end{tikzcd}$$

and let f_L, f_L', g_L, g_L', f_R, f_R', g_R and g_R' be morphisms of bifunctors (resp. suspended bifunctors) whose sources and targets will be as follows:

$$LF_1(*,-) \underset{f_L'}{\overset{f_L}{\rightleftarrows}} F_2(*,L'(-)) \qquad L'G_1(*,-) \underset{g_L}{\overset{g_L'}{\rightleftarrows}} G_2(*,L(-))$$

$$F_1(*,R'(-)) \underset{f_R}{\overset{f_R'}{\rightleftarrows}} RF_2(*,-) \qquad G_1(*,R(-)) \underset{g_R'}{\overset{g_R}{\rightleftarrows}} R'G_2(*,-)$$

Let us consider the following diagrams, in which the maps and their directions will be the only obvious ones in all the cases discussed below.

$$\begin{tikzcd}
F_2(B,L'G_1(A,-)) \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{L}"] & LF_1(B,G_1(A,-)) \\
F_2(B,G_2(A,L(-)) & L \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1(C,-) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{G_1}"] & G_1(C,RL(-)) \arrow[d, dash] \\
R'L'G_1(C,-) \arrow[r, dash] & R'G_2(C,L(-))
\end{tikzcd}$$

$$\begin{tikzcd}
F_1(B, R'G_2(A,-)) \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{R}"] & RF_2(B, G_2(A,-)) \\
F_1(B, G_1(A, R(-))) & R \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1(C,-) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{F_1}"] & F_1(C, R'L'(-)) \arrow[d, dash] \\
RLF_1(C,-) \arrow[r, dash] & RF_2(C, L'(-))
\end{tikzcd}$$

$$\begin{tikzcd}
L' & G_2(A, F_2(B, L'(-))) \arrow[l] \arrow[d, dash] \arrow[dl, phantom, "\boxed{L'}"] \\
L'G_1(A, F_1(B,-)) \arrow[u] \arrow[r, dash] & G_2(A, LF_1(B,-))
\end{tikzcd}
\qquad
\begin{tikzcd}
L'G_1(C, R(-)) \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{G_2}"] & G_2(C, LR(-)) \arrow[d] \\
L'R'G_2(C,-) \arrow[r] & G_2(C,-)
\end{tikzcd}$$

$$\begin{tikzcd}
R' & G_1(A, F_1(B, R'(-))) \arrow[l] \arrow[d, dash] \arrow[dl, phantom, "\boxed{R'}"] \\
R'G_2(A, F_2(B,-)) \arrow[u] \arrow[r, dash] & G_1(A, RF_2(B,-))
\end{tikzcd}
\qquad
\begin{tikzcd}
LF_1(C, R'(-)) \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\boxed{F_2}"] & F_2(C, L'R'(-)) \arrow[d] \\
LRF_2(C,-) \arrow[r] & F_2(C,-)
\end{tikzcd}$$

Then

1. *Let (G_i, F_i), $i = 1, 2$ be ACBs (resp. suspended ACBs). Let g_L (resp. f_L) be given, then there is a unique (suspended) f_L (resp. g_L) such that $\boxed{L}$ and $\boxed{L'}$ are commutative for any morphism $f : B \to A$. Let g_R (resp. f_R) be given, then there is a unique (suspended) f_R (resp. g_R) such that $\boxed{R}$ and $\boxed{R'}$ are commutative for any morphism $f : B \to A$.*
2. *Let (L, R) and (L', R') be adjoint couples (resp. suspended adjoint couples). Let f_L (resp. f'_R) be given, then there is a unique (suspended) f'_R (resp. f_L) such that $\boxed{F_1}$ and $\boxed{F_2}$ are commutative. Let g'_L (resp. g_R) be given, then there is a unique (suspended) g_R (resp. g'_L) such that $\boxed{G_1}$ and $\boxed{G_2}$ are commutative.*
3. *Assuming (G_i, F_i), $i = 1, 2$, are ACBs (resp. suspended ACBs), (L, R) and (L', R') are (suspended) adjoint couples, and g_L and $g'_L = g_L^{-1}$ are given (resp. f_R and $f'_R = f_R^{-1}$). By 1 and 2, we obtain f_L and g_R (resp. g_R and f_L). We then may construct f_R and f'_R (resp. g_L and g'_L) which are inverse to each other.*

Proof: The proof is the same as for Theorem 1.9, but using Lemma 1.19 (or 1.40) instead of Lemma 1.7 for Points 1 and 2 and using the lifting of Lemma 1.11 for Point 3. □

Remark 1.42. We didn't state the analogues of Points 1' and 2' of Theorem 1.9 in this context because we don't need them.

1.6. **Dualities.** We now start using our main subject of interest: duality. As before, we state everything for the suspended (or triangulated) case and the usual case in a uniform way.

Definition 1.43. A category with duality is a triple $(\mathcal{C}, D, \varpi)$ where $\mathcal{C}$ is a category with an adjoint couple $(D, D^o, \varpi, \varpi^o)$ from $\mathcal{C}$ to $\mathcal{C}^o$, which is an equivalence of categories (ϖ is an isomorphism). A suspended (resp. triangulated) category with duality is defined in the same way, but (D, d) is a suspended (resp. δ-exact) functor on a suspended (resp. triangulated) category $\mathcal{C}$, and the adjunction is suspended.

Remark 1.44. Observe that the standard condition $(D^o\varpi^o) \circ (\varpi^o_D) = id_{D^o}$ is satisfied by the definition of an adjunction. The only difference between our definition and Balmer's definition [3, Def. 2.2] is that we don't require the isomorphism $d : DT \to T^{-1}D$ to be an equality in the suspended or triangulated case. Assuming that duality and suspension strictly commute is as bad as assuming that the internal Hom and the duality strictly commute, or (by adjunction) that the tensor

product and the suspension strictly commute. This is definitely a too strong condition when checking strict commutativity of diagrams in some derived category. Dropping all signs in this setting when defining these isomorphisms just by saying "take the canonical ones" may even lead to contradictions as the results of the Appendix show. When $d = id$, we say that the duality is strict.

Definition 1.45. Let $(\mathcal{C}, D, \varpi)$ be a triangulated category with duality for which (D, d) is δ-exact. By Definition 1.34 we get a shifted adjoint couple $T(D, D^o) = (TD, D^oT^{-1}, \varpi', (\varpi')^o)$. We define the suspension of $(\mathcal{C}, D, \varpi)$ as $T(\mathcal{C}, D, \varpi) = (\mathcal{C}, TD, -\delta\varpi')$.

Remark 1.46. This is the definition of [3, Definition 2.8] adapted to cover the non strict case, and the next one generalizes [3, Definition 2.13] to the non strict case.

Definition 1.47. For any triangulated category with duality $(\mathcal{C}, D, \varpi)$, we define its i-th Witt group W^i by $\mathrm{W}^i(\mathcal{C}, D, \varpi) := W(T^i(\mathcal{C}, D, \varpi))$ (extending [3, 2.4 and Definitions 2.12 and 2.13] in the obvious way). If D and ϖ are understood, we sometimes also denoted write $\mathrm{W}^i(\mathcal{C})$ for short.

Remark 1.48. In concrete terms, this means that the condition of *loc. cit.* for an element in $W^1(\mathcal{C})$ represented by some ϕ to be symmetric is that $(TdT^{-1}) \circ \phi = (D^oT\phi) \circ \varpi$ whereas in the strict case the (TdT^{-1}) may be omitted.

Definition 1.49. A duality preserving pair (of functors) between (suspended, triangulated) categories with duality $(\mathcal{C}_1, D_1, \varpi_1)$ and $(\mathcal{C}_2, D_2, \varpi_2)$ is a 4-tuple $\{F, G, f, g\}$ where G (resp. F) is a (suspended, δ-exact) functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ (resp. from $\mathcal{C}_1^o$ to $\mathcal{C}_2^o$), $f : D_2G \to FD_1$ in $\mathcal{C}_2^o$, $g : GD_1^o \to D_2^oF$ in $\mathcal{C}_2$ are isomorphisms of (suspended, δ-exact) functors as in Lemma 1.7 (resp. Theorem 1.36) when setting $J_1 = D_1$, $K_1 = D_1^o$, $J_2 = D_2$, $K_2 = D_2^o$, $H = F$ and $H' = G$. In other words, the diagrams $\boxed{H}$ and $\boxed{H'}$ of Lemma 1.7 must commute:

$$\begin{tikzcd}
G \arrow[r, "G\varpi_1"] \arrow[d, "\varpi_2 G"'] & GD_1^oD_1 \arrow[d, "gD_1"] & D_2GD_1^o \arrow[r, "D_2g"] \arrow[d, "fD_1^o"'] & D_2D_2^oF \arrow[d, "\varpi_2^oF"] \\
D_2^oD_2G \arrow[r, "D_2^of"'] & D_2^oFD_1 & FD_1D_1^o \arrow[r, "F\varpi_1^o"'] & F
\end{tikzcd}$$

Lemma 1.7 tells us that g is uniquely determined by f (and vice versa), so that this data is in fact redundant. We sometimes drop the morphisms in the notation, writing $\{F, G\}$ instead of $\{F, G, f, g\}$ when no confusion can occur.

Duality preserving pairs can be composed in an obvious way.

Definition 1.50. A duality preserving functor between (suspended, triangulated) categories with dualities is a duality preserving pair $\{G^o, G, g^o, g\}$, that we just denote by $\{G, g\}$ or sometimes even by $\{G\}$.

Again, when the duality is strict, this coincides with the usual definition (see for example [7, Definition 2.1]). The first diagram in *loc. cit.* is one of the diagrams above (the other one is identical, up to "op") and the second diagram is only used in the suspended case and corresponds to the fact that g is a morphism of suspended functor in our definition.

Duality preserving functors can be composed in an obvious way.

Definition 1.51. A morphism between (suspended, exact) duality preserving pairs $\{F, G, f, g\}$ and $\{F', G', f', g'\}$ is a pair of morphisms of (suspended, exact) functors

$$\rho : F' \to F, \qquad \sigma : G \to G'$$

such that the diagrams

$$\begin{tikzcd} GD_1^o \arrow[r, "g"] \arrow[d, "\sigma D_1^o"'] & D_2^o F \arrow[d, "D_2^o\rho"] \\ G'D_1^o \arrow[r, "g'"'] & D_2^o F' \end{tikzcd} \qquad \begin{tikzcd} D_2 G \arrow[r, "f"] \arrow[d, "D_2\sigma"'] & FD_1 \arrow[d, "\rho D_1"] \\ D_2 G' \arrow[r, "f'"'] & F'D_1 \end{tikzcd}$$

commute.

As above, this data is redundant:

Lemma 1.52. *Any morphism σ or ρ can be completed as a morphism of duality preserving pairs (ρ, σ). If one of the morphisms is an isomorphism, then so is the other one, and thus the pair.*

Proof: Left to the reader. □

Definition 1.53. A morphism between (suspended, exact) duality preserving functors (G, g) and (G', g') is a morphism (ρ, σ) between the underlying duality preserving pairs such that $\sigma^o \rho = id$. This is equivalent to the commutativity of the diagram

$$\begin{tikzcd} GD_1^o \arrow[r, "g"] \arrow[d, "\sigma D_1^o"'] & D_2^o G^o \\ G'D_1^o \arrow[r, "g'"'] & D_2^o (G')^o \arrow[u, "D_2^o \sigma^o"'] \end{tikzcd}$$

and we just denote the morphism σ in this case (forgetting about the redundant ρ in the notation).

Composing two morphisms between duality preserving pairs (resp. functors) obviously gives another one.

Definition 1.54. An adjoint couple of (suspended, triangulated) duality preserving pairs is defined as an adjoint couple in the usual sense, but replacing functors and morphisms of functors by (suspended, triangulated) duality preserving pairs and morphism of (suspended, triangulated) duality preserving pairs.

Theorem 1.55. *Lemma 1.7, Lemma 1.8 and Theorem 1.9 are still valid when we replace*

- *every functor by a (suspended, triangulated) duality preserving pair,*
- *every adjoint couple by a (suspended, triangulated) adjoint couple of duality preserving pairs,*
- *every morphism of functor by a (suspended, triangulated) morphism of duality preserving pairs.*

Proof: The proofs are the same, they just rely on compositions and commutative diagrams. □

Note that restricting the notion of an adjoint couple of duality preserving pairs to duality preserving functors gives something that is rarely satisfied in practice: we would need to have functors L and R such that L is a left *and* a right adjoint of R (because of the "op" in the pair, and the fact that "op" reverses the order of adjoint couples). Nevertheless, we have a weaker situation, explained in the following theorem, and of practical use to obtain push-forwards (see Theorem 1.75).

Theorem 1.56. *Let L and L' be (suspended, triangulated) functors from $\mathcal{C}_1$ to $\mathcal{C}_2$, and let $l' : L'D_1^o \to D_2^o L^o$ be an isomorphism of functors in $\mathcal{C}_2$. Let $R : \mathcal{C}_2 \to \mathcal{C}_1$ be a right adjoint to L and a left adjoint to L', in other words R^o is a right adjoint to $(L')^o$. Then*

1. *there is a unique l in $\mathcal{C}_2$ to complete $\{(L')^o, L, (l')^o, l\}$ as a (suspended, triangulated) duality preserving pair,*
2. *there are unique r and r' such that $\{R^o, R, r', r\}$ is a (suspended, triangulated) duality preserving pair and a right adjoint to $\{(L')^o, L, (l')^o, l\}$.*
3. *if $r' = r^o$ or equivalently (by uniqueness) the diagram*

$$\begin{tikzcd} R \arrow[r, "R\varpi_1"] \arrow[d, "\varpi_2 R"'] & RD_1^o D_1 \arrow[d, "rD_1"] \\ D_2^o D_2 R \arrow[r, "D_2^o r^o"'] & D_2^o R^o D_1 \end{tikzcd}$$

commutes, then (R^o, R, r^o, r) is a duality preserving functor.

Proof: We apply Theorem 1.9 (or Theorem 1.36 in the suspended case) to $G_1 = D_1$, $F_1 = D_1^o$, $G_2 = D_2$, $F_2 = D_2^o$, L, R, $R' = R^o$, L' there is $(L')^o$ here and $g_L = (g'_L)^{-1}$ given by $(l')^o$. This proves Points 1 and 2. Point 3 is true by definition of a duality preserving functor. □

The proofs of following two propositions are straightforward (see also [7, Theorem 2.7] for a proof of the first one in the strict case).

Proposition 1.57. *A 1-exact duality preserving functor induces a morphism on Witt groups (this is not true for duality preserving pairs).*

Proposition 1.58. *An isomorphism between 1-exact duality preserving functors ensures that they induce the same morphism on Witt groups.*

1.7. **Tensor product and internal Hom.** We now recall a few notions on tensor products and internal Hom functors (denoted by $[-,*]$) and prove very basic facts related to the suspension. A category satisfying the axioms of this section deserves being called a "rigid (or closed) triangulated monoidal category".

Let $(\mathcal{C}, \otimes)$ be a monoidal category (see [14, Chapter VII]) with an internal Hom $[\ ,\ ]$ adjoint to the tensor product. In the suspended case, we assume that we have a suspended bifunctor $(- \otimes *, tp_1, tp_2)$ (see Definition 1.31). We further assume that $(- \otimes *, [*, -])$ is an ACB. By Proposition 1.39, we get suspended bifunctors th_1 and th_2 that make $([*, -], th_1, th_2)$ a suspended bifunctor and $(- \otimes *, [*, -])$ a suspended ACB (Definition 1.37). Recall that the morphisms th_1 and th_2 obtained are as follows:

$$th_1 : [T^{-1}(*), -] \to T[*, -] \qquad\qquad th_2 : [*, T(-)] \to T[*, -]$$

We say that the monoidal category $(\mathcal{C}, \otimes)$ is symmetric if we have an isomorphism $c : (- \otimes *) \to (* \otimes -)$ equal to its inverse and such that the diagram

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r] \arrow[d, "1\otimes c"'] & (A \otimes B) \otimes C \arrow[r, "c"] \arrow[d, phantom, "\boxed{sm}"] & C \otimes (A \otimes B) \arrow[d] \\ A \otimes (C \otimes B) \arrow[r] & (A \otimes C) \otimes B \arrow[r, "c\otimes 1"'] & (C \otimes A) \otimes B \end{tikzcd}$$

commutes (compare [14, p. 184]) and such that

$$\begin{tikzcd} T(-)\otimes * \arrow[r, "c"] \arrow[d, "tp_1"'] \arrow[dr, phantom, "\boxed{c}"] & *\otimes T(-) \arrow[d, "tp_2"] \\ T(-\otimes *) \arrow[r, "Tc"'] & T(*\otimes -) \end{tikzcd}$$

commutes in the suspended case.

In particular, using c, we obtain a new suspended ACB $(*\otimes -, [*,-])$ from the previous one.

The morphisms

$$\mathrm{ev}^l_{A,K} : [A,K]\otimes A \to K \qquad\qquad \mathrm{coev}^l_{A,K} : K \to [A, K\otimes A]$$

respectively

$$\mathrm{ev}^r_{A,K} : A\otimes [A,K] \to K \qquad\qquad \mathrm{coev}^r_{A,K} : K \to [A, A\otimes K]$$

induced by the counit and the unit of the (suspended) ACB $(-\otimes *, [*,-])$ (resp. $(*\otimes -, [*,-])$) are called the left (resp. right) evaluation and coevaluation.

Lemma 1.59. *The following diagrams are commutative.*

$$\begin{tikzcd} (T[A,K])\otimes A \arrow[r, "tp_1"] \arrow[dr, phantom, "\boxed{1}"] & T([A,K]\otimes A) \arrow[d, "T\mathrm{ev}^l_{A,K}"] \\ {[A,TK]\otimes A} \arrow[u, "th_2\otimes id"] \arrow[r, "\mathrm{ev}^l_{A,TK}"'] & TK \end{tikzcd} \qquad \begin{tikzcd} A\otimes (T[A,K]) \arrow[r, "tp_2"] \arrow[dr, phantom, "\boxed{2}"] & T(A\otimes [A,K]) \arrow[d, "T\mathrm{ev}^r_{A,K}"] \\ {A\otimes [A,TK]} \arrow[u, "id\otimes th_2"] \arrow[r, "\mathrm{ev}^r_{A,TK}"'] & TK \end{tikzcd}$$

$$\begin{tikzcd} (T^{-1}[A,K])\otimes TA \arrow[r, "tp_2"] \arrow[d, "T^{-1}th_{1,TA,K}\otimes id"'] \arrow[dr, phantom, "\boxed{3}"] & T(T^{-1}[A,K]\otimes A) \arrow[r, "tp_1^{-1}"] & {[A,K]\otimes A} \arrow[d, "\mathrm{ev}^l_{A,K}"] \\ {[TA,K]\otimes TA} \arrow[rr, "\mathrm{ev}^l_{TA,K}"'] & & K \end{tikzcd}$$

$$\begin{tikzcd} (TA\otimes T^{-1}[A,K]) \arrow[r, "tp_1"] \arrow[d, "id\otimes T^{-1}th_{1,TA,K}"'] \arrow[dr, phantom, "\boxed{4}"] & T(A\otimes T^{-1}[A,K]) \arrow[r, "tp_2^{-1}"] & {A\otimes [A,K]} \arrow[d, "\mathrm{ev}^r_{A,K}"] \\ {TA\otimes [TA,K]} \arrow[rr, "\mathrm{ev}^r_{TA,K}"'] & & K \end{tikzcd}$$

The following diagrams are anti-commutative.

$$\begin{tikzcd} (T^{-1}[A,K])\otimes TA \arrow[r, "tp_1^{-1}"] \arrow[d, "T^{-1}th_{1,TA,K}\otimes id"'] \arrow[dr, phantom, "\boxed{5}\quad -1"] & T^{-1}([A,K]\otimes TA) \arrow[r, "tp_2"] & {[A,K]\otimes A} \arrow[d, "\mathrm{ev}^l_{A,K}"] \\ {[TA,K]\otimes TA} \arrow[rr, "\mathrm{ev}^l_{TA,K}"'] & & K \end{tikzcd}$$

$$\begin{tikzcd} (TA\otimes T^{-1}[A,K]) \arrow[r, "tp_2^{-1}"] \arrow[d, "id\otimes T^{-1}th_{1,TA,K}"'] \arrow[dr, phantom, "\boxed{6}\quad -1"] & T^{-1}(TA\otimes [A,K]) \arrow[r, "tp_1"] & {A\otimes [A,K]} \arrow[d, "\mathrm{ev}^r_{A,K}"] \\ {TA\otimes [TA,K]} \arrow[rr, "\mathrm{ev}^r_{TA,K}"'] & & K \end{tikzcd}$$

Proof: This is a straightforward consequence of Point 1 of Definition 1.37 (see Remark 1.38) for the first two diagrams and of Point 2 of Definition 1.37 for the next two. The last two diagrams are obtained from the previous two by stacking over them the anti-commutative diagram from the Definition 1.31 of a suspended bifunctor. □

1.8. **Bidual isomorphism.** We still assume that $\mathcal{C}$ is a monoidal category with an internal Hom as in the previous section. When K is a dualizing object in $\mathcal{C}$ (see below) and the functor $D_K = [-, K]$ is exact, we show it naturally defines a duality on the $\mathcal{C}$. We also show in the suspended case that the dualities D_{TK} and TD_K are naturally isomorphic.

To form the adjoint couple $(D_K, D_K^o, \varpi_K, \varpi_K^o)$, we define the bidual morphism of functors $\varpi_K : Id \to D_K^o D_K$ as the image of the right evaluation by the adjunction $(- \otimes *, [*, -])$ isomorphism

$$\mathrm{Hom}(A \otimes [A, K], K) \xrightarrow{\sim} \mathrm{Hom}(A, [[A, K], K]) \ .$$

It is functorial in A and defines a morphism of functors from Id to $D_K^o D_K$. Note that its definition uses the adjunction $(- \otimes *, [*, -])$ and the right evaluation, which is not the counit of this adjunction but of the one obtained from it by using c; so the fact that the monoidal category is symmetric is essential, here. One cannot proceed with only one of these adjunctions. In the suspended case, D_K becomes a suspended functor via $T^{-1} th^{-1}_{1,-,K} T : D_K T \to T^{-1} D_K$.

Proposition 1.60. *In the suspended (or triangulated) case, ϖ_K is a morphism of suspended (or exact) functors.*

Proof: First note that in the exact case, whatever the sign of D_K is, $D_K^o D_K$ is 1-exact, so there is no sign involved in the diagram

$$\begin{tikzcd}
TA \arrow[r, "\varpi_K T"] \arrow[d, "T\varpi_K"'] & {[[TA, K], K]} \arrow[d, "T^{-1} th_{1,TA,K}"] \\
T[[A, K], K] \arrow[r, "th^{-1}_{1,[A,K],K}"'] & {[T^{-1}[A, K], K]}
\end{tikzcd}$$

that we have to check (see Definitions 1.29 and 1.25). It is obtained by (suspended) adjunction from the fourth diagram in Lemma 1.59 □

Definition 1.61. We say that K is a dualizing object when ϖ_K is an isomorphism of (suspended) functors.

Proposition 1.62. *When K is a dualizing object, the triple $(\mathcal{C}, D_K, \varpi_K)$ is a (suspended) category with duality. When $\mathcal{C}$ is triangulated and D_K is δ-exact, then $(\mathcal{C}, D_K, \varpi_K)$ is a triangulated category with duality which we often denote by $\mathcal{C}_K$ for short.*

Proof: We have to prove (see Definition 1.43) that $(D_K, D_K^o, \varpi_K, \varpi_K^o)$ is an adjoint couple. We already know that ϖ_K is a suspended isomorphism. Consider the following diagram, in which all vertical maps are isomorphisms. We use the notation $f^\sharp : \mathrm{Hom}(F', G) \to \mathrm{Hom}(F, G)$ and $f_\sharp : \mathrm{Hom}(G, F) \to \mathrm{Hom}(G, F')$ for the maps induced by $f : F \to F'$. The unlabeled morphisms are just adjunction

bijections, and we set $\varpi_{A,K} := (\varpi_K)_A$.

$$\begin{tikzcd}
\mathrm{Hom}([A,K],[A,K]) & \mathrm{Hom}([A,K],[[[A,K],K],K]) \arrow[l, "{([\varpi_{A,K},Id_K])_\sharp}"'] \\
\mathrm{Hom}([A,K]\otimes A,K) \arrow[u] & \mathrm{Hom}([A,K]\otimes[[A,K],K],K) \arrow[l, "{(Id_{[A,K]}\otimes\varpi_{A,K})^\sharp}"] \arrow[u] \\
\mathrm{Hom}(A\otimes[A,K],K) \arrow[u, "c^\sharp_{A,[A,K]}"] \arrow[d] & \mathrm{Hom}([[A,K],K]\otimes[A,K],K) \arrow[l, "{(\varpi_{A,K}\otimes Id_{[A,K]})^\sharp}"] \arrow[u, "c^\sharp_{[[A,K],K],[A,K]}"'] \arrow[d] \\
\mathrm{Hom}(A,[[A,K],K]) & \mathrm{Hom}([[A,K],K],[[A,K],K]) \arrow[l, "{(\varpi_{A,K})^\sharp}"]
\end{tikzcd}$$

The diagram commutes by functoriality of c and the adjunction bijections. Now $Id_{[[A,K],K]}$ in the lower right set is sent to $\varpi_{[A,K],K}$ in the upper right set, which is in turn sent to $[\varpi_{A,K},K]\circ\varpi_{[A,K],K}$ in the upper left set. But $Id_{[[A,K],K]}$ is also sent to $\varpi_{A,K}$ in the lower left set, which is sent to $Id_{[A,K]}$ in the upper left set by definition of $\varpi_{A,K}$. This proves the two required formulas (see Definition 1.1) for the composition of the unit and the counit in the adjoint couple (which are identical in this case). □

Proposition 1.63. *When D_K is δ-exact, so is D_{TK}, and the isomorphism $th_2 : [-,T(*)] \to T[-,*]$ defines a suspended duality preserving functor $\{Id_{\mathcal{C}}, th_{2,-,K}\}$ from $\mathcal{C}_{TK}$ to $T(\mathcal{C}, D_K, \delta\varpi_K)$. This functor is an isomorphism of triangulated categories with duality and therefore induces an isomorphism on Witt groups.*

Proof: Since we know by 1.28 that TD_K is $(-\delta)$-exact, the fact that the diagram $\boxed{sus}$ is anti-commutative (see Definition 1.29) shows by 1.30 that D_{TK} is δ-exact. This just follows from the fact that $([-,*], th_1, th_2)$ is a suspended bifunctor. Then, we obtain that by adjunction from $\boxed{2}$ in Lemma 1.59 the fact that th_2 defines a duality preserving functor (see Definition 1.50). □

We conclude this section by a trivial lemma for future reference.

Lemma 1.64. *Let $i : K \to M$ be an isomorphism with K a dualizing object such that D_K is δ-exact. Then M is also dualizing, D_M is δ-exact and $\{Id, I_i\}$, where $I_i : D_K \to D_M$ is induced by i, is a duality preserving functor that induces an isomorphism on Witt groups. When we chain isomorphisms, this respects composition. If $j : M \to N$ is another isomorphism, then $\{Id, I_{ji}\} = \{Id, I_j\}\{Id, I_i\}$.*

1.9. **Classical adjunctions and the projection formula isomorphism.** In this section, we will assume that we have adjoint couples (f^*, f_*) and $(f_*, f^!)$, such that the projection formula morphism is an isomorphism. We will then construct a functor between (triangulated) categories with dualities that will provide the desired push-forward.

Assume from now on that all categories considered are symmetric monoidal with a right adjoint to the tensor product $\otimes$ called internal Hom and denoted $[*,-]$, and these two form an ACB. The tensor product is assumed to be exact in both variables, and by Proposition 1.39, in the suspended (triangulated) case, we turn $[*,-]$ into a suspended (exact) bifunctor such that $(-\otimes *, [*,-])$ is a suspended ACB. We will then assume successively:

(1) the functor $f^* : \mathcal{C}_1 \to \mathcal{C}_2$ is a symmetric monoidal (suspended, exact in both variables) functor, which means that it comes equipped with an isomorphism of (suspended) bifunctors $fp : f^*(-)\otimes f^*(*) \to f^*(-\otimes *)$ and with an isomorphism

$f^*(\mathbf{1}) \simeq \mathbf{1}$ making the standard diagrams commutative (see [14, section XI.2] for the details where such functors are called *strongly monoidal*),
(2) we have a functor $f_* : \mathcal{C}_2 \to \mathcal{C}_1$ that fits into an adjoint couple $(f^*, f_*, \eta_*^*, \epsilon_*^*)$,
(3) we have a functor $f^! : \mathcal{C}_1 \to \mathcal{C}_2$ that fits into an adjoint couple $(f_*, f^!, \eta_*^!, \epsilon_*^!)$,
(4) the morphism $fh_K : f^*[-, K] \to [f^*(-), f^*K]$ from Proposition 1.68 is an isomorphism,
(5) the morphism $qp : f_*(-) \otimes * \to f_*(- \otimes f^*(*))$ from Proposition 1.71 is an isomorphism (the "projection formula" isomorphism),
(6) K is a dualizing object in $\mathcal{C}_1$ (see Definition 1.61),
(7) f^*K is a dualizing object in $\mathcal{C}_2$,
(8) $f^!K$ is a dualizing object in $\mathcal{C}_2$.

Remark 1.65. Later, we will consider five more assumptions which are introduced after Proposition 1.79, in Proposition 1.91, at the beginning of Section 1.14 and after Lemma 1.103, respectively.

Proposition 1.66. *Assume (1) and (2). Let $a : f^*(- \otimes f_*(*)) \to f^*(-) \otimes *$ be the morphism of (suspended) bifunctors defined by the composition*

$$f^*(- \otimes f_*(*)) \xrightarrow{fp^{-1}} f^*(-) \otimes f^*f_*(*) \xrightarrow{\epsilon_*^*} f^*(-) \otimes * \ .$$

Then, there are unique morphisms

$$fg : f_*(-) \otimes f_*(*) \to f_*(- \otimes *)$$

and

$$ff : f_*[*, -] \to [f_*(*), f_*(-)]$$

of (suspended) bifunctors such that the diagrams

$$\begin{tikzcd}
{[f_*A, f_*(- \otimes B)]} & f_*[A, - \otimes B] \arrow[l, "ff"'] \\
{[f_*A, f_*(-) \otimes f_*B]} \arrow[u, "fg_\sharp"] \arrow[ur, phantom, "\boxed{7}"] & f_*(-) \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
f_*(-) & {[f_*A, f_*(-)] \otimes f_*B} \arrow[l] \\
{f_*([A, -] \otimes B)} \arrow[u] \arrow[ur, phantom, "\boxed{8}"] & {f_*[A, -] \otimes f_*B} \arrow[l, "fg"'] \arrow[u, "ff \otimes id"']
\end{tikzcd}$$

$$\begin{tikzcd}
{- \otimes f_*C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{9}"] & {f_*f^*(-) \otimes f_*C} \arrow[d, "fg"] \\
{f_*f^*(- \otimes f_*C)} \arrow[r, "a"] & {f_*(f^*(-) \otimes C)}
\end{tikzcd}
\qquad
\begin{tikzcd}
{f^*(f_*(-) \otimes f_*C)} \arrow[r, "a"'] \arrow[d, "fg"'] \arrow[dr, phantom, "\boxed{10}"] & {f^*f_*(-) \otimes C} \arrow[d] \\
{f^*f_*(- \otimes C)} \arrow[r] & {- \otimes C}
\end{tikzcd}$$

commute for every morphism $A \to B$ and for every object C in $\mathcal{C}_1$.

Proof: Apply Theorem 1.41 Point 2 and then Point 1 to $L = L' = f^*$, $R = R' = f_*$ $G_1 = - \otimes f_*(*)$, $F_1 = [f_*(*), -]$, $G_2 = (- \otimes *)$, $F_2 = [*, -]$ and $g'_L = a$. □

Lemma 1.67. *The diagram*

$$\begin{tikzcd}
f_*A \otimes f_*B \arrow[r, "fg"] \arrow[d, "c(f_* \otimes f_*)"'] \arrow[dr, phantom, "\boxed{11}"] & f_*(A \otimes B) \arrow[d, "f_*c"] \\
f_*B \otimes f_*A \arrow[r, "fg"] & f_*(B \otimes A)
\end{tikzcd}$$

is commutative.

Proof: Apply Lemma 1.8 to the cube

$$\begin{tikzcd}
\mathcal{C}_2\times\mathcal{C}_2 \arrow[rr, "\otimes"] \arrow[dr, "\tau"] & & \mathcal{C}_2 \arrow[dr, "Id"] & \\
& \mathcal{C}_2\times\mathcal{C}_2 \arrow[rr, "\otimes"] \arrow[ur, Rightarrow, "c"] & & \mathcal{C}_2 \\
\mathcal{C}_1\times\mathcal{C}_1 \arrow[uu, "f^*\times f^*"] \arrow[dr, "\tau"] & & & \\
& \mathcal{C}_1\times\mathcal{C}_1 \arrow[uu, "f^*\times f^*"] \arrow[rr, "\otimes"'] \arrow[uurr, Rightarrow, "fp^{-1}"] & & \mathcal{C}_1 \arrow[uu, "f^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_2\times\mathcal{C}_2 \arrow[rr, "\otimes"] & & \mathcal{C}_2 \arrow[dr, "Id"] & \\
& & & \mathcal{C}_2 \\
\mathcal{C}_1\times\mathcal{C}_1 \arrow[uu, "f^*\times f^*"] \arrow[rr] \arrow[dr, "\tau"] & & \mathcal{C}_1 \arrow[uu, Rightarrow, "fp^{-1}"] \arrow[uu, "f^*"] \arrow[dr, "Id"] & \\
& \mathcal{C}_1\times\mathcal{C}_1 \arrow[ur, Rightarrow, "c"] \arrow[rr, "\otimes"'] & & \mathcal{C}_1 \arrow[uu, "f^*"']
\end{tikzcd}$$

front *back*

where τ is the functor exchanging the components. Note that the morphism of functors $f_*(-\otimes *) \to f_*(-)\otimes f_*(*)$ obtained on the front and back squares indeed coincides with fg by construction. □

Proposition 1.68. *Under Assumption (1), there is a unique morphism*

$$fh : f^*[-,*] \to [f^*(-), f^*(*)]$$

of (suspended) bifunctors such that the diagrams

$$\begin{tikzcd}
{[f^*A, f^*(-\otimes B)]} & f^*[A, -\otimes B] \arrow[l, "fh"'] \\
{[f^*A, f^*(-)\otimes f^*B]} \arrow[u, "fp"] \arrow[ur, phantom, "\boxed{12}"] & f^*(-) \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(-) & {[f^*A, f^*(-)]\otimes f^*B} \arrow[l] \\
{f^*([A,-]\otimes B)} \arrow[u] \arrow[ur, phantom, "\boxed{13}"] & f^*[A,-]\otimes f^*B \arrow[l, "fp"'] \arrow[u, "fh"']
\end{tikzcd}$$

commute for every morphism $A \to B$ in $\mathcal{C}_1$.

Proof: Apply Theorem 1.41 Point 1 to $L = L' = f^*$, $G_1 = -\otimes *$, $F_1 = [*,-]$, $G_2 = (-\otimes f^*(*))$, $F_2 = [f^*(*),-]$ and $g_L = (g'_L)^{-1} = fp$ (recall that in an ACB, the variable denoted $*$ is the parameter, as explained in Definition 1.12). Then define $fh = f_L$. □

Theorem 1.69. *(existence of the pull-back) Under Assumptions (1) for f^*, (6) for K, (4) and (7) for f^* and K, the isomorphism*

$$fh_K : f^*[-,K] \to [f^*(-), f^*K]$$

defines a duality preserving functor $\{f^*, fh_K\}$ of (suspended, triangulated) categories with duality from $(\mathcal{C}_1)_K$ to $(\mathcal{C}_2)_{f^*K}$.

Proof: We need to show that the diagram

$$\begin{tikzcd}
f^* \arrow[r, "f^*\varpi_K"] \arrow[d, "\varpi_K f^*"'] & f^* D_K^o D_K \arrow[d, "fh_K D_K"] \\
D^o_{f^*K} D_{f^*K} f^* \arrow[r, "D_{f^*K} fh^o_K"] & D_{f^*K}(f^*)^o D_K
\end{tikzcd}$$

commutes. This follows from the following commutative diagram (for all A in $\mathcal{C}_1$).

$$\begin{tikzcd}
f^*A \arrow[r, "\mathrm{coev}^l"] \arrow[d, "\mathrm{coev}^l"'] \arrow[dr, phantom, "\boxed{12}"] & f^*[[A,K], A\otimes[A,K]] \arrow[r, "(\mathrm{ev}^r)_\sharp"] \arrow[d, "fh"'] \arrow[dr, phantom, "\boxed{mf}"] & f^*[[A,K],K] \arrow[d, "fh"] \\
{[f^*[A,K], f^*A\otimes f^*[A,K]]} \arrow[r, "fp_\sharp"] & {[f^*[A,K], f^*(A\otimes[A,K])]} \arrow[r] & {[f^*[A,K], f^*K]} \\
{[[f^*A,f^*K], f^*A\otimes f^*[A,K]]} \arrow[u, "fh^\sharp"] \arrow[d, "fh_\sharp"'] \arrow[ur, phantom, "\wr"] \arrow[r] \arrow[dr, phantom, "\boxed{mf}"] & {[[f^*A,f^*K], f^*(A\otimes[A,K])]} \arrow[u, "fh^\sharp"'] \arrow[ur, phantom, "\wr"] \arrow[r] \arrow[dr, phantom, "\boxed{mf}"] & {[[f^*A,f^*K], f^*K]} \arrow[u, "fh^\sharp"] \arrow[ul, phantom, "\wr"] \\
{[[f^*A,f^*K], f^*A\otimes[f^*A,f^*K]]} \arrow[ur, "(\mathrm{ev}^r)_\sharp"'] \arrow[ur, phantom, "\boxed{13'}"] & &
\end{tikzcd}$$

where $\boxed{13'}$ is obtained from $\boxed{13}$ by using the compatibility of fp with c. By functoriality of coev, the counit of the adjunction of bifunctors $(-\otimes *, [-,*])$, the left hand side vertical composition is coev again, and the outer part of this diagram diagram is therefore the one we are looking for. In the suspended (or triangulated) case, fh is a morphism of suspended functors by Proposition 1.68. □

Proposition 1.70. *In the suspended case, under Assumption (2) there is a unique way of turning f_* into a suspended functor such that (f^*, f_*) is a suspended adjoint couple. If further Assumption (3) holds, then there is a unique way of turning $f^!$ into a suspended functor such that $(f_*, f^!)$ is a suspended adjoint couple.*

Proof: Both results follow directly from Proposition 1.35. □

Proposition 1.71. *Under Assumptions (1) and (2), there is a unique morphism*

$$qp : f_*(-)\otimes * \to f_*(-\otimes f^*(*))$$

and a unique isomorphism

$$qh : [*, f_*(-)] \to f_*[f^*(*), -]$$

of (suspended) bifunctors such that the diagrams

$$\begin{tikzcd}
{[A, f_*(-\otimes f^*B)]} \arrow[d, "qp"] \arrow[dr, phantom, "\boxed{14}"] & f_*[f^*A, -\otimes f^*B] \arrow[l, "qh^{-1}"'] \\
{[A, f_*(-)\otimes B]} & f_* \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^* & {[A, f_*(-)]\otimes B} \arrow[l] \\
{f_*([f^*A,-]\otimes f^*B)} \arrow[u] \arrow[ur, phantom, "\boxed{15}"] & {f_*[f^*A,-]\otimes B} \arrow[l, "qp"'] \arrow[u, "qh^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
-\otimes A \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{16}"] & f_*f^*(-)\otimes A \arrow[d, "qp"] \\
f_*f^*(-\otimes A) \arrow[r, "fp^{-1}"] & f_*(f^*(-)\otimes f^*A)
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(f_*(-)\otimes A) \arrow[r, "fp^{-1}"'] \arrow[d, "qp"'] \arrow[dr, phantom, "\boxed{17}"] & f^*f_*(-)\otimes f^*A \arrow[d] \\
f^*f_*(-\otimes f^*A) \arrow[r] & -\otimes f^*A
\end{tikzcd}$$

$$\begin{tikzcd}
{[A,-]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{18}"] & {[A, f_*f^*(-)]} \arrow[d, "qh"] \\
f_*f^*[A,-] \arrow[r, "fh"] & f_*[f^*A, f^*(-)]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*[A, f_*(-)] \arrow[r, "fh"'] \arrow[d, "qh"'] \arrow[dr, phantom, "\boxed{19}"] & {[f^*A, f^*f_*(-)]} \arrow[d] \\
f^*f_*[f^*A,-] \arrow[r] & {[f^*A,-]}
\end{tikzcd}$$

commute for any morphism $A \to B$.

Proof: Apply Point 3 of Theorem 1.41 with $L = L' = f^*$, $R = R' = f_*$, $G_1 = -\otimes *$, $G_2 = -\otimes f^*(*)$, $F_1 = [*,-]$, $F_2 = [f^*(*),-]$, $g_L = (g'_L)^{-1} = fp$. Then define $qp = g_R$ and $qh = f'_R$. □

Lemma 1.72. *The composition*

$$f_*(-)\otimes(*)\xrightarrow{Id\otimes\eta_*^*}f_*(-)\otimes f_*f^*(*)\xrightarrow{fg}f_*(-\otimes f^*(*))$$

coincides with qp.

Proof: This follows from the commutative diagram (for any A and B in $\mathcal{C}_2$)

$$\begin{tikzcd}
f_*A\otimes B \arrow[r] \arrow[d] \arrow[dd, bend right=60, "qp"'] \arrow[ddrr, bend left=60, "fg"] \arrow[dr, phantom, "\boxed{mf}"] & f_*A\otimes f_*f^*B \arrow[d] \\
f_*f^*(f_*A\otimes B) \arrow[r] \arrow[d, "fp^{-1}"'] \arrow[dr, phantom, "\boxed{mf}"] & f_*f^*(f_*A\otimes f_*f^*B) \arrow[d, "fp^{-1}"] \\
f_*(f^*f_*A\otimes f^*B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{mf}"] & f_*(f^*f_*A\otimes f^*f_*f^*B) \arrow[d] \\
f_*(A\otimes f^*B) \arrow[r] \arrow[rr, bend right=40, "Id"'] \arrow[rr, phantom, shift right=3ex, "\boxed{adj}"] & f_*(A\otimes f^*f_*f^*B) \arrow[r] & f_*(A\otimes f^*B)
\end{tikzcd}$$

in which the curved maps are indeed fg and qp by construction. □

Lemma 1.73. *The composition*

$$f_*[f^*(-),*]\xrightarrow{ff}[f_*f^*(-),f_*(*)]\xrightarrow{(\eta_*^*)^\sharp}[-,f_*(*)]$$

coincides with qh.

Proof: This follows from the commutative diagram (for any A in $\mathcal{C}_1$ and B in $\mathcal{C}_2$)

$$\begin{tikzcd}
f_*[f^*A,B] \arrow[r, "\mathrm{coev}^l"] \arrow[d, "\mathrm{coev}^l"'] \arrow[dd, bend right=60, "ff"'] \arrow[dr, phantom, "\boxed{mf}"] & {[A,f_*[f^*A,B]\otimes A]} \arrow[d, "(\eta_*^*)_\sharp"] \arrow[dd, bend left=60, "qp_\sharp"] \\
{[f_*f^*A,f_*[f^*A,B]\otimes f_*f^*A]} \arrow[r, "(\eta_*^*)^\sharp"] \arrow[d, "fg_\sharp"'] \arrow[dr, phantom, "\boxed{mf}"] & {[A,f_*[f^*A,B]\otimes f_*f^*A]} \arrow[d, "fg_\sharp"] \\
{[f_*f^*A,f_*([f^*A,B]\otimes f^*A)]} \arrow[r] \arrow[d, "\mathrm{ev}^l"'] \arrow[dr, phantom, "\boxed{mf}"] & {[A,f_*([f^*A,B]\otimes f^*A)]} \arrow[d, "\mathrm{ev}^l"] \\
{[f_*f^*A,f_*B]} \arrow[r, "(\eta_*^*)^\sharp"'] & {[A,f_*B]}
\end{tikzcd}$$

in which the left curved arrow is ff by construction, the right one is qp by Lemma 1.72 and the composition from the top left corner to the bottom right one along the upper right corner is then qh by construction. □

Proposition 1.74. *Under assumptions (1), (2), (3) and (5), there is a unique morphism*

$$sp: f^!(-)\otimes f^*(*)\to f^!(-\otimes *)$$

and a unique isomorphism

$$sh: f^![*,-]\to[f^*(*),f^!(-)]$$

of (suspended) bifunctors such that the diagrams

$$\begin{tikzcd}
{[f^*A,f^!(-\otimes B)]} \arrow[dr, phantom, "\boxed{20}"] & f^![A,-\otimes B] \arrow[l, "sh"'] \\
{[f^*A,f^!(-)\otimes f^*B]} \arrow[u, "sp"] & f^! \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^! & {[f^*A,f^!(-)]\otimes f^*B} \arrow[l] \\
f^!([A,-]\otimes B) \arrow[u] \arrow[ur, phantom, "\boxed{21}"] & f^![A,-]\otimes f^*B \arrow[l, "sp"'] \arrow[u, "sh"']
\end{tikzcd}$$

$$\begin{tikzcd}
{[f^*A,-]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{22}"] & {[f^*A, f^!f_*(-)]} \arrow[d, "sh^{-1}"] & f_*[f^*A, f^!(-)] \arrow[r, "qh^{-1}"'] \arrow[d, "sh^{-1}"'] \arrow[dr, phantom, "\boxed{23}"] & {[A, f_*f^!(-)]} \arrow[d] \\
f^!f_*[f^*A,-] \arrow[r, "qh^{-1}"] & f^![A, f_*(-)] & f_*f^![A,-] \arrow[r] & {[A,-]}
\end{tikzcd}$$

$$\begin{tikzcd}
-\otimes f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{24}"] & f^!f_*(-)\otimes f^*A \arrow[d, "sp"] & f_*(f^!(-)\otimes f^*A) \arrow[r, "qp^{-1}"'] \arrow[d, "sp"'] \arrow[dr, phantom, "\boxed{25}"] & f_*f^!(-)\otimes A \arrow[d] \\
f^!f_*(-\otimes f^*A) \arrow[r, "qp^{-1}"] & f^!(f_*(-)\otimes A) & f_*f^!(-\otimes A) \arrow[r] & -\otimes A
\end{tikzcd}$$

commute for any morphism $A \to B$ in $\mathcal{C}_1$.

Proof: Apply Point 3 of Theorem 1.41 with $L = L' = f_*$, $R = R' = f^!$, $G_1 = -\otimes f^*(*)$, $G_2 = -\otimes *$, $F_1 = [f^*(*), -]$, $F_2 = [*, -]$, $g_L = (g'_L)^{-1} = qp$. Note that we first obtain a second time the (same) morphism qh^{-1} as in Proposition 1.71. Then define $sh = f_R$ and $sp = g_R$. □

In particular, this gives isomorphisms $sh_K : f^!D^o_K \to D^o_{f^!K}(f^*)^o$ (in $\mathcal{C}_2$) for each K.

Theorem 1.75. *(existence of the push-forward) Under Assumptions (1), (2), (3) and (5) for f^*, (6) for K and (8) for $f^!K$ there are unique isomorphisms of (suspended) functors*

$$sh'_K : f^*D^o_K \to D^o_{f^!K}(f^!)^o$$

and

$$r_K : f_*D^o_{f^!K} \to D^o_K f^o_*$$

such that the diagrams

$$\begin{tikzcd}
D^o_{f^!K}(f^!)^oD_K & f^*D^o_KD_K \arrow[l, "sh'_K"'] \\
D^o_{f^!K}D_{f^!K}f^* \arrow[u, "sh^o_K"] \arrow[ur, phantom, "\boxed{26}"] & f^* \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^! \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{27}"] & D^o_{f^!K}D_{f^!K}f^! \arrow[d, "(sh'_K)^o"] \\
f^!D^o_KD_K \arrow[r, "sh_K"'] & D^o_{f^!K}(f^*)^oD_K
\end{tikzcd}$$

$$\begin{tikzcd}
D^o_Kf^o_*D_{f^!K} & f_*D^o_{f^!K}D_{f^!K} \arrow[l, "r_K"'] \\
D^o_KD_Kf_* \arrow[u, "r^o_K"] \arrow[ur, phantom, "\boxed{28}"] & f_* \arrow[l] \arrow[u]
\end{tikzcd}$$

commute (for convenience, all diagrams are displayed in $\mathcal{C}_1$ or $\mathcal{C}_2$, and not in $\mathcal{C}_1^o$ or $\mathcal{C}_2^o$). In other words,

1. *$\{(f^!)^o, f^*, sh_K, (sh'_K)^o\}$ is a duality preserving pair of (suspended, triangulated) categories with duality from $(\mathcal{C}_2)_K$ to $(\mathcal{C}_1)_{f^!K}$,*
2. *$\{f_*, r_K\}$ is a duality preserving functor of (suspended, triangulated) categories with duality from $(\mathcal{C}_2)_{f^!K}$ to $(\mathcal{C}_1)_K$.*

Proof: First note that since $(f_*, f^!)$ is an adjoint couple, so is $((f^!)^o, f^o_*)$ (beware of the order). The result is then a straightforward application of Theorem 1.56: we start with $L' = f^!$, $L = f^*$, $R = f_*$ and $l' = sh_K$ to obtain first $sh'_K = l$ by Point 1 and then $r_K = r$ by Point 2. The last result follows if we prove that the diagram in Theorem 1.56 Point 3, corresponding here to $\boxed{28}$, is commutative. Let us first prove the following.

Lemma 1.76. *The composition*

$$f_*[-, f^!K] \xrightarrow{ff} [f_*(-), f_*f^!K] \xrightarrow{(\epsilon_*^!)_\sharp} [f_*(-), K]$$

coincides with r_K.

Proof: This follows from the commutative diagram (for every A and B in $\mathcal{C}_2$)

$$\begin{tikzcd}
f_*[A, f^!K] \arrow[r] \arrow[d, "ff"'] \arrow[dr, phantom, "\boxed{mf}"] & f_*[f^*f_*A, f^!K] \arrow[r, "sh_K^{-1}"'] \arrow[d, "ff"] & f_*f^![f_*A, K] \arrow[dd, "\epsilon_*^!"] \\
{[f_*A, f_*f^!K]} \arrow[r] \arrow[dr, equal] & {[f_*f^*f_*A, f_*f^!K]} \arrow[d, "(\eta_*^*)^\sharp"] \arrow[dr, phantom, "\boxed{23}"] & \\
 & {[f_*A, f_*f^!K]} \arrow[r, "(\epsilon_*^!)_\sharp"] \arrow[ur, phantom, "\boxed{adj}"] & {[f_*A, K]}
\end{tikzcd}$$

identifying the vertical composition in the middle with qh^{-1} by Lemma 1.73 to recognize $\boxed{23}$. The composition around the top right corner is then r_K by construction. □

Diagram $\boxed{28}$ of the theorem now follows from the commutative diagram

$$\begin{tikzcd}
f_*A \arrow[r, "\mathrm{coev}^l"] \arrow[d, "\mathrm{coev}^l"'] \arrow[dr, phantom, "\boxed{7}"] & f_*[[A, f^!K], A \otimes [A, f^!K]] \arrow[r, "\mathrm{ev}^r_\sharp"] \arrow[d, "ff"'] \arrow[dr, phantom, "\boxed{mf}"] & f_*[[A, f^!K], f^!K] \arrow[d, "ff"] \\
{[f_*[A, f^!K], f_*A \otimes f_*[A, f^!K]]} \arrow[r, tail] \arrow[dr, phantom, "\boxed{mf}"] & {[f_*[A, f^!K], f_*(A \otimes [A, f^!K])]} \arrow[r, tail] \arrow[dr, phantom, "\boxed{mf}"] & {[f_*[A, f^!K], f_*f^!K]} \\
{[[f_*A, K], f_*A \otimes f_*[A, f^!K]]} \arrow[r] \arrow[u, "r^\sharp", "\wr"'] \arrow[d, "ff_\sharp"'] \arrow[dr, phantom, "\boxed{8'}"] & {[[f_*A, K], f_*(A \otimes [A, f^!K])]} \arrow[r] \arrow[u, "r^\sharp", "\wr"'] & {[[f_*A, K], f_*f^!K]} \arrow[u, "r^\sharp", "\wr"'] \arrow[d, "(\epsilon_*^!)_\sharp"] \\
{[[f_*A, K], f_*A \otimes [f_*A, f_*f^!K]]} \arrow[d, "(\epsilon_*^!)_\sharp"'] \arrow[dr, phantom, "\boxed{mf}"] \arrow[urr, "(\mathrm{ev}^r)_\sharp"'] & & {[[f_*A, K], K]} \\
{[[f_*A, K], f_*A \otimes [f_*A, K]]} \arrow[urr, "(\mathrm{ev}^r)_\sharp"'] & &
\end{tikzcd}$$

where $\boxed{8'}$ is obtained from $\boxed{8}$ by using $\boxed{11}$. The two last maps of the vertical composition on the left coincide with $r_\sharp$ by Lemma 1.76. By functoriality of coev, the left vertical composition is thus equal to $\mathrm{coev}_{[f_*A,K],f_*A}$, so going from the top left corner to the bottom right corner of the diagram counterclockwise gives the morphism $\varpi_{f_*A,K}$. On the other hand, the composition on the top is $f_*\varpi_{f^!K}$, so it just remains to show that we can complete the commutative diagram on the right to get the last two maps $r_K D_{f^!K}$ and $D_K^o r_K$. This is achieved by the following commutative diagram, using again Lemma 1.73 for the triangle.

$$\begin{tikzcd}
f_*[[A, f^!K], f^!K] \arrow[d, "ff"'] \arrow[dr, "r_K D_{f^!K}"] & \\
{[f_*[A, f^!K], f_*f^!K]} \arrow[r] \arrow[dr, phantom, "\boxed{mf}"] & {[f_*[A, f^!K], K]} \\
{[[f_*A, K], f_*f^!K]} \arrow[u, "r_K^\sharp"] \arrow[r, "(\epsilon_*^!)_\sharp"] & {[f_*[f_*A, K], K]} \arrow[u, "r_K^\sharp"']
\end{tikzcd}$$

□

1.10. **Associativity of products.** We now establish a few commutative diagrams related to the associativity of the tensor product. For simplicity, we assume that it is strictly associative.

Since f^* is monoidal, the diagram (involving fp)

$$\begin{tikzcd}
f^*A \otimes f^*B \otimes f^*C \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{29}"] & f^*(A \otimes B) \otimes f^*C \arrow[d] \\
f^*A \otimes f^*(B \otimes C) \arrow[r] & f^*(A \otimes B \otimes C)
\end{tikzcd}$$

is commutative.

Proposition 1.77. *Under Assumptions (2) (for the first diagram), (3), and (5), the following diagrams are commutative.*

$$\begin{tikzcd}
f_*A \otimes B \otimes C \arrow[r, "qp"'] \arrow[d, "qp"'] \arrow[dr, phantom, "\boxed{30}"] & f_*(A \otimes f^*B) \otimes C \arrow[d, "qp"] \\
f_*(A \otimes f^*(B \otimes C)) \arrow[r, "fp^{-1}"] & f_*(A \otimes f^*B \otimes f^*C)
\end{tikzcd}
\qquad
\begin{tikzcd}
f^!A \otimes f^*B \otimes f^*C \arrow[r, "sp"] \arrow[d, "fp"] \arrow[dr, phantom, "\boxed{31}"] & f^!(A \otimes B) \otimes C \arrow[d, "sp"] \\
f^!A \otimes f^*(B \otimes C) \arrow[r, "sp"] & f^!(A \otimes B \otimes C)
\end{tikzcd}$$

Proof: Consider the commutative diagram

$$\begin{tikzcd}
& f_*A \otimes B \otimes C \arrow[r, "qp"] \arrow[d, "\eta_*^*"'] \arrow[dr, phantom, "mf"] & f_*(A \otimes f^*B) \otimes C \arrow[d, "\eta_*^*"] \\
& f_*f^*(f_*A \otimes B \otimes C) \arrow[r] \arrow[d, "fp^{-1}"] \arrow[dr, phantom, "mf"] \arrow[dl, "fp^{-1}"'] & f_*f^*(f_*(A \otimes f^*B) \otimes C) \arrow[d, "fp^{-1}"] \\
& f_*(f^*(f_*A \otimes B) \otimes f^*C) \arrow[r] \arrow[d, "fp^{-1}"] \arrow[dr, phantom, "\boxed{29}"] & f_*(f^*f_*(A \otimes f^*B) \otimes f^*C) \arrow[d, "\epsilon_*^*"] \\
f_*(f^*f_*A \otimes f^*(B \otimes C)) \arrow[r, "fp^{-1}"] \arrow[dr, "\epsilon_*^*"'] & f_*(f^*f_*A \otimes f^*B \otimes f^*C) \arrow[r, "\epsilon_*^*"] \arrow[d, phantom, "mf"] & f_*(A \otimes f^*B \otimes f^*C) \\
& f_*(A \otimes f^*(B \otimes C)) \arrow[ur, "fp^{-1}"'] &
\end{tikzcd}$$

whose outer part is diagram $\boxed{30}$ by construction of the morphisms involved. Similarly, we prove $\boxed{31}$ by using the commutative diagram

$$\begin{tikzcd}
& f^!A \otimes f^*B \otimes f^*C \arrow[r, "sp"] \arrow[dl, "fp"'] \arrow[d] \arrow[dr, phantom, "mf"] & f^!(A \otimes B) \otimes f^*C \arrow[d, "\eta_*^!"] \\
f^!A \otimes f^*(B \otimes C) \arrow[d, "\eta_*^!"'] \arrow[dr, phantom, "mf"] & f^!f_*(f^!A \otimes f^*B \otimes f^*C) \arrow[r] \arrow[d, "qp^{-1}"] \arrow[dl, "fp"] \arrow[dr, phantom, "mf"] & f^!f_*(f^!(A \otimes B) \otimes f^*C) \arrow[d, "qp^{-1}"] \\
f^!f_*(f^!A \otimes f^*(B \otimes C)) \arrow[dr, "qp^{-1}"'] \arrow[dr, phantom, "\boxed{30}"] & f^!(f_*(f^!A \otimes f^*B) \otimes C) \arrow[r] \arrow[d, "qp^{-1}"] \arrow[dr, phantom, "\boxed{25}"] & f^!(f_*f^!(A \otimes B) \otimes C) \arrow[d, "\epsilon_*^!"] \\
& f^!(f_*f^!A \otimes B \otimes C) \arrow[r, "\epsilon_*^!"] & f^!(A \otimes B \otimes C)
\end{tikzcd}$$

□

This result automatically implies that all higher associativity diagrams involving more than three factors commute. For example the following cube (involving only

fp and sp) commutes, just because all its faces commute.

$$\begin{tikzcd}
f^!A \otimes f^*B \otimes f^*C \otimes f^*D \arrow[rr] \arrow[dd] \arrow[dr] & & f^!(A \otimes B) \otimes f^*C \otimes f^*D \arrow[dr] \arrow[dd] & \\
& f^!A \otimes f^*(B \otimes C) \otimes f^*D \arrow[rr] \arrow[dd] & & f^!(A \otimes B \otimes C) \otimes f^*D \arrow[dd] \\
f^!A \otimes f^*B \otimes f^*(C \otimes D) \arrow[rr] \arrow[dr] & & f^!(A \otimes B) \otimes f^*(C \otimes D) \arrow[dr] & \\
& f^!A \otimes f^*(A \otimes B \otimes C) \arrow[rr] & & f^!(A \otimes B \otimes C \otimes D)
\end{tikzcd}$$

We now define two more classical morphisms.

Definition 1.78. Consider objects $K, M \in \mathcal{C}_1$ such that Assumption (6) from Section 1.9 is satisfied for K, M and $K \otimes M$. We define the morphism $d_{K,M}$: $D_K \otimes D_M \to D_{K\otimes M}$ by the composition (for any A_1 and A_2 in $\mathcal{C}$)

$$[A_1, K] \otimes [A_2, M] \to [A_1 \otimes A_2, ([A_1, K] \otimes [A_2, M]) \otimes (A_1 \otimes A_2)]$$

$$\to [A_1 \otimes A_2, ([A_1, K] \otimes A_1) \otimes ([A_2, M] \otimes A_2)] \to [A_1 \otimes A_2, K \otimes M].$$

where the first map is induced by the unit of the adjoint couple $(- \otimes *, [*, -])$, the second is induced by c (the commutativity of $\otimes$) and the third by the tensor product of the left evaluation maps $\mathrm{ev}^l_{A,K} \otimes \mathrm{ev}^l_{B,M}$.

The proofs of the next two results are left to the reader. They are not difficult although they require large commutative diagrams.

Proposition 1.79. *The morphism $d_{K,M}$ is a morphism of suspended bifunctors.*

Let us consider the following Assumption:
(9) The morphism $d_{K,M}$ is an isomorphism.

The proof of the following lemma uses the unit of the tensor product.

Lemma 1.80. *Assumptions (6) and (9) for K and M imply (6) for $K \otimes M$. Assumptions (4) and (9) for K and M imply (4) for $K \otimes M$.*

Definition 1.81. We define an "internal adjunction" morphism of functors

$$ad_{A,B,C} : [A \otimes B, C] \to [A, [B, C,]]$$

by the composition

$$[A \otimes B, C] \xrightarrow{\mathrm{coev}_A} [A, [A \otimes B, C] \otimes A] \xrightarrow{(\mathrm{coev}_B)_\sharp} [A, [B, [A \otimes B, C] \otimes A \otimes B]]$$

$$\xrightarrow{(\mathrm{ev}^l_{A\otimes B})_\sharp} [A, [B, C]]$$

It is well known that this morphisms is an isomorphism (see for example [24, I, § 3.2]).

Definition 1.82. We define an "internal composition" morphism of functor

$$comp_{A,B,C} : [A, B] \otimes [B, C] \to [A, C]$$

by the composition

$$[A, B] \otimes [B, C] \xrightarrow{\mathrm{coev}^r_A} [A, A \otimes [A, B] \otimes [B, C]] \xrightarrow{(\mathrm{ev}^r_A \otimes Id)_\sharp} [A, B \otimes [B, C]] \xrightarrow{(\mathrm{ev}^r_B)_\sharp} [A, C] \ .$$

1.11. **Units of tensor products.** In the same spirit as in Section 1.10, we now establish commutative diagrams related to the unit of the tensor product. All proofs are easy and similar to the ones from Section 1.10, so we leave them to the reader.

Recall that a monoidal category $\mathcal{C}$ has a unit object $\mathbf{1}_{\mathcal{C}}$ and right and left unit isomorphism of functors

$$Id_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}} \to Id_{\mathcal{C}} \qquad\qquad \mathbf{1}_{\mathcal{C}} \otimes Id_{\mathcal{C}} \to Id_{\mathcal{C}}$$

satisfying some compatibilities with the associativity and symmetry morphisms. A monoidal functor $f^* : \mathcal{C}_1 \to \mathcal{C}_2$ comes equipped with an isomorphism $f^*\mathbf{1}_{\mathcal{C}_1} \to \mathbf{1}_{\mathcal{C}_2}$ making the diagram

$$\begin{tikzcd} f^*A \otimes f^*\mathbf{1}_{\mathcal{C}_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{32}"] & f^*(A \otimes \mathbf{1}_{\mathcal{C}_1}) \arrow[d] \\ f^*A \otimes \mathbf{1}_{\mathcal{C}_2} \arrow[r] & f^*A \end{tikzcd}$$

commutative for any object A of $\mathcal{C}_1$, and similarly for the left unit isomorphism.

From diagram $\boxed{32}$, one deduces that the following diagrams are commutative (under the suitable assumptions discussed previously for the existence of the maps).

$$\begin{tikzcd} f_*A \otimes \mathbf{1}_{\mathcal{C}_1} \arrow[r, "qp"] \arrow[d] \arrow[dr, phantom, "\boxed{33}"] & f_*(A \otimes f^*\mathbf{1}_{\mathcal{C}_1}) \arrow[d] \\ f_*A & f_*(A \otimes \mathbf{1}_{\mathcal{C}_2}) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} f^!A \otimes f^*\mathbf{1}_{\mathcal{C}_1} \arrow[r, "sp"] \arrow[d] \arrow[dr, phantom, "\boxed{34}"] & f^!(A \otimes \mathbf{1}_{\mathcal{C}_1}) \arrow[d] \\ f^!A \otimes \mathbf{1}_{\mathcal{C}_2} \arrow[r] & f^!A \end{tikzcd}$$

By adjunction from the map $A \otimes \mathbf{1} \to A$, we obtain a map $A \to [\mathbf{1}, A]$. This map is an isomorphism with inverse given by the composition

$$[\mathbf{1}, A] \longrightarrow \mathbf{1} \otimes [\mathbf{1}, A] \xrightarrow{\mathrm{ev}^r} A\ .$$

From this, we obtain two more commutative diagrams.

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{35}"] & f^*[\mathbf{1}_{\mathcal{C}_1}, A] \arrow[d] \\ {[\mathbf{1}_{\mathcal{C}_2}, f^*A]} \arrow[r] & {[f^*\mathbf{1}_{\mathcal{C}_1}, f^*A]} \end{tikzcd} \qquad \begin{tikzcd} {[\mathbf{1}, K] \otimes [\mathbf{1}, M]} \arrow[r, "d"] \arrow[dr, phantom, "\boxed{36}"] & {[\mathbf{1} \otimes \mathbf{1}, K \otimes M]} \\ K \otimes M \arrow[r] \arrow[u] & {[\mathbf{1}, K \otimes M]} \arrow[u] \end{tikzcd}$$

Proposition 1.83. *Under the suitable assumptions for their existence explained above, the morphisms*

$$f_*A \otimes \mathbf{1}_{\mathcal{C}_1} \xrightarrow{qp} f_*(A \otimes f^*\mathbf{1}_{\mathcal{C}_1})\ ,$$

$$f^!A \otimes f^*\mathbf{1}_{\mathcal{C}_1} \xrightarrow{sp} f^!(A \otimes \mathbf{1}_{\mathcal{C}_1})\ ,$$

$$f^*[\mathbf{1}_{\mathcal{C}_1}, A] \xrightarrow{fh} [f^*\mathbf{1}_{\mathcal{C}_1}, f^*A]$$

and

$$[\mathbf{1}, K] \otimes [\mathbf{1}, M] \xrightarrow{d} [\mathbf{1} \otimes \mathbf{1}, K \otimes M]$$

are always isomorphisms.

Proof: The other arrows in the diagrams $\boxed{33}$, $\boxed{34}$, $\boxed{35}$ and $\boxed{36}$ are isomorphisms. □

Lemma 1.84. *The isomorphism $\mathbf{1} \to [\mathbf{1}, \mathbf{1}]$ defined above (with $A = \mathbf{1}$) is a symmetric form, which is a unit for the product on Witt groups.*

Proof: Left to the reader. Note that a form $A \to [A, K]$ is symmetric if and only if it is adjoint to a morphism $A \otimes A \to K$ invariant by exchanging the two copies of A by c. □

We next give a few interesting facts concerning invertible objects.

Definition 1.85. An object K is invertible if there exists an object K^{-1} and an isomorphism $K^{-1} \otimes K \xrightarrow{inv} \mathbf{1}$.

Let A be any object. We obtain a map $\mathbf{1} \to [A, A]$ by the composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}^l} [A, \mathbf{1} \otimes A] \simeq [A, A] \ .$$

Using the isomorphism $A \to [\mathbf{1}, A]$ from the previous section as well as d, one also obtains a map $[A, B] \otimes C \to [A, B \otimes C]$ by the composition

$$[A, B] \otimes C \xrightarrow{\sim} [A, B] \otimes [\mathbf{1}, C] \longrightarrow [A \otimes \mathbf{1}, B \otimes C] \xrightarrow{\sim} [A, B \otimes C] \ .$$

Proposition 1.86. (1) *Let K be an invertible object and let K^{-1} be an inverse of K. Then there is a natural isomorphism $K^{-1} \xrightarrow{\sim} [K, \mathbf{1}]$ such that the diagram*

$$\begin{tikzcd} K^{-1} \otimes K \arrow[rr, "\sim"] \arrow[dr, "inv"'] & & {[K, \mathbf{1}] \otimes K} \arrow[dl, "\mathrm{ev}^l"] \\ & \mathbf{1} & \end{tikzcd}$$

commutes.

(2) *An object K is invertible if and only if the morphism $\mathrm{ev}^l_{K,\mathbf{1}} : [K, \mathbf{1}] \otimes K \to \mathbf{1}$ is an isomorphism.*

(3) *If K is invertible, then the map $\mathbf{1} \to [K, K]$ defined above is an isomorphism.*

(4) *If K is invertible, then the map $[A, \mathbf{1}] \otimes K \to [A, K \otimes \mathbf{1}] \simeq [A, K]$ defined above is an isomorphism.*

(5) *If $\mathbf{1}$ is dualizing and K is invertible, then K is dualizing.*

Proof: For Point 1, the required morphism is given by the composition

$$K^{-1} \xrightarrow{\mathrm{coev}^l_K} [K, K^{-1} \otimes K] \xrightarrow{inv_\sharp} [K, \mathbf{1}] \ .$$

One may check that inverse is given, up to an automorphism of order two of K (induced by the signature of K, see [24, I, § 2.5]) by the composition

$$[K, \mathbf{1}] \xrightarrow{\sim} \mathbf{1} \otimes [K, \mathbf{1}] \xrightarrow{inv^{-1}} K^{-1} \otimes K \otimes [K, \mathbf{1}] \xrightarrow{\mathrm{ev}^r} K^{-1} \ .$$

Point 2 is then an immediate consequence of Point 1. For Point 3, the required inverse is

$$[K, K] \xrightarrow{\sim} [K, K] \otimes \mathbf{1} \xrightarrow{(\mathrm{ev}^r_K)^{-1}} [K, K] \otimes K \otimes [K, \mathbf{1}] \xrightarrow{\mathrm{ev}^l_K \otimes Id} K \otimes [K, \mathbf{1}] \xrightarrow{\mathrm{ev}^r_K} \mathbf{1} \ .$$

For Point 4, the inverse of the map is given by the composition

$$[A, K] \xrightarrow{\sim} [A, K] \otimes \mathbf{1} \xrightarrow{(\mathrm{ev}^l)^{-1}} [A, K] \otimes [K, \mathbf{1}] \otimes K \xrightarrow{comp} [A, \mathbf{1}] \otimes K$$

For Point 5, the inverse of the bidual isomorphism $\varpi_{A,K}$ is given by

$$[[A, K], K] \xrightarrow{\sim} [[A, \mathbf{1}] \otimes K, K] \xrightarrow{ad} [[A, \mathbf{1}], [K, K]] \xrightarrow{\sim} [[A, \mathbf{1}], \mathbf{1}] \xrightarrow{\varpi^{-1}_{A,\mathbf{1}}} A$$

□

1.12. **Projection formula.** In [10], Gille and Nenashev define two natural products for Witt groups. These products coincide up to a sign. We just choose one of them (the left product, for example), and refer to it as *the* product, but everything works fine with the other one too. In this section, we show that a projection formula is satisfied for the product. Let us first recall the basic properties of the product, rephrasing [10] in our terminology.

Theorem 1.87. *([10, Definition 1.11 and Theorem 2.9]) Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be triangulated categories with dualities D_1, D_2 and D_3. Let $(B, b_1, b_2) : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ be a suspended bifunctor (see Definition 1.31) and $d : B(D_1^o \times D_2^o) \to D_3^o B^o$ be an isomorphism of suspended bifunctors (see Definition 1.32) that makes $\{B, d\}$ a duality preserving functor (see Definition 1.50, here $\mathcal{C}_1 \times \mathcal{C}_2$ is endowed with the duality $D_1 \times D_2$). Then $\{B, d\}$ induces a product*

$$\mathrm{W}(\mathcal{C}_1) \times \mathrm{W}(\mathcal{C}_2) \to \mathrm{W}(\mathcal{C}_3).$$

The following proposition is not stated in [10], but easily follows from the construction of the product.

Proposition 1.88. *Let $\rho : B \to B'$ be an isomorphism of suspended bifunctors that is duality preserving. Then B and B' induce the same product on Witt groups.*

Let us now apply this to our context.

Proposition 1.89. *(existence of the product) Under the assumptions (6) and (9) for K and M (which imply (6) for $K \otimes M$ by Lemma 1.80), $d_{K,M}$ turns $\{- \otimes *, d_{K,M}\}$ into a duality preserving functor from $(\mathcal{C} \times \mathcal{C}, D_K \times D_M, \varpi_K \times \varpi_M)$ to $(\mathcal{C}, D_{K\otimes M}, \varpi_{K\otimes M})$. By Theorem 1.87, it therefore induces a product*

$$\mathrm{W}(\mathcal{C}_K) \times \mathrm{W}(\mathcal{C}_M) \to \mathrm{W}(\mathcal{C}_{K\otimes M})$$

on Witt groups.

We now also assume that (1), (2), (3), (4), (5), (7) and (8) are satisfied for the objects K and M in $\mathcal{C}_1$, and that (9) is satisfied for the couples (K, M) and (f^*K, f^*M). We have already seen in Theorem 1.69 that $\{f^*, fh_{K\otimes M}\}$ is a duality preserving functor between (suspended, triangulated) categories with dualities. Let $I_{fp} : (\mathcal{C}_2)_{f^*K\otimes f^*M} \to (\mathcal{C}_2)_{f^*(K\otimes M)}$ be the duality preserving functor induced by fp (using Lemma 1.64). We have the following diagram of duality preserving functors.

$$\begin{tikzcd}
(\mathcal{C}_1)_K \times (\mathcal{C}_1)_M \arrow[rr, "\{-\otimes *, d_{K,M}\}"] \arrow[d, "\{f^*\times f^*, fh_K \times fh_M\}"'] & & (\mathcal{C}_1)_{K\otimes M} \arrow[d, "\{f^*, fh_{K\otimes M}\}"] \\
(\mathcal{C}_2)_{f^*K} \times (\mathcal{C}_2)_{f^*M} \arrow[r, "\{-\otimes *, d_{f^*K, f^*M}\}"] & (\mathcal{C}_2)_{f^*K\otimes f^*M} \arrow[r, "I_{fp}"] & (\mathcal{C}_2)_{f^*(K\otimes M)}
\end{tikzcd}$$

Proposition 1.90. *(the pull-back respects the product) Under the assumptions for the existence of the pull-back (1.69) for K, M and f^* and the existence of the product (1.89) for (K, M) and for (f^*K, f^*M), the isomorphism of suspended bifunctors $fp : f^*(-) \otimes f^*(*) \to f^*(- \otimes *)$ is a morphism of duality preserving functors between the two functors defined by the compositions above. Thus, they induce the same pairing on Witt groups: $f^*(x.y) = I_{fp}(f^*(x).f^*(y))$ for all $x \in \mathrm{W}((\mathcal{C}_1)_K)$ and $y \in \mathrm{W}((\mathcal{C}_1)_M)$.*

Proof: Let us consider the commutative diagram

$$
\begin{tikzcd}
(a) \arrow[r] \arrow[rrrr

We have also seen in Theorem 1.75 Point 1 that $\{(f^!)^o, f^*\}$ is a duality preserving pair. Let $I_{sp} : (\mathcal{C}_2)_{f^!K\otimes f^*M} \to (\mathcal{C}_2)_{f^*(K\otimes M)}$ be the duality preserving functor induced by sp (using Lemma 1.64). We have the following diagram of duality preserving pairs.

$$\begin{tikzcd}
(\mathcal{C}_1)_K \times (\mathcal{C}_1)_M \arrow[rr, "\{-\otimes *\}"] \arrow[d, "\{(f^!)^o\times(f^*)^o, f^*\times f^*\}"'] & & (\mathcal{C}_1)_{K\otimes M} \arrow[d, "\{(f^!)^o, f^*\}"] \\
(\mathcal{C}_2)_{f^!K} \times (\mathcal{C}_2)_{f^*M} \arrow[r, "\{-\otimes *\}"] & (\mathcal{C}_2)_{f^!K\otimes f^*M} \arrow[r, "I_{sp}"] & (\mathcal{C}_2)_{f^!(K\otimes M)}
\end{tikzcd}$$

Proposition 1.91. *If*
(10) *the morphism of suspended bifunctors* $sp : f^!(-) \otimes f^*(*) \to f^!(- \otimes *)$ *is an isomorphism*

then it is an isomorphism of duality preserving pairs between the two pairs defined above.

Proof: The proof involves exactly the same diagrams as in Proposition 1.90 after replacing the isomorphism $f^*(-)\otimes f^*(*) \to f^*(-\otimes *)$ by $f^!(-)\otimes f^*(*) \to f^!(-\otimes *)$. □

This last diagram of duality preserving pairs may be rewritten as follows (the horizontal maps are pairs that are in fact functors):

$$\begin{tikzcd}
(\mathcal{C}_2)_{f^!K} \times (\mathcal{C}_1)_M \arrow[rr, "I_{sp}\{Id\otimes f^*\}"] & & (\mathcal{C}_2)_{f^!(K\otimes M)} \\
(\mathcal{C}_1)_K \times (\mathcal{C}_1)_M \arrow[rr, "\{Id\otimes Id\}"'] \arrow[u, "\{(f^!\times Id)^o, f^*\times Id\}"] & & (\mathcal{C}_1)_{K\otimes M} \arrow[u, "\{(f^!)^o, f^*\}"']
\end{tikzcd}$$

and by Theorem 1.55, taking $\{f_* \times Id, f_* \times Id\}$ and $\{f_*, f_*\}$ as the right adjoints of the vertical pairs, we obtain the diagram

$$\begin{tikzcd}
(\mathcal{C}_2)_{f^!K} \times (\mathcal{C}_1)_M \arrow[rr, "I_{sp}\{Id\otimes f^*\}"] \arrow[d, "\{f_*\times Id\}"'] & & (\mathcal{C}_2)_{f^!(K\otimes M)} \arrow[d, "\{f_*\}"'] \\
(\mathcal{C}_1)_K \times (\mathcal{C}_1)_M \arrow[rr, "\{Id\otimes Id\}"'] & & (\mathcal{C}_1)_{K\otimes M}
\end{tikzcd}$$

only involving functors (and not pairs), and an isomorphism of duality preserving pairs between the two compositions. To prove that this isomorphism of pairs is an isomorphism of functors, we just have to check the condition of Definition 1.53. By construction of the pair of morphisms, this amounts to check that

$$qp : f_*A \otimes B \to f_*(A \otimes f^*B)$$

and

$$f_*(A \otimes f^*B) \to f_*(f^!f_*A \otimes f^*B) \to f_*f^!(f_*A \otimes B) \to f_*A \otimes B$$

are inverse to each other. This follows from the commutative diagram

$$\begin{tikzcd}
f_*A \otimes B \arrow[r, "qp"] \arrow[d, "\eta^!_*"] \arrow[dd, bend right=60, "id"'] \arrow[dr, phantom, "\boxed{mf}"] & f_*(A \otimes f^*B) \arrow[d, "\eta^!_*"] \\
f_*f^!f_*A \otimes B \arrow[r, "qp", "\sim"'] \arrow[d, "\epsilon^!_*"] \arrow[dr, phantom, "\boxed{25}"] & f_*(f^!f_*A \otimes f^*B) \arrow[d, "sp"] \\
f_*A \otimes B & f_*f^!(f_*A \otimes B) \arrow[l, "\epsilon^!_*"]
\end{tikzcd}$$

Summarizing, we have therefore proved the following.

Theorem 1.92. *(projection formula) Under the assumptions of the existence of pull-backs for f^* and M (1.69), the existence of push-forward for f^* and K (1.75), the existence of the product for (K, M) and $(f^!K, f^*M)$ (1.89) and Assumption (10) for f^* and $f^!$, the maps f^* and f_* between Witt groups as defined in 1.69 and 1.75 satisfy*

$$f_*(I_{sp}(x.f^*(y))) = f_*(x).y$$

for $x \in \mathrm{W}((\mathcal{C}_2)_{f^!K})$ and $y \in \mathrm{W}((\mathcal{C}_1)_M)$, where

$$I_{sp} : \mathrm{W}((\mathcal{C}_2)_{f^!K \otimes f^*M}) \to \mathrm{W}((\mathcal{C}_2)_{f^!(K \otimes M)})$$

is the map induced by the isomorphism sp.

1.13. **Composition.** This section studies the behavior of pull-backs and push-forwards with respect to composition. Let $\mathcal{K}$ be a category whose objects are (suspended, triangulated) categories, and whose morphisms are (suspended, exact) functors. Let $\mathcal{B}$ be another category, and let $(-)^*$ be a weak contravariant functor from $\mathcal{B}$ to $\mathcal{K}$, *i.e.* a functor, except that instead of having equalities $f^*g^* = (gf)^*$ when f and g are composable, we only have isomorphisms of (suspended) functors $f^*g^* \to (gf)^*$. We also require that $(-)^*$ sends the identity of an object to the identity. When, moreover, the diagram

$$\begin{tikzcd} f^*g^*h^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{37}"] & (gf)^*h^* \arrow[d] \\ f^*(hg)^* \arrow[r] & (hgf)^* \end{tikzcd}$$

is commutative, we say that the weak functor is associative.

Remark 1.93. An example for this setting is to take for $\mathcal{B}$ the category of schemes (or regular schemes) and $(X)^* = D^b(Vect(X))$ (or $(X)^* = D^b(O_X - Mod)$).

We assume that $\mathcal{B}$ and all categories in $\mathcal{K}$ are monoidal, with a right adjoint to the tensor product, as explained in Section 1.7. We require that $(-)^*$ is a monoidal associative weak functor, which means that each f^* is symmetric monoidal (Assumption (1)) and that the diagram

$$\begin{tikzcd} g^*f^* \otimes g^*f^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{38}"] & g^*(f^* \otimes f^*) \arrow[r] & g^*f^*(- \otimes *) \arrow[d] \\ (fg)^* \otimes (fg)^* \arrow[rr] & & (fg)^*(- \otimes *) \end{tikzcd}$$

commutes for any two composable f and g.

Let X be an object in $\mathcal{B}$ and K an object in $(X)^*$ satisfying (6) (K is dualizing and D_K is exact in the triangulated case). We denote by $\mathcal{C}_{X,K}$ the (suspended, triangulated) category with duality obtained by Proposition 1.62. Let $\mathcal{D}ual$ denote the category whose objects are the $\mathcal{C}_{X,K}$ and whose morphisms are the exact functors which are duality preserving as in Definition 1.50. We then define a new category $\mathcal{B}^*$ whose objects are pairs (X, K) with $X \in \mathcal{B}$, K as above, and whose morphisms from (X, K) to (Y, M) are pairs (f, ϕ) where $f : X \to Y$ and $\phi : K \to f^*M$ is an isomorphism in $(X)^*$. In particular, f^*M must be dualizing (Assumption (7)). For $(g, \psi) : (Y, M) \to (Z, N)$, the composition $(g, \psi)(f, \phi)$ is defined as (gf, χ), where χ is the composition

$$K \xrightarrow{\phi} f^*M \xrightarrow{f^*(\psi)} f^*g^*N \longrightarrow (gf)^*N \ .$$

It is obvious that the composition is associative (using 37), hence this defines a category. We then have a weak functor

$$\{-\}^* : \mathcal{B}^o \to \mathcal{D}ual$$

that sends (X, K) to $\mathcal{C}_{X,K}$ and (f, ϕ) to the duality preserving functor from $\mathcal{C}_{Y,M}$ to $\mathcal{C}_{X,K}$ obtained by composing the duality preserving functor $\{f^*, fh_M\}$ from Theorem 1.69 with the duality preserving functor $\{Id, \phi\}$. We denote $\mathrm{W}\{f, \phi\}^*$ by $(f, \phi)^*$ and $\mathrm{W}(\mathcal{C}_{X,K})$ by $\mathrm{W}(X, K)$.

Theorem 1.94. *(composition of pull-backs) For any two composable (f, ϕ) and (g, ψ), the isomorphism of functors $f^*g^* \to (gf)^*$ induces a morphism between duality preserving functors from $\{f, \phi\}^*\{g, \psi\}^*$ to $\{(g, \psi)(f, \phi)\}^*$. Therefore, the composition $(-)^*$ of the weak functor $\{-\}^*$ by the functor W is a functor (i.e. strictly associative) with values in the category of Abelian groups.*

Proof: First, we assume the isomorphisms ϕ and ψ are both the identity. Then the claim amounts to check that the diagram

$$\begin{tikzcd}
f^*g^*[-, K] \arrow[r] \arrow[d] & f^*[g^*(-), g^*K] \arrow[r] & {[f^*g^*(-), f^*g^*K]} \arrow[d] \\
(gf)^*[-, K] \arrow[r] & {[(gf)^*(-), (gf)^*K]} \arrow[r] & {[f^*g^*(-), (gf)^*K]}
\end{tikzcd}$$

is commutative. This follows from diagram 38 by adjunctions. Now the general case follows easily. □

Definition 1.95. Let $(-)^*$ and $(-)_*$ be (suspended, triangulated) weak functors with the same source and target, having opposite variances. We say that $((-)^*, (-)_*)$ is an adjoint couple of weak functors if (f^*, f_*) is a (suspended, triangulated) adjoint couple for every f (in particular $(-)^*$ and $(-)_*$ coincide on objects), and the diagrams

$$\begin{tikzcd}
Id \arrow[r] \arrow[d] \arrow[dr, phantom, "39"] & (fg)_*(fg)^* \\
f_*f^* \arrow[r] & f_*g_*g^*f^* \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
g^*f^*f_*g_* \arrow[r] \arrow[d] \arrow[dr, phantom, "40"] & g^*g_* \arrow[d] \\
(fg)^*(fg)_* \arrow[r] & Id
\end{tikzcd}$$

commute for any composable f and g.

As usual, the right (or left) adjoint is unique up to unique isomorphism.

Lemma 1.96. *Assume that for any f^*, we are given a right (suspended, exact) adjoint f_* (which is the identity when f is the identity), then there is a unique collection of isomorphisms $(gf)_* \to g_*f_*$ such that $((-)^*, (-)_*)$ forms an adjoint couple of weak functors.*

Proof: Apply Theorem 1.9 (or Theorem 1.36 in the suspended case) to $(L, R) = (f^*, f_*)$, $(F_1, G_1) = (g^*, g_*)$, $(F_2, G_2) = (Id, Id)$ and $(L', R') = ((fg)^*, (fg)_*)$. This gives the required isomorphism $f_*g_* \to (fg)_*$ and the diagrams 39 and 40. □

Lemma 1.97. *The right (resp. left) adjoint of an associative weak functor is associative.*

Proof: Left to the reader. □

Let us now consider the subcategory $\mathcal{B}'$ of $\mathcal{B}$ with the same objects, but whose morphisms are only the f such that there exists a right adjoint for f^*, and this right adjoint itself again has a right adjoint. We choose these successive right adjoints f_* and $f^!$ for each morphism f (they are unique up to unique isomorphism), and by Lemma 1.96, using $(-)^*$, we turn them into weak functors

$$(-)_* : \mathcal{B}' \to \mathcal{K} \qquad\qquad (-)^! : \mathcal{B}' \to \mathcal{K}$$

that are associative by Lemma 1.97.

By Theorem 1.75, under Assumptions (1) (2), (3), (5), (6) and (8) for M, there is an adjoint couple of duality preserving pairs $(\{(f^!)^o, f^*\}, \{f_*\})$ from $\mathcal{C}_{X,f^!M}$ to $\mathcal{C}_{Y,M}$. By using the same for g and by composition, we get another adjoint couple

$$(\{(f^!g^!)^o, f^*g^*\}, \{g_*f_*\})$$

from $\mathcal{C}_{Z,K}$ to $\mathcal{C}_{f^!g^!K}$, if we choose $M = g^!K$. We can compose again by the obvious duality preserving functor to get to $\mathcal{C}_{(gf)^!K}$. We denote the couple obtained this way by

$$(\{(f^!g^!)^o, f^*g^*\}, \{g_*f_*\})'.$$

By Lemma 1.52, we can complete the isomorphism $f^*g^* \to (gf)^*$ into an isomorphism of duality preserving pair from $\{(f^!g^!)^o, f^*g^*\}'$ to $\{((gf)^!)^o, (gf)^*\}$. By construction, the isomorphism $f^!g^! \to (gf)^!$ coincides with the one obtained above when we turned $(-)^!$ into a weak functor, but we now also have the morphisms that make this pair duality preserving. We therefore have two adjoint couples of duality preserving pairs

$$(\{(f^!g^!)^o, f^*g^*\}, \{g_*f_*, g_*f_*\})'$$

and

$$(\{((gf)^!)^o, (gf)^*\}', \{(gf)_*\})$$

whose first terms are isomorphic. We thus get an isomorphism of duality preserving pairs $\{g_*f_*\} \to \{(gf)_*\}$ by the uniqueness of the right adjoint. Of course, it coincides by construction with the one obtained when turning $(-)_*$ into a weak functor, but we now know that it is duality preserving. It remains to check the condition of Definition 1.53 in order to have a morphism of duality preserving functors. This is ensured by the following commutative diagram.

$$\begin{tikzcd}
(gf)_* \arrow[r] \arrow[d] \arrow[dd, bend right, "id"'] \arrow[dr, phantom, "\boxed{mf}"] & g_*f_* \arrow[d] \\
(gf)_*(gf)^!(gf)_* \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{40'}"] & g_*f_*(gf)^!(gf)_* \arrow[d] \\
(gf)_* & g_*f_*f^!g^!(gf)_* \arrow[l]
\end{tikzcd}$$

in which $\boxed{40'}$ is $\boxed{40}$ for the adjoint couple of weak functors $((-)_*, (-)^!)$ composed with $(gf)_*$ from the right.

We use the following notation: Let $\mathcal{B}^!$ be the category whose objects are the same pairs (X, K) as in $\mathcal{B}^*$, but the morphisms are $(f, \phi) : (X, K) \to (Y, M)$ such that $f \in \mathcal{B}'$ and ϕ is an isomorphism $K \to f^!M$ (so again, $f^!$ and M have to satisfy Assumption (8)). The composition is defined as for $\mathcal{B}^*$ and we have a (covariant) weak functor

$$\{-\}_* : \mathcal{B}^! \to \mathcal{D}ual$$

defined in a similar way as $\{-\}^*$, but using the push-forward of Theorem 1.75. As before, we denote by $(-)_*$ the composition of $-_*$ with W. Then the above discussion may be summarized as follows.

Theorem 1.98. *(composition of push-forwards) For any two composable (f,ϕ) and (g,ψ) in $\mathcal{B}^!$, the isomorphism of functors $g_*f_* \to (gf)_*$ induces a morphism between duality preserving functors from $\{g,\psi\}_*\{f,\phi\}_*$ to $\{(g,\psi)(f,\phi)\}_*$. Therefore, $(-)_*$, defined as the composition of the weak functor $\{-\}_*$ with the functor* W *is a functor (i.e. strictly associative).*

Remark 1.99. The proof that we gave was when the isomorphisms ϕ and ψ are the identity. Again, the general case follows easily.

Using the categories $\mathcal{B}^!$ and $\mathcal{B}^*$, Theorem 1.92 may be rephrased as follows.

Theorem 1.100. *(projection formula) Let $(f,\phi):(X,K)\to(Y,M)$ be a morphism in $\mathcal{B}^*$ and $(f,\psi):(X,L)\to(Y,N)$ a morphism in $\mathcal{B}^!$. Then, under Assumption (10), we have an equality*

$$(f, sp_{N,M}(\phi\otimes\psi))_*(x.(f,\phi)^*(y)) = (f,\psi)_*(x).y$$

in $W^{i+j}(Y, N\otimes M)$ for all $x\in \mathrm{W}^i(X,L)$ and $y\in W^j(Y,M)$ (recall the Definition 1.47 of graded Witt groups).

1.14. **Base change.** The last fundamental theorem that we will prove is base change.

Consider a commutative diagram in $\mathcal{B}$ with g and $\bar{g}$ in $\mathcal{B}'$.

$$\begin{tikzcd} V \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

Using Theorem 1.9 Point 2 (or its suspended version 1.36) with $F_1 = f^*$, $F_2 = \bar{f}^*$, $L = \bar{g}^*$, $R = \bar{g}_*$, $L' = g^*$, $R' = g_*$ and the isomorphism of (suspended) functors

$$\bar{g}^*f^* \to (f\bar{g})^* = (g\bar{f})^* \to \bar{f}^*g^*$$

for f_L, we obtain a morphism

$$\varepsilon : f^*g_* \to \bar{g}_*\bar{f}^*$$

Assuming
(11) the morphism of functors ε is an isomorphism,
and applying the same theorem to $F_1 = \bar{f}^*$, $F_2 = f^*$, $L = \bar{g}_*$, $R = \bar{g}^!$, $L' = g_*$, $R' = g^!$ and $f_L = \varepsilon^{-1}$, we obtain a morphism

$$\gamma : \bar{f}^*g^! \to \bar{g}^!f^*.$$

We also obtain several commutative diagrams, of which only the one corresponding to $\boxed{F_2}$

$$\begin{tikzcd} \bar{g}_*\bar{f}^*g^! \arrow[d, "\gamma"'] \arrow[dr, phantom, "\boxed{41}"] & f^*g_*g^! \arrow[l, "\varepsilon"'] \arrow[d] \\ \bar{g}_*\bar{g}^!f^* \arrow[r] & f^* \end{tikzcd}$$

will be important for us. We now assume as well that
(12) the morphism of functors γ is an isomorphism.
We therefore have a duality preserving functor $I_\gamma : \mathcal{C}_{V,\bar{f}^*g^!K} \to \mathcal{C}_{V,\bar{g}^!f^*K}$ for any dualizing object K, whose underlying functor is just the identity. We now apply

Theorem 1.9 to duality preserving pairs (as explained in Theorem 1.55) and choose $L = \{(\bar{g}^!)^o, \bar{g}^*\}$, $R = \{\bar{g}_*\}$, $L' = \{(g^!)^o, g^*\}$, $R' = \{g_*\}$, $F_1 = \{f^*\}$, $F_2 = I_\gamma\{\bar{f}^*\}$ (all of which were described in the previous section). To obtain the isomorphism f'_L, we complete the morphism γ as a morphism of duality preserving pairs

$$I_\gamma\{(\bar{f}^* g^!)^o, \bar{f}^* g^*\} \to \{(\bar{g}^! f^*)^o, \bar{g}^* f^*\}$$

as explained in Lemma 1.52 Point 1. The theorem thus gives a morphism of duality preserving pairs

$$\{f^*\}\{g_*\} \to \{\bar{g}_*\}\{\bar{f}^*\}.$$

As usual, we want to show that it is a morphism of duality preserving functors (Definition 1.53). By construction, we thus have to show that the maps ε and

$$\bar{g}_*\bar{f}^* \longrightarrow \bar{g}_*\bar{f}^* g^! g_* \xrightarrow{\gamma} \bar{g}_*\bar{g}^! f^* g_* \longrightarrow f^* g_*$$

are inverse to each other. This follows from this commutative diagram:

$$\begin{tikzcd}
f^* g_* \arrow[r] \arrow[d] \arrow[dd, bend right=60, "id"'] \arrow[dr, phantom, "\boxed{mf}"] & \bar{g}_*\bar{f}^* \arrow[d] \\
f^* g_* g^! g_* \arrow[r, "\varepsilon", "\sim"'] \arrow[d] \arrow[dr, phantom, "\boxed{41}"] & \bar{g}_*\bar{f}^* g^! g_* \arrow[d] \\
f^* g_* & \bar{g}_*\bar{g}^! f^* g_* \arrow[l]
\end{tikzcd}$$

We have therefore proved the following.

Theorem 1.101. *(base change) Let (f, ϕ) and $(\bar{f}, \bar{\phi})$ be morphisms in $\mathcal{B}^*$, (g, ψ), and $(\bar{g}, \bar{\psi})$ be morphisms in $\mathcal{B}^!$ fitting in the diagram*

$$\begin{tikzcd}
\mathcal{C}_{V,N} \arrow[r, "(\bar{g},\bar{\psi})"] \arrow[d, "(\bar{f},\bar{\phi})"'] & \mathcal{C}_{Y,L} \arrow[d, "(f,\phi)"] \\
\mathcal{C}_{X,M} \arrow[r, "(g,\psi)"'] & \mathcal{C}_{Z,K}
\end{tikzcd}$$

such that $f\bar{g} = g\bar{f} \in \mathcal{B}$, such that Assumptions (11) and (12) are satisfied and that the diagram

$$\begin{tikzcd}
 & \bar{f}^* M \arrow[r, "\bar{f}^*(\psi)"] & \bar{f}^* g^! K \arrow[d, "\gamma"] \\
N \arrow[ur, "\bar{\phi}"] \arrow[dr, "\bar{\psi}"'] & & \\
 & \bar{g}^! L \arrow[r, "\bar{g}^!(\phi)"'] & \bar{g}^! f^* K
\end{tikzcd}$$

is commutative. Then the maps $(f, \phi)^*$, $(\bar{f}, \bar{\phi})^*$, $(g, \psi)_*$ and $(\bar{g}, \bar{\psi})_*$ satisfy

$$(\bar{g}, \bar{\psi})_*(\bar{f}, \bar{\phi})^* = (f, \phi)^*(g, \psi)_*$$

(as morphisms of Witt groups).

Remark 1.102. As for Theorems 1.94 and 1.98, we have just proved the case where ϕ and ψ are identities, and the general case easily follows.

1.15. $f^!$ **of unit objects.** In this section, we use the unit objects in the monoidal categories to formulate the main theorems in a different way. In the application to algebraic geometry, this will relate f^* and $f^!$ using canonical sheaves.

Recall that f^* is assumed to be a monoidal functor for every $f : X \to Y$. We assume for simplicity in this section that the coherence map $\mathbf{1}_X \to f^*(\mathbf{1}_Y)$ is the identity, where $\mathbf{1}_X$ and $\mathbf{1}_Y$ denote the unit objects in the monoidal categories $\mathcal{C}_X$ and $\mathcal{C}_Y$.

For any morphism $f : X \to Y$ in $\mathcal{B}'$, we define

$$\omega'_f = f^!(\mathbf{1}_Y)$$

To every object X we associate a number d_X and set $d_f = d_X - d_Y$ (in applications d_X will be the relative dimension of X over a base scheme). We then define

$$\omega_f = T^{d_f} f^!(\mathbf{1}_Y).$$

For any two composable morphisms f and g in $\mathcal{B}'$, let us denote by $i'_{g,f}$ the composition

$$\omega'_{gf} = (gf)^!(\mathbf{1}_Z) \simeq f^! g^!(\mathbf{1}_Z) = f^!(\omega'_g) \simeq f^!(\mathbf{1}_Y) \otimes f^*(\omega'_g) = \omega'_f \otimes f^*(\omega'_g)$$

(where we used Point 2 of Proposition 1.83) and

$$i_{g,f} : \omega_{gf} \xrightarrow{\sim} \omega_f \otimes f^*(\omega_g)$$

the composition obtained by the same chain of isomorphisms and then desuspending.

Lemma 1.103. *For any composable morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} V$ *, the diagram of isomorphisms*

$$\begin{tikzcd}
 & \omega_f \otimes f^*(\omega_{hg}) \arrow[r, "id \otimes f^* i_{h,g}"] & \omega_f \otimes f^*(\omega_g \otimes g^*(\omega_h)) \arrow[d, "\wr"] \\
\omega_{hgf} \arrow[ur, "i_{hg,f}"] \arrow[dr, "i_{h,gf}"'] & & \\
 & \omega_{gf} \otimes (gf)^*(\omega_h) \arrow[r, "i_{g,f} \otimes id"'] & \omega_f \otimes f^*(\omega_g) \otimes (gf)^*(\omega_h)
\end{tikzcd}$$

is commutative. In other words, $i_{g,f}$ *(as well as* $i'_{g,f}$*) satisfies a cocycle condition.*

Proof: Left to the reader. □

We define the category $\mathcal{B}_*$ as the subcategory of $\mathcal{B}^*$ having the same objects, and whose morphisms $(f, \phi) : (X, M) \to (Y, N)$ are such that $f \in \mathcal{B}'$ and $\phi : M \to f^*(N)$ is an isomorphism. For any such morphism (f, ϕ) in $\mathcal{B}_*$, we obtain a push-forward

$$\mathrm{W}(X, \omega'_f \otimes M) \to \mathrm{W}(Y, N) \tag{1}$$

by Theorem 1.75, applying Lemma 1.64 to the isomorphism

$$\omega'_f \otimes M = f^! \mathbf{1}_Y \otimes M \xrightarrow{id \otimes \phi} f^! \mathbf{1}_Y \otimes f^* N \xrightarrow{sp} f^!(\mathbf{1}_Y \otimes N) \xrightarrow{\sim} f^! N.$$

Let us now assume that
(13) $\mathcal{B}'$ has a final object called Pt, and we choose $d_{\mathrm{Pt}} = 0$.

For each object X, we denote by π_X the unique morphism $X \to \mathrm{Pt}$ and define

$$\omega'_X = \omega_{\pi_X} \qquad\qquad \omega_X = T^{d_X} \omega'_X.$$

Since $\pi_Y f = \pi_X$, the morphism $i'_{\pi_Y, f}$ gives us the isomorphism

$$\omega'_X \simeq \omega'_f \otimes f^*(\omega'_Y)$$

and $i_{\pi_Y,f}$ is also an isomorphism:

$$\omega_X \simeq \omega_f \otimes f^*(\omega_Y).$$

From (1), using this last isomorphism in Lemma 1.64 and Lemma 1.63 to deal with the shifting, we obtain, for any morphism $(f,\phi):(X,M)\to(Y,N)$ in $\mathcal{B}_*$, an "absolute" version of the push-forward

$$\mathrm{W}^{i+d_X}(X,\omega_X\otimes M)\to \mathrm{W}^{i+d_Y}(Y,\omega_Y\otimes N) \tag{2}$$

or,

$$\mathrm{W}^{i+d_X}(c(X,M))\to \mathrm{W}^{i+d_Y}(c(Y,N)) \tag{3}$$

if we put $c(X,M)=(X,\omega_X\otimes M)$.

Theorems 1.98, 1.92 and 1.101 then easily translate as follows:

Theorem 1.104. *(composition of push-forwards) For any composition*

$$X \xrightarrow{(f,\phi)} Y \xrightarrow{(g,\psi)} Z$$

in $\mathcal{B}_*$, the induced morphisms on Witt groups satisfy

$$(g,\psi)_*(f,\phi)_* = ((g,\psi)(f,\phi))_*$$

from $\mathrm{W}^{i+d_X}(c(X,M))$ to $\mathrm{W}^{i+d_Y}(c(Y,N))$.

Theorem 1.105. *(projection formula) Let*

$$(f,\phi):(X,M)\to(Y,N)$$

be a morphism in $\mathcal{B}^*$ and

$$(f,\psi):(X,M)\to(Y,N)$$

be a morphism in $\mathcal{B}_*$ (same f). Then

$$(f,\phi\otimes\psi)_*(x.(f,\phi)^*(y)) = (f,\psi)_*(x).y \in \mathrm{W}^{i+j+d_Y}(c(Y,L\otimes N))$$

for any $x\in \mathrm{W}^{i+d_X}(c(X,K))$ and $y\in \mathrm{W}^j(Y,N)$.

Let us remark that when f, g, $\bar f$ and $\bar g$ satisfy Assumption (12), we have an isomorphism

$$\omega'_{\bar g} = \bar g^!(\mathbf{1}_Y) \xrightarrow{\sim} \bar g^! f^*(\mathbf{1}_Z) \xrightarrow[\sim]{\gamma} \bar f^* g^!(\mathbf{1}_Z) = \bar f^*(\omega'_g)$$

and similarly

$$\omega_{\bar g} \simeq \bar f^*(\omega_g).$$

Theorem 1.106. *(base change) Let (f,ϕ) and $(\bar f,\bar\phi)$ be morphisms in $\mathcal{B}^*$, (g,ψ) and $(\bar g,\bar\psi)$ be morphisms in $\mathcal{B}_*$ with sources and targets as on the diagram*

$$\begin{array}{ccccc}
c(V,N) & & (V,N) & \xrightarrow{(\bar g,\bar\psi)} & (Y,L) & & c(Y,L) \\
{\scriptstyle(\bar f,\bar\phi)}\downarrow & & & & & & \downarrow{\scriptstyle(f,\phi)} \\
c(X,M) & & (X,M) & \xrightarrow[(g,\psi)]{} & (Z,K) & & c(Z,K)
\end{array}$$

such that $f\bar{g} = g\bar{f} \in \mathcal{B}$, such that (12) and (11) are satisfied and the diagram

$$\begin{tikzcd}
\omega_V \otimes N \arrow[r, "\bar{\phi}"] \arrow[d, "id\otimes\bar{\psi}"'] & \bar{f}^*(\omega_X \otimes M) \arrow[r, "\bar{f}^*(id\otimes\psi)"] & \bar{f}^*(\omega_X \otimes g^*K) \arrow[d] \\
\omega_V \otimes \bar{g}^*L \arrow[d] & & \bar{f}^*(\omega_g \otimes g^*(\omega_Z) \otimes g^*K) \arrow[d] \\
\omega_{\bar{g}} \otimes \bar{g}^*(\omega_Y) \otimes \bar{g}^*L \arrow[d] & & \bar{f}^*(\omega_g) \otimes \bar{f}^*\bar{g}^*(\omega_Z \otimes K) \arrow[d] \\
\omega_{\bar{g}} \otimes \bar{g}^*(\omega_Y \otimes L) \arrow[r] & \bar{f}^*(\omega_g) \otimes \bar{g}^*(\omega_Y \otimes L) \arrow[r, "id\otimes\bar{g}^*(\phi)"'] & \bar{f}^*(\omega_g) \otimes \bar{g}^*f^*(\omega_Z \otimes K)
\end{tikzcd}$$

is commutative. Then we have an equality of morphisms

$$(\bar{g}, \bar{\psi})_*(\bar{f}, \bar{\phi})^* = (f, \phi)^*(g, \psi)_*$$

from $\mathrm{W}^{i+d_X}(c(X, M))$ *to* $\mathrm{W}^{i+d_Y}(c(Y, L))$.

We will also need the following.

Lemma 1.107. *Let f, $\bar{f}$, g and $\bar{g}$ be as in Theorem 1.106 and assume ω'_Z is invertible (see Proposition 1.86). Then we have isomorphisms*

$$\omega'_V \simeq \bar{f}(\omega'_X) \otimes (g\bar{f})^*(\omega'_Z)^{-1} \otimes \bar{g}^*(\omega'_Y)$$

and

$$\omega_V \simeq \bar{f}(\omega_X) \otimes (g\bar{f})^*(\omega_Z)^{-1} \otimes \bar{g}^*(\omega_Y).$$

Proof: The first isomorphism is given by the chain of isomorphisms

$$\begin{aligned}
\omega'_V \simeq \omega_{\bar{g}} \otimes \bar{g}^*(\omega'_Y) \simeq \bar{g}^!f^*(\mathbf{1}_Z) \otimes \bar{g}^*(\omega'_Y) \simeq \bar{f}^*g^!(\mathbf{1}_Z) \otimes \bar{g}^*(\omega'_Y) = \bar{f}^*(\omega_g) \otimes \bar{g}^*(\omega'_Y) \\
\simeq \bar{f}^*(\omega'_X \otimes g^*(\omega'_Z)^{-1}) \otimes \bar{g}^*(\omega'_Y) \simeq \bar{f}^*(\omega'_X) \otimes \bar{f}^*g^*(\omega_Z)^{-1} \otimes \bar{g}^*(\omega'_Y) \\
\simeq \bar{f}^*(\omega'_X) \otimes (g\bar{f})^*(\omega_Z)^{-1} \otimes \bar{g}^*(\omega'_Y)
\end{aligned}$$

and the second is just a shifted version of the first. □

2. Application to the coherent Witt groups of schemes

2.1. **Grothendieck duality.** In this section, we introduce a few results from the theory of Grothendieck duality, as developed in [11] or [26]. The reader should be aware that we do not care about the sign conventions in these references, since we only use results about the existence or invertibility of some functors which are independent of the sign conventions.

Remark 2.1. We use homological complexes, because this is the convention usually chosen in the articles on Witt groups following Balmer. Virtually all articles (including [11] and [26]) on derived categories of sheaves use cohomological complexes. When quoting results from these articles, we have applied the standard equivalence from the category of cochains to the category of chains (re-indexing $(\ )^n$ as $(\ )_{-n}$ with no additional signs, e.g. "bounded above" becomes "bounded below"). Nevertheless we stick to the notation D^b and terms like "finite cohomological dimension" rather than D_b and "finite homological dimension".

Recall that for any exact category $\mathcal{E}$ (e.g. $\mathcal{E} = \mathcal{O}_X$-modules on a given scheme X), the canonical functor from the derived category of bounded complexes $D^b(\mathcal{E})$ to the subcategory of the unbounded derived category $D(\mathcal{E})$ of complexes with bounded cohomology is an equivalence of categories (see e.g. [12, Lemma 11.7]). Thus, we shall use the same symbol for this latter category as well, and it is this variant we work with when using the previous section.

Let $\mathcal{R}eg$ be the full subcategory of the category of schemes in which the objects are separated Noetherian regular schemes of finite Krull dimension, such that the global sections of X contain $1/2$.

Let $D^?_*(X)$, where $? \in \{\emptyset, +, -, b\}$ and $* \in \{\emptyset, qc, c\}$ denote the derived category (resp. cohomologically bounded above, below, above and below) of the category of sheaves of O_X-modules (resp. of quasi-coherent cohomology, coherent cohomology). For X Noetherian, the inclusion $D^b(Qcoh(X)) \xrightarrow{\simeq} D^b_{qc}(X)$ is an equivalence of triangulated categories (see [11, Corollary II.7.19] and [25, section B.4], inducing an equivalence $D^b_c(Qcoh(X)) \xrightarrow{\simeq} D^b_c(X)$. It is the category $D^b_c(Qcoh(X))$ used in the original definition of coherent Witt groups [7]. Similarly, one shows that $D^b(Coh(X)) \xrightarrow{\simeq} D^b_c(X)$ is an equivalence of triangulated categories. We freely use these equivalences without mentioning them.

In this section, we construct pull-backs (with respect to arbitrary maps between regular Noetherian schemes of finite Krull dimension) and transfer maps between (Grothendieck-)Witt groups with respect to proper morphisms and establish some properties such as the base change and the projection formula. In contrast to K_0, the transfer maps for (Grothendieck-)Witt groups will shift the degree and twist the duality. These transfer maps and their properties are then used for the construction of the categories of Grothendieck-Witt motives and Witt motives.

Everything in the sequel is true both for GW and W, so we just state everything for W.

We observe that in some very special cases there are already constructions that deserve the name transfer map. In particular, for any projection map $\pi : \mathbf{P}^n \times X \to X$, Walter establishes maps $W^i(\mathbf{P}^n \times X, \pi^* L(-n-1)) \to W^{i-n}(X, L)$ [27, p. 24] which using in particular Theorem 5.6, Proposition 5.11 and p.23/24 of *loc. cit.* can be seen to be natural with respect to X. Also, there seems to be work in progress by C. Walter on the construction of transfer maps in a very general setting (which should presumably yield the same transfer maps as those we constructed). There are also transfer constructions for Witt groups with respect to certain finite maps and closed embeddings in the affine space [9], [29], but not for other projective morphisms which is what we need.

One always has pullbacks for locally free (that is defined using complexes of vector bundles) Witt groups. Recall that as X is Noetherian regular of finite Krull dimension, the inclusion $(D^b(Vect(X)) \to (D^b_c(X))$ together with the choice of an injective resolution of a line bundle L on X gives rise to an equivalence $(D^b(Vect(X)), Hom(\quad, L)) \xrightarrow{\simeq} (D^b_c(X), RHom(\quad, L))$ of triangulated categories with duality, inducing a non-canonical isomorphism between the associated Witt groups (the proof of [7, Corollary 2.17.2] for $\mathcal{O}_X$ carries over to arbitrary line bundles L). One also has a map $f^* : W^*(Y, O_Y) \to W^*(X, \mathcal{O}_X)$ between coherent Witt groups for $f : X \to Y$ a flat morphism [8, p. 221].

2.1.1. *Functors on the bounded derived category.*

Proposition 2.2. *Let $X \in \mathcal{R}eg$.*

(1) *The tensor product on the category of complexes (see Appendix A) admits a left (in both variables) derived functor which restricts to an exact bifunctor*

$$(-\otimes^L *) : D^b_c(X) \times D^b_c(X) \to D^b_c(X).$$

It comes naturally equipped with a unit, an associativity and a symmetry isomorphism satisfying all necessary axioms, thus turning $D^b_c(X)$ into a symmetric monoidal category. The tensor product on the category of complexes also admits a suspended bifunctor structure, which induces one on $\otimes^L$.

(2) *The internal Hom on the category of complexes (see Appendix A) admits a right (in both variables) derived functor which restricts to a suspended bifunctor*
$$\mathrm{RHom}(*,-) : D^b_c(X)^o \times D^b_c(X) \to D^b_c(X).$$
(3) *The adjunction between the usual tensor product of sheaves and the internal Hom induce a suspended ACB (see Definition 1.37)* $(-\otimes^L *, \mathrm{RHom}(*,-))$.

Proof: By [11, Proposition II.4.3], the tensor product on the category of complexes induces a bifunctor $D^b_c(X) \times D^b_c(X) \to D_c(X)$, and since $X \in \mathcal{R}eg$, every coherent sheaf has a finite locally free resolution (see [23, §7, Point 1]), so the bifunctor takes values in $D^b_c(X)$. The associativity, unit $(\mathcal{O}_X)$ and symmetry isomorphisms are defined on the category of complexes (see Appendix A) and induce the symmetric monoidal category structure on $D^b_c(X)$. The suspended bifunctor structure is also induced from the category of complexes as introduced in Appendix A. This proves Point 1.
The internal Hom on the category of complexes admits a right (in both variables) derived functor $D^b_c(X) \times D^b_c(X) \to D_c(X)$, by [11, Proposition II.3.3], and again, it takes values in $D^b_c(X)$ because $X \in \mathcal{R}eg$. (Recall that there are two canonically isomorphic definitions of RHom, namely $R_I R_{II} Hom$ and $R_{II} R_I Hom$, see [11, p. 65/66 and 91]; we use the first one.) The existence of the suspended structure follows from Proposition 1.39. This proves Point 2.
The suspended adjunction of Point 3 is induced by the one on complexes (see Appendix A). □

Proposition 2.3. *Let $X, Y \in \mathcal{R}eg$ and let $f : X \to Y$. Then*

(1) *f^* admits a left derived functor which restricts to*
$$Lf^* : D^b_c(Y) \to D^b_c(X).$$
(2) *The canonical isomorphism $f^*(A \otimes B) \to f^*(A) \otimes f^*(B)$ on sheaves induces an isomorphism of suspended functors*
$$fp : Lf^*(-) \otimes^L Lf^*(*) \to Lf^*(-\otimes^L *)$$
and an isomorphism
$$Lf^*(\mathcal{O}_Y) \to \mathcal{O}_X$$
which turns Lf^* into a symmetric monoidal functor.
(3) *For any composable morphisms f and g, we have an isomorphism $Lf^* Lg^* \simeq L(gf)^*$. This turns $L(-)^*$ into an associative weak functor.*

Proof: By [11, Proposition II.4.4], f^* has a left derived functor from $D^+_c(X)$ to $D^+_c(X)$. Since $Y \in \mathcal{R}eg$, by [23, §7 Point 1] every coherent sheaf has a finite resolution by locally free sheaves, which stay locally free by pull-back, so we can restrict the functor from $D^b_c(X)$ to $D^b_c(Y)$. This proves Point 1. The definition of the morphism fp in Point 2 is Proposition II.5.9 in *loc. cit.* and checking that Lf^* is symmetric monoidal is easy. Point 3 is [11, Proposition II.5.4] for the existence of the isomorphism, and then the remark p. 60, Point 1 in *loc. cit.*. □

Proposition 2.4. *Let $X, Y \in \mathcal{R}eg$ and let $f : X \to Y$ be a proper morphism. Then we have:*

(1) *The functor f_* admits a right derived functor that restricts to*
$$Rf_* : D^b_c(X) \to D^b_c(Y).$$
(2) *Rf_* is a right adjoint of Lf^*.*
(3) *The functor Rf_* admits a right adjoint*
$$f^! : D^b_c(Y) \to D^b_c(X).$$

Proof: By [11, Proposition II.2.2], f_* has a right derived functor $D_c^-(X) \to D_c^-(Y)$. It restricts to a functor $D_c^b(X) \to D_c^b(Y)$ since f_* has finite cohomological dimension (see e. g. *loc. cit.*, p. 87). The fact that it is a right adjoint of Lf^* is Corollary II.5.11 in *loc. cit.* For Point 3, we use the following, which is the main result of Grothendieck duality theory.

Theorem 2.5. [26, Theorem 1 (existence theorem)] *Let $f : X \to Y$ be a proper morphism of Noetherian schemes of finite Krull dimension. Then there is a functor $f^! : D_{qc}^-(Y) \to D_{qc}^-(X)$ and a morphism of functors Tr_f such that for all $F \in D_{qc}(X)$ and $G \in D_{qc}^-(Y)$ the morphism induced by Tr_f*

$$\mathrm{Ext}_X^p(F, f^!G) \to \mathrm{Ext}_Y^p(Rf_*F, G)$$

is an isomorphism for all p.

In particular, for $p = 0$, this gives the adjunction.

We now finish the proof of Point 3 of the proposition. The fact that $f^!$ restricts to D_c is [26, Lemma 1], and that it restricts to D_c^b is obtained in the following way: first, $f^!(\mathcal{O}_Y)$ is a dualizing complex by Corollary 3 of *loc. cit.*, and since dualizing complexes are shifted line bundles (see Proposition 2.10 below), it is bounded. Then, Corollary 2 of *loc. cit.* gives us an isomorphism

$$f^!(\mathcal{O}_Y) \otimes^L Lf^*(G) \to f^!G$$

so the fact that Lf^* and $\otimes^L$ restrict to D^b suffices to conclude (we don't need f to be of finite flat dimension in Corollary 2, because X and Y are regular). □

Remark 2.6. Being part of an adjoint pair, the functor $f^! : D_{qc}^+(X) \to D_{qc}^+(Y)$ and the natural transformation Tr_f are unique up to unique isomorphism, see [26, p. 394]. There are at least two different ways to construct them and to prove the isomorphism of the theorem (see also [18] for still another approach). One is to use residual complexes as Hartshorne [11] does. The other is to use the techniques of Verdier as done by Deligne in the appendix of [11].

We will need the following lemma to prove that some morphisms of functors are isomorphisms.

Lemma 2.7. *Let X be in $\mathcal{R}eg$.*

- *Assume X is affine. The subcategory of complexes of coherent sheaves in $D_c^b(X)$ is generated as a thick triangulated category (i.e. stable under suspension, triangles and direct factors) by $\mathcal{O}_X$.*
- *Let F and F' be triangulated functors from $D_c^b(X)$, X affine, to some other triangulated category. Any triangulated morphism of functors from F to F' is an isomorphism if it is an isomorphism on $\mathcal{O}_X$.*

Proof: For Point 1, denote by $\mathcal{L}$ the category generated by $\mathcal{O}_X$. Free O_X-modules (of finite rank) are obviously in $\mathcal{L}$, so locally free sheaves (that is vector bundles) are in $\mathcal{L}$ as direct factors of free sheaves as X is affine. Next, any bounded complex of locally free sheaves is in $\mathcal{L}$ by induction on the length of the complex: there is an obvious triangle with the object of smallest degree in a complex, the complex and the complex with the first object truncated. Since X is regular, we can replace any coherent sheaf by a finite resolution of locally free sheaves, so any coherent sheaf is in $\mathcal{L}$. Finally, again by induction, any complex of coherent sheaves is in $\mathcal{L}$. Point 2 is an obvious consequence of Point 1, using [11, Proposition I.7.1, (i)]. □

Proposition 2.8. *Let $f : Y \to Z$ be a morphism in $\mathcal{R}eg$, proper for Points 3 and 4. In Point 2, K and M are dualizing objects. In Point 5, f, $\bar{f}$, $g : X \to Z$ and $\bar{g}$ are in $\mathcal{R}eg$ and form a Cartesian diagram with g and $\bar{g}$ proper and f and $\bar{f}$ flat. Then the following morphisms of functors are isomorphisms:*

1. *the morphism $fh : Lf^*\mathrm{RHom}(-,*) \to \mathrm{RHom}(Lf^*(-), Lf^*(*))$, defined as in Proposition 1.68, using Assumption (1), satisfied by Proposition 2.3 Point 2,*
2. *the morphism $d : \mathrm{RHom}(-,K)\otimes^L\mathrm{RHom}(*,M) \to \mathrm{RHom}(-\otimes^L *, K\otimes^L M)$ defined as in Definition 1.78,*
3. *the morphism $qp : Lf_*(-)\otimes^L * \to Lf_*(-\otimes^L f^*(*))$, defined as in Proposition 1.71, using moreover Assumption (2), satisfied by Proposition 2.4 Point 2,*
4. *the morphism $sp : f^!(-)\otimes^L Lf^*(*) \to f^!(-\otimes^L *)$, defined as in Proposition 1.74, using moreover Assumption (3), satisfied by Proposition 2.4 Point 3, and Assumption (5) by the previous point,*
5. *the morphisms $\varepsilon : Lf^*Rg_* \to R\bar{g}_*L\bar{f}^*$ and $\gamma : L\bar{f}^*g^! \to \bar{g}^!Lf^*$ defined as at the beginning of Section 1.14.*

Proof: We prove Point 4, a similar but easier proof applies to Points 1, 2 and 3. For a fixed object A, we show that $f^!(A)\otimes^L Lf^*(*) \to f^!(A\otimes^L *)$ is an isomorphism. First, as we have a morphism between two functors with values complexes of sheaves which commute with the restriction to open subschemes, it suffices to check that the morphism is locally an isomorphism. Thus we may restrict to the case where Y is affine. Then we apply Lemma 2.7, Point 2 to restrict to the case $* = \mathcal{O}_Y$, which follows from Proposition 1.83. To prove Point 5, one observes that ε coincides with the morphism defined in [2, p. 84, p. 285] and is thus an isomorphism by [1, p. 290]. The morphism γ coincides with the one considered in [26, p. 401] and thus is an isomorphism by [26, Theorem 2]. □

Remark 2.9. For the applications considered in this article (the construction of Witt motives and the proof of dévissage), the hypotheses of Point 5 of Proposition 2.8 are satisfied. Nevertheless there might be other interesting situations where f is not flat but ε and γ are still isomorphisms, and consequently Theorem 2.17 and Corollary 2.18 still hold. By [1, p. 290], one knows that more generally ε is an isomorphism provided $Tor^n_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for all $n > 0$. Using the notations and results of Section 1.15 and reducing to $\mathcal{O}_Z$ as in the proof of Point 4 above, one may show that γ is an isomorphism if $\bar{f}^*\omega_g \simeq \omega_{\bar{g}}$.

2.1.2. *Dualizing complexes.*

Proposition 2.10. *(see [11, Theorem V.3.1]) Let $X \in \mathcal{R}eg$. Then $\mathcal{O}_X$ is a dualizing object and, furthermore, the dualizing objects in $D^b_c(X)$ are exactly the ones isomorphic to shifted line bundles, that is considered as complexes concentrated in a single degree.*

Corollary 2.11. *Let K be a dualizing object. Then Lf^*K is dualizing.*

Proof: The pull-back of a shifted line bundle is a shifted line bundle, and $Lf^*K = f^*K$ as K is locally free. □

Corollary 2.12. *Let K be a dualizing object and $f \in \mathcal{R}eg$ be proper. Then $f^!K$ is dualizing.*

Proof: By the isomorphism sp (Proposition 2.8 Point 4) and the previous corollary, we restrict to the case where $K = \mathcal{O}_Y$, which is [26, Corollary 3]. □

2.2. **Application to Witt groups.** We now apply the formalism of Section 1, using Grothendieck duality theory. Recall that we always work with *coherent* Witt groups, which are non-canonically isomorphic to the usual Witt groups for regular schemes.

2.2.1. *Regular schemes.* We choose $\mathcal{B} = \mathcal{R}eg$ and we use the weak functor $L(-)^*$ described in Proposition 2.3, Point 3, to construct the category $\mathcal{R}eg^* = \mathcal{B}^*$, as in Section 1.13.

Theorem 2.13. *(pull-backs) Let $(f, \phi) : (X, K) \to (Y, M)$ be a morphism in $\mathcal{R}eg^*$. It induces a pull-back*

$$(f, \phi)^* : \mathrm{W}(Y, M) \to \mathrm{W}(X, K)$$

respecting composition. Furthermore, if M is a dualizing object,

$$(f, id) : (X, Lf^*M) \to (Y, M)$$

is a morphism in $\mathcal{R}eg^*$ so it induces a pull-back

$$f^* := (f, id)^* : \mathrm{W}(Y, M) \to \mathrm{W}(X, Lf^*M).$$

Proof: All assumptions of Theorem 1.69 and Theorem 1.94 are satisfied by Proposition 2.3 and Proposition 2.8, Point 1. To prove that (f, id) is indeed in $\mathcal{R}eg^*$, we just need to prove that $Lf^*M = f^*M$ is dualizing, which is just Corollary 2.11. □

The product on Witt groups is defined thanks to Proposition 1.89, using Proposition 2.8 Point 2.

Theorem 2.14. *(compatibility with products) The pull-back satisfies $(f, \phi)^*(x.y) = (f, \phi)^*(x).(f, \phi)^*(y)$.*

Proof: This is exactly Proposition 1.90. □

We define the category $\mathcal{R}eg^! = \mathcal{B}^!$ as in Section 1.13.

Theorem 2.15. *(push-forwards) To any morphism $(f, \phi) : (X, K) \to (Y, M)$ in $\mathcal{R}eg^!$, we associate a push-forward*

$$(f, \phi)_* : \mathrm{W}(X, K) \to \mathrm{W}(Y, M)$$

by Theorem 1.75. It respects composition by Theorem 1.98. Let $f : X \to Y$ be a proper morphism with $X, Y \in \mathcal{R}eg$, and M be a dualizing object on Y. Then $(f, id) : (X, f^!M) \to (Y, M)$ is a morphism in $\mathcal{R}eg^!$. In particular, it gives a push-forward

$$f_* := (f, id)_* : \mathrm{W}(X, f^!M) \to \mathrm{W}(Y, M)$$

Proof: For a proper morphism f, Lf^* admits a right adjoint Rf_*, which admits a right adjoint $f^!$ by 2.4, Points 2 and 3. This establishes Assumptions (1)-(3). Assumption (5) needed in Theorem 1.75 is Point 3 in Proposition 2.8. Finally, $f^!M$ is dualizing by Corollary 2.12. □

We thus have achieved the definition of push-forwards maps for Witt groups.

Theorem 2.16. *(projection formula) The pull-back and push-forward satisfy the projection formula (see Theorem 1.92).*

Proof: The morphism $sp : f^!(-) \otimes^L Lf^*(*) \to f^!(- \otimes^L *)$ is an isomorphism by Proposition 2.8, Point 4. □

Theorem 2.17. *(base change) Let $f, \bar{f}, g, \bar{g} \in \mathcal{R}eg$ with f flat and g proper such that the diagram*

$$\begin{tikzcd} V \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

is Cartesian. Assume that the four schemes are equipped with dualizing complexes such that the morphisms above extend to (f, ϕ), $(\bar{f}, \bar{\phi})$,(g, ψ) and $(\bar{g}, \bar{\psi})$ as in Theorem 1.101, making the pentagon there commutative. Then we have an equality

$$(\bar{g}, \bar{\psi})_*(\bar{f}, \bar{\phi})^* = (f, \phi)^*(g, \psi)_*$$

between morphisms of Witt groups.

Proof: This follows from Theorem 1.101 as Assumptions (11) and (12) are satisfied by Proposition 2.8, Point 5. □

In practice, this reads as follows.

Corollary 2.18. *Assume that we have a Cartesian square in $\mathcal{R}eg$ as in Theorem 2.17, and let K be a line bundle on Z. Then we have a commutative square of Witt groups*

$$\begin{tikzcd} \mathrm{W}^*(V, L\bar{f}^* g^! K) \arrow[r, "\{\bar{g},\gamma\}_*"] & \mathrm{W}^*(Y, Lf^* K) \\ \mathrm{W}^*(X, g^! K) \arrow[u, "\bar{f}^*"] \arrow[r, "g_*"'] & \mathrm{W}^*(Z, K) \arrow[u, "f^*"'] \end{tikzcd}$$

Proof: This follows immediately from Theorem 2.17. □

2.2.2. *Smooth schemes over a regular base.* Let $S \in \mathcal{R}eg$ be a connected scheme. We denote by $\mathcal{S}m/S$ the full subcategory of the category of schemes over S whose objects are smooth and equidimensional over S. In particular, every object of $\mathcal{S}m/S$ is also in $\mathcal{R}eg$. For every scheme $X \in \mathcal{S}m/S$, we define d_X as the relative dimension of the structural morphism of X over S.

Proposition 2.19. *Let $X \in \mathcal{S}m/S$. Let ω_X be the canonical sheaf of X, i.e. the maximum exterior power of the sheaf Ω^1_X of differentials of X (which is free because X is smooth over S). Then, $(\pi_X)^!(\mathcal{O}_S) \simeq T^{d_X}\omega_X$.*

Proof: By a theorem of Nagata [16], [17], [13] the structural morphism π_X admits a factorization $X \xrightarrow{i} \mathbf{P}^N_S \xrightarrow{\pi_{\mathbf{P}^N}} S$ for some suitable N, where i is an open immersion. Now the claim follows from [26, Theorem 3] (see also [11, p. 144]). □

We are therefore in the situation of Section 1.15, and the sheaf ω_X found there can be identified with the canonical sheaf ω_X for any scheme $X \in \mathcal{S}m/S$. We can define the category $(\mathcal{S}m/S)_*$ as $\mathcal{B}_*$ in Section 1.15.

Theorem 2.20. *Equation 2 of Section 1.15 defines a push-forward along any morphism f in $\mathcal{S}m/S$*

$$f_* : \mathrm{W}^{i+d_X}(X, \omega_X) \to \mathrm{W}^{i+d_Y}(Y, \omega_Y)$$

respecting composition, the projection formula and base change as in Theorems 1.104, 1.105 and 1.106.

2.3. **Witt motives.** We now define the category of (pure) Witt motives, inspired by [15]. In this section, we will drop certain canonical isomorphisms between dualizing objects from the notation. The isomorphisms of triangulated categories with dualities that are thus hidden are completely trivial, namely those arising from Lemma 1.64. This makes the proofs more synthetic and readable. The results and constructions hold in full generality, without those identifications. We also drop the derived notation in the tensor product as we work with line bundles anyway.

Let L, L', M, M' and N, N' be line bundles over X, Y and Z, respectively, and assume we have morphisms $p_{X/Z} : X \to Z$ and $p_{Y/Z} : Y \to Z$. Let $V = X \times_Z Y$ be the Cartesian product of X by Y over Z. We denote $L \boxtimes_N M$ the vector bundle $p^*_{V/X}(L) \otimes (p_{V/X} \circ p_{X/Z})^*(N) \otimes p^*_{V/Y}(M)$ over $X \times_Z Y$. When we write $L \boxtimes M$, we mean that Z is the point and that N is trivial. We therefore get a vector bundle over $X \times Y$. We identify

- $(L \boxtimes_N M) \otimes (L' \boxtimes_{N'} M') = (L \otimes L') \boxtimes_{N \otimes N'} (M \otimes M')$
- $\omega_{X \times_Z Y} = \omega_X \boxtimes_{\omega_Z^{-1}} \omega_Y$ (see Lemma 1.107)

When $f : X' \to X$ and $g : Y' \to Y$, we also identify

- $f^*(\mathcal{O}_X) = \mathcal{O}_{X'}$
- $(f \times g)^*(L \boxtimes M) = f^*L \boxtimes g^*M$.

Finally, we denote by L^{-1} the dual line bundle $\mathrm{RHom}(L, \mathcal{O}_X)$ of L and we identify

- $L^{-1} \otimes L = \mathcal{O}_X$ (see Proposition 1.86).

Definition 2.21. Let $\mathcal{P}Sm/S$ be the full subcategory of $\mathcal{S}m/S$ whose objects are proper over S. As at the beginning of Section 1.13, let $(\mathcal{P}Sm/S)^*$ be the category of couples (X, L) with X an object of $\mathcal{P}Sm/S$ and L a dualizing complex on X (which is isomorphic to a shifted line bundle by Proposition 2.10). By definition, the category $\mathbf{W}PSm/S$ has the same objects as $(\mathcal{P}Sm/S)^*$, and the set of morphisms (called W-*correspondences*) between two objects is the Abelian group defined by

$$\mathrm{Hom}_{\mathbf{W}PSm/S}((X, L), (Y, M)) = \mathrm{W}^{d_X}(X \times Y, (\omega_X \otimes L^{-1}) \boxtimes M).$$

For $a \in \mathrm{Hom}_{\mathbf{W}PSm/S}((X, L), (Y, M))$ and $b \in \mathrm{Hom}_{\mathbf{W}PSm/S}((Y, M), (Z, N))$ the composition ba is defined as

$$(\pi_{XZ}, Id_{L^{-1} \boxtimes \mathcal{O}_Y \boxtimes (\omega_Z^{-1} \otimes N)})_*$$
$$((\pi_{YZ}, Id_{\mathcal{O}_X \boxtimes (\omega_Y \otimes M^{-1}) \boxtimes N})^*(b).(\pi_{XY}, Id_{(\omega_X \otimes L^{-1}) \boxtimes M \boxtimes \mathcal{O}_Z})^*(a)).$$

Proposition 2.22. *The above composition law in $\mathbf{W}PSm/S$ is associative and any object admits an identity automorphism, so $\mathbf{W}PSm/S$ really is a category.*

Proof: The proof of associativity is the usual proof of the associativity of correspondences, as in [15, §2, Lemma p. 446]. It just uses the composition of the pull-backs and push-forwards, the base change and the projection formula. The identity of (X, L) is given by $(\Delta_X, Id_{\omega_X^{-1}})_*(1_X)$ (recall that 1_X is the class in $W_0(X, \mathcal{O}_X)$ of the one dimensional canonical form on $\mathcal{O}_X$ described in Lemma 1.84). Again, the proof that it is an identity is a generalization of the classical one. In fact, it is a particular case of the existence of graphs (see Proposition 2.23 below). □

We can now construct the graph functor.

Proposition 2.23. *There is a contravariant functor Γ from the category $(\mathcal{P}Sm/S)^*$ to $\mathbf{W}PSm/S$. It is the identity on objects, and it sends a morphism $(f, \phi) : (X, L) \to (Y, M)$ to $(\gamma_f, (\phi^\vee)^{-1} \otimes Id_L \otimes Id_{\omega_X^{-1}})_*(1_X) \in \mathrm{W}^{d_Y}(Y \times X, (\omega_Y \otimes M^{-1}) \boxtimes L) = \mathrm{Hom}((Y, M), (X, L))$, where $\gamma_f : X \to Y \times X$ is the graph morphism (it is always proper as all considered varieties are separated). By $\phi^\vee$, we mean the morphism dual to ϕ, going from L^{-1} to $f^*(M)^{-1}$.*

Proof: This functor respects the composition. This follows from standard arguments, as in [15, §2, Proposition p. 447]. □

We now define a realization functor to the category of graded Abelian groups.

Definition 2.24. We define the covariant functor R from $\mathbf{W}PSm/S$ to the category of Abelian groups by setting $R(X,L) = \mathrm{W}^0(X,L)$ and

$$R(c) = (x \mapsto (p_Y)_*(p_X^*(x).c))$$

for an element $c \in \mathrm{Hom}((X,L),(Y,M))$.

Remark 2.25. The functor R respects the composition because it coincides with the functor $\mathrm{Hom}(\mathrm{Pt},-)$, where Pt is the base scheme S equipped with the trivial line bundle. In particular, any motivic isomorphism induces an isomorphism on the realizations. Observe also that the composition $R \circ \Gamma$ sends a morphism (f,ϕ) to $(f,\phi)^*$.

We can define a monoidal structure on this category by setting $(X,L)\otimes(Y,M) = (X\times Y, L\boxtimes M)$. It is rigid, and the dual of an object (X,L) is given by $(X,L)^\vee = (X, \omega_X \otimes T^{-d_X}L^{-1})$.

Remark 2.26. Let $T^i(X,L)$ denote the object (X,T^iL), for $i \in \mathbf{Z}$. We have $R(T^i(X,L)) \simeq \mathrm{W}^i(X,L)$. It is interesting to note that $T\mathrm{Pt}$ behaves like a Tate object in the category $\mathbf{W}PSm/S$, *i.e.*, tensoring with it shifts the degree of the Witt groups, since $T\mathrm{Pt}\otimes(X,L) \simeq T(X,L)$. Its inverse $T^{-1}\mathrm{Pt}$ also lies in $\mathbf{W}PSm/S$.

2.4. **Effective Witt Motives.** We now define the category $(\mathbf{W}PSm/S)_{eff}$ of effective Witt motives. It is just the pseudo-Abelianized completion of the previous category. For a definition of the pseudo-Abelian completion, see e. g. [15, §5]. Recall that the objects are just pairs $((X,L),p)$ where p is an idempotent in $\mathrm{End}(X,L)$ and the morphisms between $((X,L),p)$ and $((Y,L),q)$ are given by the quotient of the subgroup of $\mathrm{Hom}_{\mathbf{W}PSm/S}((X,L),(Y,L))$ given by the elements f such that $fp = qf$ by the subgroup of elements f such that $fp = qf = 0$. It contains $\mathbf{W}PSm/S$ as the full subcategory of objects for which $p = Id$.

Remark 2.27. We can extend the realization functor R to $(\mathbf{W}PSm/S)_{eff}$ because of the universal property of the pseudo-Abelian completion. More precisely, we set $R((X,L),p) = \ker R(p)$ on objects.

3. Dévissage

Assume that $f : Z \to X$ is a closed embedding of equidimensional varieties in $\mathcal{R}eg$ of codimension d and L is a line bundle on X. Then by Theorem 1.75

$$\{Rf_*, r_L\} : D^b_c(Z)_{f^!L} \to D^b_c(X)_L$$

is a functor of triangulated categories with duality. The map $\{Rf_*, r_L\}$ obviously factors through the full triangulated subcategory with duality $D^b_{c,Z}(X)_{f^!L}$ which by definition consists of complexes whose homology has support on Z. We denote its Witt groups by $W^*_Z(X,L)$. The goal of this section is to prove the following dévissage theorem for Witt groups.

Theorem 3.1. *(dévissage) In the above situation, the map*

$$f_* : W^*(Z, f^!L) \to W^*_Z(X,L)$$

induced by the functor of triangulated categories with duality

$$\{Rf_*, r_L\} : D^b_c(Z)_{f^!L} \to D^b_{c,Z}(X)_L$$

is an isomorphism.

Proof: We roughly follow the classical strategy (as for K-theory) as it is developed for Witt groups in [7, section 4]. First one replaces [7, Theorem 4.2] by Theorem 2.15. Then one considers the corresponding two long exact sequences arising from the filtration by the codimension of support as in [7, p. 130]. Of course, one needs to twist correctly the dualities (by L for X and $f^!L$ for Z) in this sequence, as well as everywhere else, but these twists don't change anything in the proof. We are thus reduced to show the claim on the top of page 131 of *loc. cit.* with B/J and B replaced by Z and X. Replace [7, Lemma 4.3] by Theorem 1.106 (which applies as closed embeddings are proper and localizations are flat). We therefore obtain the following commutative diagram

$$\begin{array}{ccc} W^{p+q}((D^b_c(Z)^{(p)}/D^b_c(Z)^{(p+1)})_{f^!L}) & \xrightarrow{\gamma_Z} & \oplus_{x\in Z^{(p)}} W^{p+q}_{m_x}(O_{Z,x}, f^!L) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \oplus(f_x)_*} \\ W^{p+q}(D^b_{c,Z}(X)^{(p+d)}/D^b_{c,Z}(X)^{(p+d+1)}, *_L) & \xrightarrow{\gamma_X} & \oplus_{x\in Z\cap X^{(p+d)}} W^{p+q}_{m_x}(O_{X,x}, L) \end{array}$$

which is similar to the one of [7, 4.2.3]. In particular, $D^b(X)^{(p)}$ denotes complexes with cohomology support in codimension $\geq p$ and $X^{(p)}$ denotes points in codimension p. The maximal ideals of the regular local rings $O_{X,x}$ and $O_{Z,x}$ are both denoted by m_x (observe that $k := O_{X,x}/m_x \cong O_{Z,x}/m_x$) and γ_X and γ_Z are induced by localization. They are isomorphisms by [7, Theorem 3.12], so we are reduced to show that $\oplus(f_x)_*$ is an isomorphism. This follows from Lemma 3.2 below applied to $R = O_{Z,x}$ and to $R = O_{X,x}$ and the closed embeddings $g : Spec(k) \to Spec(O_{Z,x})$ and $(f_x \circ g) : Spec(k) \to Spec(O_{X,x})$. Indeed, this shows that $f_{x,*}$ fits in a commutative triangle of transfer morphisms

$$\begin{array}{ccc} W^{p+q}(k, (f_x\circ g)^!L) & \xrightarrow{g_*} & W^{p+q}_{m_x}(O_{Z,x}, f_x^!L) \\ & \searrow^{(f_x\circ g)_*} & \downarrow{\scriptstyle (f_x)_*} \\ & & W^{p+q}_{m_x}(O_{X,x}, L) \end{array}$$

in which the two other morphisms are isomorphisms. Observe that we can't apply [7, Lemma 4.4] directly as the transfer morphism of *loc. cit.* is not obviously the same as ours. □

Lemma 3.2. *Let $R \in \mathcal{R}eg$ be a local ring of dimension d with maximal ideal m and residue field $R/m = k$ and L a line bundle (i. e., a free module of rank one) on $Spec(R)$. Then the transfer morphism with respect to the closed embedding $h : Spec(k) \to Spec(R)$ induces an isomorphism*

$$h_* : W^*(k, h^!L) \xrightarrow{\cong} W^*_m(R, L).$$

Proof: We have full inclusions of categories $M^{ss}_{fl}(R) \subset M_{fl}(R) \subset M(R)$, where $M(R)$ is the category of R-modules of finite type, $M_{fl}(R)$ the subcategory of objects of finite length, and $M^{ss}_{fl}(R)$ those that are semi-simple (*i.e.* a finite direct sum of objects of length 0, *i.e.* finite dimensional vector spaces over k). The morphism of Witt groups Rh_* is given (up to the canonical equivalences $D^b(M(k)) \xrightarrow{\cong} D^b_c(Spec(k))$ and $D^b_m(M(R)) \xrightarrow{\cong} D^b_{c,m}(Spec(R))$) by the following composition of morphisms of triangulated categories with duality:

$$D^b(M(k), \mathrm{RHom}(-, h^!L)) \xrightarrow{(Rh_*, r_L)} D^b(M^{ss}_{fl}(R), \mathrm{RHom}(-, L)) \xrightarrow{i_1}$$

$$D^b(M_{fl}(R), \mathrm{RHom}(-, L)) \xrightarrow{i_2} D^b_m(M(R), \mathrm{RHom}(-, L))$$

where i_1 and i_2 are the obvious inclusions. Here we have used that the duality $\mathrm{RHom}(-,L)$ restricts to $D^b(M_{fl}(R))$ and to $D^b(M^{ss}_{fl}(R))$ by [7, Lemma 3.8]. Note that the duality $\mathcal{X}_{fg}$ in *loc. cit.* is by definition the same as $\mathrm{RHom}(-,L)$ as injective resolutions are unique up to unique chain homotopy equivalence. (The signs in the bidual isomorphism are the same by the choices made in Appendix A.) Thus the morphism i_2 is an isomorphism because it coincides with the one in [7, Lemma 3.9]. The morphism h_* is an isomorphism already on the underlying triangulated (and even exact) categories. We are left with i_1, for which we use the following:

Theorem 3.3. *Let $(\mathcal{A}, D, \varpi)$ be an Abelian category with exact duality D, and let $\mathcal{A}^{ss}$ be the full subcategory of semi-simple objects. Then D restricts to $\mathcal{A}^{ss}$, and the inclusion $\mathcal{A}^{ss} \subset \mathcal{A}$ induces an isomorphism on the i-th Witt group of the derived categories (with the induced duality) for all i.*

Proof: The result follows from the classical dévissage theorem of Quebbemann-Scharlau-Schulte [5, Proposition 5.1] for the even Witt groups, since the "derived" Witt groups coincide with the classical ones by [4, Theorem 4.3]. The odd Witt groups are zero in both cases by [5, Proposition 5.2], so there is nothing to prove. □

To apply this in our context, up to renumbering of the Witt groups, it suffices to show that $\mathrm{RHom}(-,T^dL)$ (and the bidual isomorphism) is induced by a duality on the underlying Abelian categories. This follows from [7, Theorem 3.10]. □

We write $j : X - Z \to X$ for the open inclusion of the complement. As usual, dévissage implies (or improves) a localization exact sequence.

Corollary 3.4. *(localization) In the above situation, we have a long exact sequence*

$$\cdots \xrightarrow{\partial} W^n(Z, f^!L) \xrightarrow{f_*} W^n(X,L) \xrightarrow{j^*} W^n(X-Z, j^*L) \xrightarrow{\partial} W^{n+1}(Z, f^!L) \to \cdots$$

Proof: By definition resp. construction, we have a short exact sequence of triangulated categories with dualities $D^b_{c,Z}(X)_L \longrightarrow D^b_c(X)_L \xrightarrow{j^*} D^b_c(X-Z)_{j^*L}$. Hence Balmer's abstract localization theorem [3] and our dévissage theorem yield the claim. □

4. Cellular varieties

This section provides the computation of Witt groups of cellular varieties, that is varieties admitting a filtration as in Definition 4.1 below. The proof requires the transfer maps and the dévissage theorem established in the previous sections. Examples of cellular varieties include projective homogeneous varieties G/P where G is a linear algebraic group (not necessarily split) and P a parabolic subgroup (see [21, 4.3] and the references given there). Our proof is inspired by the one of [21], and we will assume that the reader who tries to understand the proof below has a copy of their article at hand. (That it might be possible to combine their methods with ours to obtain such computations was already mentioned at the beginning of section 7 of *loc. cit.*.)

Definition 4.1. A *cellular decomposition* of a smooth projective variety X over a field k is a filtration $X = X_0 \supset X_1 \supset ...X_i \supset X_{i+1}... \supset X_n \supset 0$ by closed subvarieties such that each complement $E_i := X_i - X_{i+1}$ is the total space of an affine fibration $p_i : E_i \to Y_i$ where Y_i is a smooth projective variety.

Theorem 4.2. *Let X be a smooth projective variety over k with a cellular decomposition as in Definition 4.1 and assume that $char(k) = 0$. Let further L be a dualizing complex on X. Assume that on each Y_i there is a dualizing complex M_i*

such that $\bar{p_i}^* M_i \cong (j_i f_i)^! L$ *where* $j_i f_i$ *and* $\bar{p}_i$ *are the morphisms introduced in the proof below. Then we have an isomorphism of Witt groups*

$$W^*(X, L) \cong \oplus_{i=0}^n W^*(Y_i, M_i).$$

Proof: Let $h_i : E_i \to X - X_{i+1}$ and $u_i : X - X_{i+1} \to X$ be the obvious embeddings. Using localization for (coherent) Witt groups with supports [8, Theorem 2.4.b)] (the proof given there works for other dualizing complexes than the structural sheaf), we obtain a long exact sequence $... \to W^*_{X_{i+1}}(X, L) \to W^*_{X_i}(X, L) \xrightarrow{u_i^*} W^*(X - X_{i+1}, u_i^* L) \to ...$ Applying the dévissage Theorem 3.1 and strong homotopy invariance [8, Theorem 4.2] (which again holds for dualizing complexes in general) one deduces similar to Step I of the proof of [21, Theorem 4.4] a long exact sequence $... \to W^*_{X_{i+1}}(X, L) \to W^*_{X_i}(X, L) \xrightarrow{\alpha_i} W^*(Y_i, M_i) \to ...$ where $\alpha_i := (p_i^*)^{-1}(h_i)_*^{-1} u_i^*$. As in *loc. cit.*, if we can show that each α_i admits a splitting Θ_i, then the claim follows. For this, consider the following diagram consisting of two Cartesian squares and a commutative triangle as in Step II of *loc. cit.*

$$\begin{tikzcd}
X & X - X_{i+1} \arrow[l, "u_i"'] \\
X_i \arrow[u, "j_i"] & E_i \arrow[l, "q_i"] \arrow[u, "h_i"'] \\
W_i \arrow[u, "f_i"] \arrow[dr, "\bar{p}_i"'] & E_i \arrow[l, "g_i"'] \arrow[u, "id"'] \arrow[d, "p_i"] \\
& Y_i
\end{tikzcd}$$

in which all varieties except possibly X_i are smooth. From Corollary 2.18 we deduce a commutative diagram

$$\begin{tikzcd}
W^*_{X_i}(X, L) \arrow[r, "u_i^*"] & W^*_{E_i}(X - X_{i+1}, u_i^* L) \\
W^*(W_i, (j_i f_i)^! L) \arrow[u, "(j_i f_i)_*"] \arrow[r, "g_i^*"] & W^*(E_i, g_i^*(j_i f_i)^! L) \arrow[u, "\simeq"', "\{h_{i*}, \gamma\}"] \\
& W^*(Y_i, M_i) \arrow[ul, "\bar{p_i}^*"] \arrow[u, "\simeq"', "p_i^*"]
\end{tikzcd}$$

which shows that $\Theta_i := (j_i f_i)_* \bar{p_i}^*$ yields the desired splitting. □

Given a concrete example of a cellular variety, the twists and shifts of the duality arising from $f_i^!$ may be computed using Proposition 2.8 Point 4 and Proposition 2.19. Observe also that Theorem 4.2 may be rephrased as a decomposition of the Witt motive of X in $\mathbf{W}PSm/k$ similar to [21, section 5].

Applying Theorem 4.2 it is possible to recover certain computations about projective spaces and Nenashev's [19] computations of Witt groups of completely split quadrics. Observe also that in many cases the W_i may be constructed without assuming resolution of singularities (that is $char(k) = 0$), compare [21, section 7].

Appendix A. Signs in the category of complexes

In this section, we explain how signs in the category of complexes of an Abelian category $\mathcal{A}$ with a tensor product $\bullet$ adjoint to an internal Hom (denoted by h)

have to be chosen in order to obtain an exact tensor product adjoint to an internal Hom on the derived category. We use homological complexes, as it is the usual convention in the articles about Witt groups, so the differential of a complex is

$$d_i^A : A_i \to A_{i-1}.$$

The suspension functor T translates as in

$$(TA)_n = A_{n-1}.$$

The groups in the tensor product and the internal Hom are given by

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \bullet B_j$$

and

$$[A, B]_n = \prod_{j-i=n} h(A_i, B_j).$$

In table 1, we give a possible choice of signs for the translation functor, tensor product, the symmetry functor and the adjunction morphism (denoted by ath), and what it induces on the internal Hom using Proposition 1.39. Balmer, Gille and

Definition of	Sign	Choice	Locus
TA	ϵ_i^T	-1	$d_{i+1}^{TA} = \epsilon_i^T d_i A$
$A \otimes B$	$\epsilon_{i,j}^{1\otimes}$ $\epsilon_{i,j}^{2\otimes}$	1 $(-1)^i$	$\epsilon_{i,j}^{1\otimes} d_i^A \bullet id_{B_j}$ $\epsilon_{i,j}^{2\otimes} id_{A_i} \bullet d_j^B$
$tp_{1,A,B}$	$\epsilon_{i,j}^{tp1}$	1	$\epsilon_{i,j}^{tp1} id_{A_i \bullet B_j}$
$tp_{2,A,B}$	$\epsilon_{i,j}^{tp2}$	$(-1)^i$	$\epsilon_{i,j}^{tp2} id_{A_i \bullet B_j}$
$c_{A,B}$	$\epsilon_{i,j}^{c}$	$(-1)^{ij}$	$\epsilon_{i,j}^{c}(A_i \bullet B_j \to B_j \bullet A_i)$
$ath_{A,B,C}$	$\epsilon_{i,j}^{ath}$	$(-1)^{i(i-1)/2}$	$\epsilon_{i,j}^{ath}(\mathrm{Hom}(A_i \bullet B_j, C_{i+j})$ $\to \mathrm{Hom}(A_i, h(B_j, C_{i+j})))$
$[A, B]$	$\epsilon_{i,j}^{1h}$ $\epsilon_{i,j}^{2h}$	1 $(-1)^{i+j+1}$	$\epsilon_{i,j}^{1h}(d_{i+1}^A)^\sharp$ $\epsilon_{i,j}^{2h}(d_j^B)_\sharp$
$th_{1,A,B}$	$\epsilon_{i,j}^{th1}$	1	$\epsilon_{i,j}^{th1} id_{h(A_i,B_j)}$
$th_{2,A,B}$	$\epsilon_{i,j}^{th2}$	$(-1)^{i+j}$	$\epsilon_{i,j}^{th2} id_{h(A_i,B_j)}$

TABLE 1. Sign definitions

Nenashev [3], [4],[7], [10] always consider strict dualities, that is $\epsilon^{th1} = 1$. The signs chosen in [4, §2.6] imply that $\epsilon_{i,0}^{1h} = 1$. The choices made by [10, Example 1.4] are $\epsilon_{i,j}^{1\otimes} = 1$ and $\epsilon_{i,j}^{2\otimes} = (-1)^i$. In [7, p. 111] the signs $\epsilon_{i,j}^{1h} = 1$ and $\epsilon_{i,j}^{2h} = (-1)^{i+j+1}$ are chosen. Finally, the sign chosen for ϖ in [7, p. 112] corresponds via our definition of ϖ (see Section 1.8) to the equality $\epsilon_{j-i,i}^{ath}\epsilon_{i,j-i}^{ath}\epsilon_{j-i,i}^{c} = (-1)^{j(j-1)/2}$. It is possible to choose the signs in a way compatible with all these choices and our formalism. It is given in the third column of Table 1. More precisely, we have the following theorem.

Theorem A.1. *Let $a, b \in \{+1, -1\}$. Then*

$$\begin{array}{ll} \epsilon^{1\otimes}_{i,j} = 1 & \epsilon^{tp1}_{i,j} = a \\ \epsilon^{2\otimes}_{i,j} = (-1)^i & \epsilon^{tp2}_{i,j} = a(-1)^i \\ \epsilon^{1h}_{i,j} = 1 & \epsilon^{th1}_{i,j} = 1 \\ \epsilon^{2h}_{i,j} = (-1)^{i+j+1} & \epsilon^{th2}_{i,j} = a(-1)^{i+j} \\ \epsilon^{ath}_{i,j} = b(-1)^{i(i-1)/2} & \epsilon^{c}_{i,j} = (-1)^{ij} \\ \epsilon^{T}_{i} = -1 & \end{array}$$

satisfies all equalities of Table 2 as well as $\epsilon^{ath}_{j-i,i}\epsilon^{ath}_{i,j-i}\epsilon^{c}_{j-i,i} = (-1)^{j(j-1)/2}$. Therefore, for any exact category $\mathcal{E}$ the category of chain complexes $Ch(\mathcal{E})$ and its bounded variant $Ch^b(\mathcal{E})$ may be equipped with the the entire structure discussed in the first seven subsections of the first section. Moreover, all signs may be chosen in a compatible way with all the above sign choices of Balmer, Gille and Nenashev.

Proof: Straightforward. □

In the following table 2, we state the compatibility that these signs must satisfy for that all axioms considered in section 1 become true. We assume that the associativity isomorphism in the monoidal category of complexes comes without sign, so the pentagon of [14, p. 162] trivially commutes.

	compatibility	reason
1	$\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j-1}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i-1,j} = -1$	$A \otimes B$ is a complex
2	$\epsilon^{1h}_{i,j}\epsilon^{1h}_{i,j-1}\epsilon^{2h}_{i,j}\epsilon^{2h}_{i+1,j} = -1$	$[A, B]$ is a complex
3	$\epsilon^{T}_{i}\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i-1,j} = 1$	$tp_{1,A,B}$ is a morphism
4	$\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j-1} = 1$	
5	$\epsilon^{T}_{j}\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i,j-1} = 1$	$tp_{2,A,B}$ is a morphism
6	$\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i-1,j} = 1$	
7	$\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i+1,j} = -1$	the square in Definition 1.31 anti-commutes
8	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{i,j}\epsilon^{1h}_{j-1,i+j-1}\epsilon^{ath}_{i,j-1}\epsilon^{ath}_{i-1,j} = -1$	ath is well defined
9	$\epsilon^{1\otimes}_{i,j}\epsilon^{2h}_{j,i+j}\epsilon^{ath}_{i-1,j}\epsilon^{ath}_{i,j} = 1$	
10	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i-1,j} = 1$	$c_{A,B}$ is a morphism
11	$\epsilon^{1\otimes}_{j,i}\epsilon^{2\otimes}_{i,j}\epsilon^{c}_{i,j}\epsilon^{c}_{i,j-1} = 1$	
12	$\epsilon^{c}_{i,j}\epsilon^{c}_{j,i} = 1$	c is self-inverse
13	$\epsilon^{c}_{j,k}\epsilon^{c}_{i,k} = \epsilon^{c}_{i+j,k}$	the square $\boxed{sm}$ commutes
14	$\epsilon^{tp1}_{i,j}\epsilon^{tp2}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i+1,j} = 1$	the square $\boxed{c}$ commutes

TABLE 2. Sign definitions

If the Abelian (or more generally exact) category $\mathcal{E}$ one considers has enough injective and projective objects, one obtains a left derived functor of the tensor product and a right derived functor of the internal Hom. These are studied in further detail in section 2.

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